Limit theorems for unbounded observables via operator perturbation techniques

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Expansions in the Central limit theorems in the i.i.d. case

Let $r \geq 0$ and $\mathbb{X} = \mathbb{R}$ or $\mathbb{Z}$. Let $(X_k)_{k\geq 0}$ be a sequence of $\mathbb{X}$-valued centered independent identically distributed random variables. Assume $\mathbb{E}[|X_1|^{r+2}] < \infty$, $\sigma^2 := \mathbb{E}[X_1^2] > 0$. Set $S_n := \sum_{k=1}^n X_k$. Under additional technical assumptions:[1]

- *Edgeworth expansions of order r (Central Limit Theorem)*

$$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \leq x\right) = \int_{-\infty}^{x} \frac{e^{-\frac{t^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dt + \sum_{k=1}^{r} \frac{\mathcal{R}_k(x)}{n^{\frac{k}{2}}} + o(n^{-\frac{r}{2}})$$

- *Expansions of order r in the Local Limit Theorem*

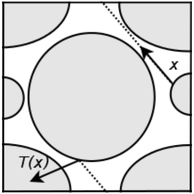
$$\mathbb{E}[g(S_n)] = \frac{I(g)}{\sqrt{2\pi\sigma^2\, n}} + \sum_{k=1}^{\lfloor r/2 \rfloor} \frac{a_k}{n^{\frac{1}{2}+k}} + o(n^{-\frac{r}{2}})$$

with $I(g) := \sum_{k\in\mathbb{Z}} g(k)$ if $\mathbb{X} = \mathbb{Z}$ (e.g. $g(k) = \mathbf{1}_{k=0}$)
and $I(g) := \int_{\mathbb{R}} g(t)\, dt$ if $\mathbb{X} = \mathbb{R}$ (e.g. $g \in \mathcal{S}$ Schwartz space).

[1] [Dolgopyat,Fernando], [Fernando,Liverani] [Breuillard]

Motivations

Probability preserving dynamical system $(\Omega, \mathbb{P}, T)$
E.g. Sinai billiard system with finite horizon
$\Omega = \{\text{post-collisional vectors}\}$



$f : \Omega \to \mathbb{X}$ (with $\mathbb{X} = \mathbb{R}, \mathbb{Z}$) centered (dynamically Hölder or Hölder$/d(\cdot, \partial\Omega)^\alpha$ for $\alpha > 0$ small). $S_n = \sum_{k=0}^{n-1} f \circ T^k$.

- *r order Edgeworth expansions (Central Limit Theorem)*

$$(\psi.\mathbb{P})\left(\frac{S_n}{\sqrt{n}} \leq x\right) = \int_{-\infty}^{x} \frac{e^{-\frac{t^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dt + \sum_{k=1}^{r} \frac{\mathcal{R}_k(x)}{n^{\frac{k}{2}}} + o(n^{-\frac{r}{2}})$$

- *r order Expansions in Mixing Local Limit Theorem (MLLT)*

$$\mathbb{E}[\psi.g(S_n).\xi \circ T^n] = \frac{I(g)\mathbb{E}[\psi]\mathbb{E}[\xi]}{\sqrt{2\pi\sigma^2\, n}} + \sum_{k=1}^{\lfloor r/2 \rfloor} \frac{a_k}{n^{\frac{1}{2}+k}} + o(n^{-\frac{r}{2}})$$

$I(g) := \sum_{k\in\mathbb{Z}} g(k)$ if $\mathbb{X} = \mathbb{Z}$ and $I(g) := \int_{\mathbb{R}} g(t)\, dt$ if $\mathbb{X} = \mathbb{R}$.

A general result based on Fourier Analysis (with K. Fernando, 2020)

$(\Omega, \mathbb{P}, T)$, $S_n = \sum_{k=1}^n f \circ T^k$.
Let $t \mapsto \lambda_t \in \mathbb{C}$ C^{r+2} with $\lambda_t = 1 - \frac{t^2\sigma^2}{2} + o(s^2)$.

- Assume $\boxed{H_n(t) := \lambda_t^{-n}\mathbb{E}(\psi e^{itS_n}\xi \circ T^n)}$ C^{r+2} satisfies: $\forall p > 0$

$$H_n^{(j)}(0) = B_j + \mathcal{O}(n^{-p}),$$

$$|\lambda_t^n H_n^{(j)}(t)| \le K(n^{\frac{1}{p}} e^{-\frac{\sigma^2 t^2}{8}} + n^{-p}) \quad \text{for } |t| < \delta .$$

- additional technical assumptions

Then:
r order Edgworth expansion for $(\psi\mathbb{P})(\frac{S_n}{\sqrt{n}} \le x)$
r order expansion in the MLLT for $\mathbb{E}[\psi g(S_n)\xi \circ T^n]$

