PS Advanced probability, SS 2015, sheet 5

28. **Lebesgue's density theorem** is a classical (and very useful) result in real analysis. A point $x \in (0,1]$ is said to be a *density point of* $A \in \mathcal{B}_{(0,1]}$ if $\lim_{\delta \searrow 0} \lambda((x-\delta/2, x+\delta/2] \cap A)/\delta = 1$. Let $D(A)$ denote the set of all such points. Lebesgue's theorem asserts that up to a set of measure zero, $D(A)$ coincides with A, so that $1_A = 1_{D(A)}$ a.s.

    a) Prove Lebesgue's theorem using the "Fundamental Theorem of Calculus" for the Lebesgue integral (take an F which has derivative 1_A).
    b) For $n \geq 1$ let Π_n be a (finite or) countable partition of $(0,1]$ into subintervals. Assume that Π_{n+1} refines Π_n. Show that $\sigma(\Pi_1, \Pi_2, \ldots) = \mathcal{B}_{(0,1]}$.
    c) For $(\Pi_n)_{n\geq 1}$ as in b) let $\Pi_n(x)$ be the element of Π_n containing the point x. Use the upward convergence theorem for conditional expectations to prove the following variant of the density theorem: For a.e. x, $\lim_{n\to\infty} \lambda(\Pi_n(x) \cap A)/\lambda(\Pi_n(x)) = 1_A(x)$.

29. **Ergodicity of irrational rotations.** Let $(\Omega, \mathcal{A}, \mathrm{P}) = ((0,1], \mathcal{B}_{(0,1]}, \lambda)$, fix some $\alpha \in (0,1]$ and consider the map $T : (0,1] \to (0,1]$ with $Tx := x + \alpha$ (mod 1). If Ω is identified with the circle $\mathbb{S}^1$ ($x \leftrightarrow e^{2\pi i x}$), then T simply rotates $\mathbb{S}^1$ by an angle $2\pi\alpha$ (so that $e^{2\pi i x} \mapsto e^{2\pi i (x+\alpha)}$). Therefore T preserves the measure, $\lambda = \lambda \circ T^{-1}$.

    Assume now that α *is irrational*. We claim that in this case T is *ergodic*: If $A \in \mathcal{B}_{(0,1]}$ is T-invariant, $A = T^{-1}A$, and nontrivial, $\lambda(A) > 0$, then it has to be almost all of $(0,1]$ in that $\lambda(A) = 1$.

    a) Recall Kronecker's theorem which ensures that for every x the orbit $(T^n x)_{n\geq 0} = (x + n\alpha \text{ (mod 1)})_{n\geq 0}$ is dense in $(0,1]$.
    b) Fix some T-invariant set $A \in \mathcal{B}_{(0,1]}$ with $\lambda(A) > 0$, and any $\varepsilon > 0$. Since A has density points and we can find an interval B such that $\lambda(B) = 1/m$ (some $m \geq 1$) and $\lambda(B \cap A)/\lambda(B) > 1 - \varepsilon$ (see Exercise 28). The proportion of A in any image $T^n B$, that is $\lambda(T^n B \cap A)/\lambda(T^n B)$, coincides with $\lambda(B \cap A)/\lambda(B)$.
    c) Partition $(0,1]$ into subintervals $I_j = (j/m, (j+1)/m]$. Take any of them, $I = I_j$. By Kronecker's theorem, there is some n such that $T^n B$ is almost the same as I, $\lambda(I \cap T^n B)/\lambda(I) > 1 - \varepsilon$. Conclude that $\lambda(I \cap A)/\lambda(I)$ is large, and hence that $\lambda(A)$ is large.
    d) Formulate the strong law of large numbers for irrational rotations.

30. **Independence and conditional expectations.** Let $\mathcal{F}, \mathcal{G}$ be sub-σ-algebras in the proba space $(\Omega, \mathcal{A}, \mathrm{P})$, and X an integrable random variable. If $\mathcal{F}$ is independent of $\sigma(\mathcal{G}, \sigma(X))$, then

$$\mathbb{E}[X \mid \sigma(\mathcal{F}, \mathcal{G})] = \mathbb{E}[X \mid \mathcal{G}].$$

    (This was used in the martingale proof of the strong law.)

31. **A warning.** Let $\mathcal{F}$ and $\mathcal{G}_n$, $n \geq 0$, be sub-σ-algebras in the proba space $(\Omega, \mathcal{A}, \mathrm{P})$, with $(\mathcal{G}_n)_{n\geq 0}$ non-increasing. Then

$$\bigcap_{n\geq 0} \sigma(\mathcal{F}, \mathcal{G}_n) = \sigma\left(\mathcal{F}, \bigcap_{n\geq 0} \mathcal{G}_n\right)$$

need not be true! Scary, isn't it?

Hint: Let $(Y_n)_{n\geq 0}$ be iid coins, $\mathrm{P}[Y_n = \pm 1] = 1/2$, and $X_n := Y_0 Y_1 \cdots Y_n$, $n \geq 1$. Set $F := \sigma(Y_1, Y_2, \ldots)$ and $\mathcal{G}_n := \sigma(X_{n+1}, X_{n+2}, \ldots)$ for $n \geq 0$. Check that $(X_n)_{n\geq 1}$ is an independent sequence, and show that Y_0 is measurable for $\bigcap_{n\geq 0} \sigma(\mathcal{F}, \mathcal{G}_n)$, but not for $\sigma(\mathcal{F}, \bigcap_{n\geq 0} \mathcal{G}_n)$.

32. **More martingales?** Let $(Y_n)_{n\geq 1}$ be iid variables with $\mathbb{E}[Y_1] = 0$, set $X_0 := 0$ and $X_n := Y_1 + \ldots + Y_n$. We know that $(X_n)_{n\geq 0}$ is a martingale for the filtration with $\mathcal{F}_n := \sigma(Y_1, \ldots, Y_n)$.

    Assume now that $\mathbb{E}[Y_1^2] =: \sigma^2 < \infty$, and let $M_n := X_n^2 - n\sigma^2$, $n \geq 0$. Is $(M_n)_{n\geq 0}$ a martingale for $(\mathcal{F}_n)$? What can be said about $(X_n^2)_{n\geq 0}$?

33. **Doob's decomposition theorem** states that a sequence (X_n) of integrable random variables is a submartingale (for some $(\mathcal{F}_n)$) iff it can be represented as $X_n = M_n + A_n$ where (M_n) is a martingale and (A_n) is previsible with $0 = A_0 \leq A_1 \leq \ldots$ Check this by trying (A_n) with $A_{n+1} - A_n = \mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]$. What about uniqueness?

34. **Lévy's distance.** For probability distribution functions F, G define

$$\mathrm{d}(F, G) := \inf\{\varepsilon > 0 : G(t - \varepsilon) - \varepsilon \leq F(t) \leq G(t + \varepsilon) + \varepsilon\}.$$

Does this define a metric? Show that $F_n \Rightarrow F$ iff $\mathrm{d}(F_n, F) \to 0$.