Flexibility of CLT in ergodic theory

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based on joint work with C. Dong, D. Dolgopyat and A. Kanigowski

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Flexibility of statistical properties

T, T^{-1} transformations

Proofs: CLT, zero entropy, $T/\ln^{1/4} T$ normalization

Proofs: other cases

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- Bernoulli: There exists A_0 (possibly with infinite range) so that A_n are iid and generate the σ-algebra.

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- PM / EM F mixes polynomially/exponentially (PM/EM) if for all $A_0, B_0 \in \mathcal{C}^r_0(M)$ the following holds with a polynomial/exponential function $\psi(n)$:

$$\text{Cov}|(A_0, B_n)| \leq \|A_0\|_{\mathcal{C}^r} \|B_0\|_{\mathcal{C}^r} \psi(n).$$

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	Erg	**WM/M**	**PE**	**K/B**	**CLT**	**PM**	**EM**
Erg	♣	(1)	(1)	(1)	(1)	(1)	(1)
WM/M	Y	♣	(2)	(2)	(5)	(5)	(5)
PE	(3)	(3)	♣	(3)	(3)	(3)	(3)
K/B	Y	Y	Y	♣	(5)	(5)	(5)
CLT	Y	(6)	(4)	(6)	♣	(6)	(6)
PM	Y	Y	(2)	(2)	(2)	♣	(2)
EM	Y	Y	??	??	??	Y	♣

(1) irrational rotation; (2) horocycle flow; (3) Anosov diffeo $\times$ identity; (4): new, see later; (5) skew products on $\mathbb{T}^2 \times \mathbb{T}^2$ of the form $(Ax, y + \alpha\tau(x))$ where A is linear Anosov map, α is Liouvillian and τ is not a coboundary; (6) Skew product of Anosov diffeo and Diophantine rotation.

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	WM	**M**	**PE**	**K**	**B**	**PM**
WM	♣	(8)	(9)	(9)	(9)	(10)
M	♣	♣	(9)	(9)	(9)	(10)
PE	(6)	(6)	♣	(6)	(6)	(6)
K	♣	♣	♣	♣	(7)	??
B	♣	♣	♣	♣	♣	??
PM	♣	♣	(9)	(9)	(9)	♣

Examples (1) - (6) as before. Examples (7) - (10) are new.

Main results

Theorem (Dong, Dolgopyat, Kanigowski, N. '20)

(i) *For each* $m \in \mathbb{N}$ *there exists an analytic diffeomorphism* F_m *which is mixing at rate* n^{-m} *but is not Bernoulli. Moreover,* F_m *is* K *and satisfies the classical CLT.* (7)

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(v) *There exists a polynomially mixing flow, which is not K and satisfies the classical CLT.* (9)

Flexibility of statistical properties

T, T^{-1} transformations

Proofs: CLT, zero entropy, $T / \ln^{1/4} T$ normalization

Proofs: other cases

Random walks in random scenery (RWRS)

Let $\xi_z, z \in \mathbb{Z}^d$ be bounded iid random variables with finite range. Let T_n be a simple random walk independent from ξ_z's. RWRS is

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Kesten, Spitzer '79, Bolthausen '89:

- $d = 1$: $S_N / N^{3/4}$ has a weak limit
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Heuristics ($d = 1$): Each site $k \asymp \sqrt{N}$ is visited $\asymp \sqrt{N}$ times. Thus $S_N \asymp \sqrt{N} \sum_{k=-\sqrt{N}}^{\sqrt{N}} \xi_k \asymp N^{3/4}$.

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- $X = \{-1, 1\}^{\mathbb{Z}}$, $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})^{\mathbb{Z}}$, $f : X \to X$ left shift.
- $\tau(x) = x(0)$
- $(Y, g, \nu) = (X, f, \mu)$

$F : X \times Y \to X \times Y$, $F(x, y) = (f(x), g^{\tau(x)}(y))$ preserves $\zeta = \mu \times \nu$.

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K/Bernoulli properties

Kalikow '82: $d = 1$: F is K but not Bernoulli.

den Hollander, Steif '97: F is Bernoulli iff $d \geq 3$.

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continuous T, T^{-1} transformations:

$$F_T(x, y) = (h_T(x), G_{\tau_T(x)}(y)) \quad \tau_T(x) = \int_0^T \tau(h_t(x))dt$$

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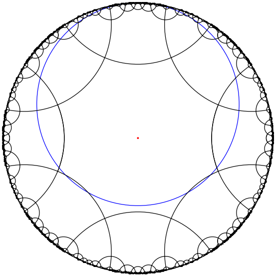

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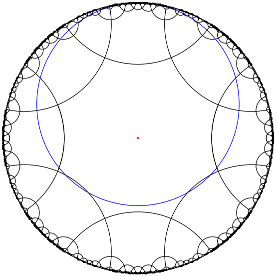

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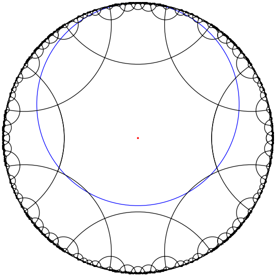

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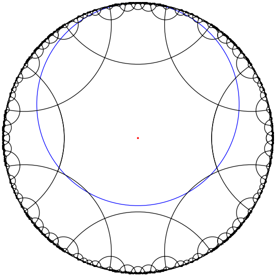

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Zero entropy follows from Abramov-Rokhlin formula.

Convergence of second moment

$H : X \times Y \to \mathbb{R}$ smooth, mean zero and $H_T = \int_0^T H \circ F_t dt$. Let us explain why $\zeta(H_T^2) \asymp T^2/\sqrt{\ln T}$.