Additional assumptions for the MLLT

- **Non lattice/Non arithmeticity**
  For all $0 < a < b < \infty$ if $\mathbb{X} = \mathbb{R}$ or $0 < a < b \le \pi$ if $\mathbb{X} = \mathbb{Z}$

  $$\forall p > 0, \quad \sup_{a<|t|<b} |\mathbb{E}(\psi e^{itS_n} \xi \circ T^n)| \le O(n^{-p}) .$$

  In the iid case: if X is not supported on a sublattice of $\mathbb{X}$, then $\sup_{a<|t|<b} |\mathbb{E}(e^{itX_1})| \le \rho < 1$, so $\mathbb{E}(e^{itS_n})| = O(\rho^n)$.
- Case $\mathbb{X} = \mathbb{Z}$: $\sum_{n\in\mathbb{Z}} |n|^{r+1}|g(n)| < \infty$.
- Case $\mathbb{X} = \mathbb{R}$:
  - (α, α_1)-diophantine Condition: For $|t| > K$,

    $$|\mathbb{E}(\psi e^{itS_n} \xi \circ T^n)| \le K_1(n^{-p} + |t|^{1+\alpha} e^{-c_0 n^{\alpha_1}|t|^{-\alpha}}) .$$

    In the iid case: $|\mathbb{E}[e^{itX_1}]| < 1 - \frac{C}{|t|^\alpha} \Rightarrow |\mathbb{E}(e^{itS_n})| < e^{-nC|t|^{-\alpha}}$.
    Dynamical case: Dolgopyat inequality, non existence of approximate eigenfunctions [Dolgopyat], [Baladi, Vallée], [Avila, Gouëzel, Yoccoz], [Melbourne]
  - $\int_{\mathbb{R}}(1 + |x|^{r+1})|g(x)|\, dx < \infty$ and g is C^{q+2} with integrable derivatives for some $q > \alpha(1 + \frac{r+1}{2\alpha_1})$.

Use of the transfer operator

- Let $S_n = \sum_{k=1}^n f \circ T^k$, $H_n(t) := \lambda_t^{-n}\mathbb{E}(\psi e^{isS_n}\xi \circ T^n)$.
- Goal: λ and H_n are C^{r+2}, $\lambda_t = 1 - \frac{t^2\sigma^2}{2} + o(t^2)$, control $H_n^{(j)}(t)$, $H_n^{(j)}(0) \approx B_j$
- P transfer operator: $\mathbb{E}[P(h).g] = \mathbb{E}[h.g \circ T]$.
  Set $P_t(h) = P(e^{itf}h)$. Then

$$\mathbb{E}[P_t(h).g] = \mathbb{E}[e^{itf}h.g \circ T]$$

$$\boxed{H_n(t) = \lambda_t^{-n}\mathbb{E}(\psi e^{itS_n}\xi \circ T^n) = \lambda_t^{-n}\mathbb{E}[\xi P_t^n(\psi)]}.$$

Use of the classical Nagaev-Guivarch's method

P transfer operator, $P_t h = P(e^{itf} h)$, $H_n(t) := \lambda_t^{-n}\mathbb{E}(\xi P_t^n(\psi))$

Goal: H_n and λ C^{r+2}, $\lambda_t = 1 - \frac{t^2\sigma^2}{2} + o(t^2)$, control $H_n^{(j)}(t)$ for small t, $H_n^{(j)}(0) \approx B_j$.

- Main idea: Assuming $P^n = \mathbb{E}[\cdot]\mathbf{1} + \mathcal{O}(\alpha^n)$ in $\mathcal{L}(\mathcal{B})$, $\alpha \in (0,1)$
  prove $P_t^n = \lambda_t^n \Pi_t + \mathcal{O}(\alpha_0^n)$ for t small
  $\Pi_0 = \mathbb{E}[\cdot]\mathbf{1}$, $\lambda_0 = 1$; λ, Π have same regularity as $t \mapsto P_t$
- Observe that $P_t^{(k)}(h) = i^k P(f^k h)$.
- Nagaev-Guivarch's method: If $P^n = \mathbb{E}[\cdot]\mathbf{1} + \mathcal{O}(\alpha^n)$ in $\mathcal{L}(\mathcal{B})$,
  with $\mathbf{1}, \psi \in (\mathcal{B}, \|\cdot\|)$ Banach space $\hookrightarrow L^p$ and $\alpha \in (0,1)$,
  if $t \mapsto P_t \in \mathcal{L}(\mathcal{B})$ is C^{r+2},
  Then $P_t^n = \lambda_t^n \Pi_t + \mathcal{O}(\alpha_0^n)$ in $\mathcal{L}(\mathcal{B})$ with $t \mapsto \Pi_t$, $t \mapsto \lambda_t$ C^{r+2}
  $$\text{So } \boxed{H_n^{(j)}(t) = \mathbb{E}[\xi \Pi_t^{(j)}(\psi)] + \mathcal{O}(\alpha_0^n)} \quad \text{if } \xi \in \mathcal{B}^*$$
- Examples: $\mathcal{B}$ set of Lipschitz functions, $f \in \mathcal{B}$, then
  $h \mapsto fh \in \mathcal{L}(\mathcal{B})$ and so $t \mapsto P_t \in \mathcal{L}(\mathcal{B})$ is C^∞.
- Problem: If $f \notin L^m$, $h \mapsto fh \notin \mathcal{L}(\mathcal{B})$
  $t \mapsto P_t \in \mathcal{L}(B)$ not C^1, often even not C^0

Nagaev-Guivarc'h method via the Keller Liverani theorem

P transfer operator, $P_t h = P(e^{itf} h)$, $H_n(t) := \lambda_t^{-n}\mathbb{E}(\xi P_t^n(\psi))$
Goal: H_n and λ C^{r+2}, $\lambda_t = 1 - \frac{t^2\sigma^2}{2} + o(t^2)$, control $H_n^{(j)}(t)$ for small t, $H_n^{(j)}(0) \approx B_j$.

Problematic case: $t \mapsto P_t \in \mathcal{L}(B)$ not C^0 (e.g. $f \notin L^m$)

- Keller-Liverani theorem: $P^n = \mathbb{E}[\cdot]\mathbf{1} + \mathcal{O}(\alpha^n)$ in $\mathcal{L}(\mathcal{B})$ with $\mathbf{1}, \psi \in (\mathcal{B}, \|\cdot\|) \hookrightarrow L^p$ and $\alpha \in (0,1)$, if $P_t \in \mathcal{L}(\mathcal{B}) \cap \mathcal{L}(L^p)$, $t \mapsto P_t \in \mathcal{L}(\mathcal{B} \to L^p)$ is C^0, $\|P_t^n h\| \le c\alpha_1^n \|h\| + c\|h\|_{L^p}$, and with $0 < \alpha < 1$, then $P_t^n = \lambda_t^n \Pi_t + \mathcal{O}(\alpha_0^n)$ in $\mathcal{L}(\mathcal{B})$; $t \mapsto \Pi_t \in \mathcal{L}(\mathcal{B} \to L^p)$ C^0
  So $\boxed{H_n(t) = \mathbb{E}[\xi \Pi_t(\psi)] + \mathcal{O}(\alpha_0^n)}$ if $\xi \in L^{\frac{p}{p-1}}$
- Derivatives?

Nagaev-Guivarc'h method via the Keller Liverani theorem

P transfer operator, $P_t h = P(e^{itf} h)$, $H_n(t) := \lambda_t^{-n}\mathbb{E}(\xi P_t^n(\psi))$
Goal: H_n and λ C^{r+2}, $\lambda_t = 1 - \frac{t^2\sigma^2}{2} + o(t^2)$, control $H_n^{(j)}(t)$ for small t, $H_n^{(j)}(0) \approx B_j$.

- Derivatives (with L. Hervé)
  Let $\mathcal{X}_0 \hookrightarrow \mathcal{X}_0^+ \hookrightarrow \ldots \hookrightarrow \mathcal{X}_{r+2} \hookrightarrow \mathcal{X}_{r+2}^+ \hookrightarrow L^p$ be such that
  - $P_t \in \bigcap_{a=0}^{r+2}(\mathcal{L}(\mathcal{X}_a) \cap \mathcal{L}(\mathcal{X}_a^+))$
  - $t \mapsto P_t \in \mathcal{L}(\mathcal{X}_a \to \mathcal{X}_a^+)$ is continuous
  - $j \geq 1$, $t \mapsto P_t \in \mathcal{L}(\mathcal{X}_a^+ \to \mathcal{X}_{a+j})$ is C^j, with $P_t^{(j)}(h) = iP_t(fh)$.
  - $\|P_t^n h\| \leq c\alpha_1^n\|h\| + c\|h\|_{L^p}$ is valid for $\mathcal{B} = \mathcal{X}_a$ and $\mathcal{B} = \mathcal{X}_a^+$.