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$\int_0^T 1_{\tau_t x=k} 1_{h_t\in A} dt \sim \frac{T}{\sqrt{\ln T}} \varphi\left(\frac{k-s_T(x)}{\sqrt{\ln T}}\right) \mu(A)$

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$$\approx C_H \sum_{\ell = -10^6}^{10^6} \frac{T^2}{\ln T} \varphi^2\left(\frac{\ell}{\sqrt{\ln T}}\right) \asymp C_H \frac{T^2}{\sqrt{\ln T}} \int_{\mathbb{R}} \varphi^2(z)dz.$$

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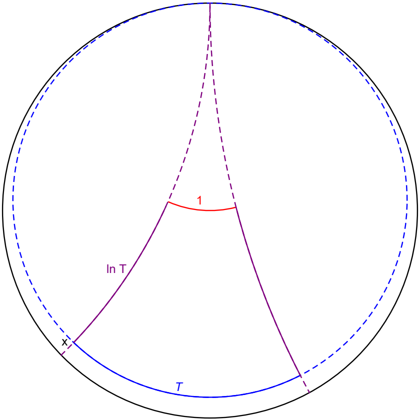

Figure: Temporal limit theorem for horocycle flow ≈ central limit theorem for the geodesic flow (Dolgopyat, Sarig'17)

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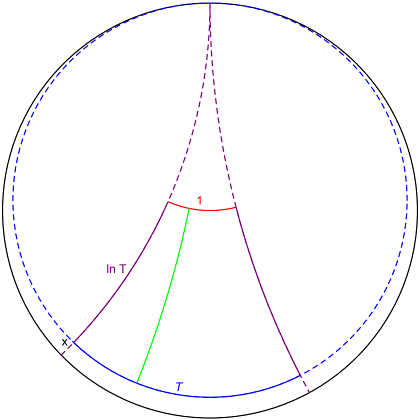

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Finishing the proof

- General observable H: write $H(x,y) = \hat{H}(x) + \tilde{H}(x,y)$, where $\int \tilde{H}(x,y)d\nu(y) = 0$ for all x. Then $\hat{H}_T = O(T^{<1})$ by Flaminio, Forni '03. For $\tilde{H}$ use exponential mixing of G.

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Remark: F_T cannot mix polynomially by the following lemma.

Finishing the proof

- General observable H: write $H(x,y) = \hat{H}(x) + \tilde{H}(x,y)$, where $\int \tilde{H}(x,y)d\nu(y) = 0$ for all x. Then $\hat{H}_T = O(T^{<1})$ by Flaminio, Forni '03. For $\tilde{H}$ use exponential mixing of G.
- Convergence of higher moment: as 2nd moment...
  Björklund, Gorodnik '20 CLT for exponentially mixing systems

Remark: F_T cannot mix polynomially by the following lemma.
Lemma: Let $X_1, ..., X_n$ be a stationary sequence of random variables with $|E(X_i X_j)| \le C|i-j|^{-\beta}$. and $S_N = \sum_{n=1}^N X_n$. Then $S_N / n^{\alpha+\varepsilon} \to 0$ almost surely, where

$$\alpha = \begin{cases} 1/2 & \text{if } \beta \ge 1 \\ 1-\beta/2 & \text{if } \beta < 1 \end{cases}$$

Flexibility of statistical properties

T, T^{-1} transformations

Proofs: CLT, zero entropy, $T/\ln^{1/4} T$ normalization

Proofs: other cases

Further choices

We always assume that G_t is mixing of all orders.
Canonical examples for $d \geq 2$:

1. $\mathbb{Z}^d$ action Cartan actions: ergodic actions of $\mathbb{Z}^d$ on $\mathbb{T}^{d+1}$ by hyperbolic automorphisms.
2. $\mathbb{R}^d$ action Weyl chamber flows: Action of the diagonal group by left translations on $SL(d+1, \mathbb{R})/\Gamma$, where Γ is a co-compact lattice in $SL(d+1, \mathbb{R})$.

Theorem (iii): $\mathcal{C}^r$ diffeo with zero entropy and classical CLT

Proposition

Suppose that $f : X \to X$ satsifies:

- D1 *Ergodic sums of all zero mean smooth observables on X grow slower than $N^{1/2}$.*
- D2

$$\mu(\| \sum_{n=1}^{N} \tau_n \| < \log^{1+\varepsilon} N) < \frac{C}{N^5}$$

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Proposition

Fix $\kappa, r, \mathbf{m}$ with $\kappa/2 < r < \mathbf{m}$. Then there is a $d \geq 0$ so that the following holds. Let $X = \mathbb{T}^{\mathbf{m}}$, $f(x) = x + \alpha$ where α is Diophantine (i.e. $|\langle k, \alpha\rangle| \geq D|k|^{-\kappa}$). Then D1 holds for all $A_0 \in \mathcal{C}^r(\mathbb{T}^{\mathbf{m}}, \mathbb{R})$ and D2 holds for some $\tau \in \mathcal{C}^r(\mathbb{T}^{\mathbf{m}}, \mathbb{R}^d)$.

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Difficult part: F is not Bernoulli (cf. symbolic actions, den Hollander - Steif)

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Theorem (v): Base: suspension over irrational rotation with polynomial singularities.

References:

1. D. Dolgopyat, C. Dong, A. Kanigowski, P.N., Flexibility of statistical properties for smooth systems satisfying the central limit theorem
   https://arxiv.org/abs/2006.02191
2. D. Dolgopyat, C. Dong, A. Kanigowski, P.N., Mixing properties of generalized T, T^{-1} transformations
   https://arxiv.org/abs/2004.07298