  Then $(P_t^n)^{(j)} = (\lambda_t^n \Pi_t)^{(j)} + \mathcal{O}(\alpha_0^n)$ in $\mathcal{L}(\mathcal{X}_a \to \mathcal{X}_{a+j}^+)$,
  $t \mapsto \Pi_t \in \mathcal{L}(\mathcal{X}_a \to \mathcal{X}_{a+j}^+)$ is C^j, λ is C^{r+2},
  $\lambda_t = 1 - \frac{\sigma^2 t^2}{2} + o(t^2)$ with $\sigma^2 = \lim_{n\to+\infty} \mathbb{E}[(S_n)^2]/n$.
- if $\psi \in \mathcal{X}_0$, $\xi \in (\mathcal{X}_{r+2}^+)^*$, then $H_n(t) = \lambda_t^n \mathbb{E}[\xi P_t^n(\psi)]$ C^{r+2} and

$$\boxed{H_n^{(j)}(t) \approx \mathbb{E}[\xi \Pi_t^{(j)}(\psi)]}.$$

Applications to Young towers (with K. Fernando)

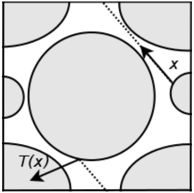
- Expanding Young tower $\Omega = \{(x,k) \in Y \times \mathbb{N} \,:\, k \le R(x)\}$
  $T(x,k) = (x, k+1)$ if $k < R(x)-1$, $\quad T(x, R(x)-1) = (F(x), 0)$
- Set $\ell(x,k) = k$. $\mathbb{E}[e^{\varepsilon_0 \ell}] = \sum_{k \ge 0} k \mathbb{P}(Y \cap \{R > k\}) < \infty$, $\varepsilon_0 > 0$
- Generalized Young spaces

$$\mathcal{B}_{\varepsilon_1, \varepsilon_1'} = \{h : \Omega \to \mathbb{C} \,:\, \|h e^{-\varepsilon_1 \ell}\|_\infty + Lip(e^{-\varepsilon_1' \ell} h) < \infty\} \subset L^{\frac{\varepsilon_0}{\varepsilon_1}}$$

- $f \in \mathcal{B}_{\varepsilon_1, \varepsilon_1'}$ with $(r+2)\varepsilon_1 + 2\gamma\varepsilon_1' < \varepsilon_0$ and $\varepsilon_1' \ge 0$ small
- If $\varepsilon_1' = 0$, $(1+\eta)^{-1}\varepsilon_0 < (r+2)\varepsilon_1 < \varepsilon_0$ and $\mathbb{E}[e^{\varepsilon_0(1+\eta)\ell}] = \infty$, then $e^{\varepsilon_1 \ell} \in (\mathcal{B} \subset L^{r+2}) \setminus L^{(r+2)(1+\eta)^2}$
- $P_t(f\cdot) \in \mathcal{L}(\mathcal{B}_{\varepsilon, \varepsilon+\varepsilon_1'} \to \mathcal{B}_{\varepsilon+\varepsilon_1, \varepsilon+\varepsilon_1'+\varepsilon_1})$
- Take $\mathcal{X}_j = \mathcal{B}_{a_j, a_j+\varepsilon_1'}$, $\mathcal{X}_j^+ = \mathcal{B}_{a_j^+, a_j^+ + \varepsilon_1'}$
  with $0 < a_0 = \varepsilon$, $a_j < a_j^+ < a_j^+ + \varepsilon_1 < a_{j+1}$, $a_{r+2}^+ < \varepsilon_0$
- Then $\boxed{H_n^{(j)}(t) \approx \mathbb{E}[\xi \Pi_t^{(j)}(\psi)]}$ for $\psi \in \mathcal{X}_0$, $\xi \in (\mathcal{X}_{r+2}^+)^*$

Applications to Sinai billiards with finite horizon (with K. Fernando)

$\Omega = \{(q,\vec{v}) \in \bigcup_{i=1}^{I} \partial O_i \times S^1 \ : \ \langle \vec{n}_q, \vec{v} \rangle \geq 0\}$
$\vec{n}_q$ unit normal vector to $\bigcup_{i=1}^{I} \partial O_i$ at q
$d\mathbb{P}(q,\vec{v}) = \langle \vec{n}_q, \vec{v} \rangle \, d(q,\vec{v})$
$\partial\Omega = \{(q,\vec{v}) \in \Omega \ : \ \langle \vec{n}_q, \vec{v} \rangle = 0\}$



$f : \Omega \to \mathbb{R}$ centered (dynamically Hölder, or Hölder$/d(\cdot,\partial\Omega)^\alpha$ for $\alpha > 0$ small). $S_n = \sum_{k=0}^{n-1} f \circ T^k$.

- If $r = 1$ and f is non-lattice, then first order Edgeworth expansion for $(\psi\mathbb{P})(\frac{S_n}{\sqrt{n}} \leq x)$ for ψ dynamically Hölder, or Hölder$/d(\cdot,\partial\Omega)^{\alpha'}$ for $\alpha' > 0$ small.
- If f is $\mathbb{Z}$-valued non arithmetic or if f is nonlattice (without approximate eigenfunctions), then r order expansion in the MLLT for $\mathbb{E}[\psi\, g(S_n)\, \xi \circ T^n]$ with g as in the i.i.d. case and for ψ and ξ dynamically Hölder, or Hölder$/d(\cdot,\partial\Omega)^{\alpha'}$ for $\alpha' > 0$ small.