Lecture Notes

Topics in Finite Elements
Part I
Introduction to Discontinuous Galerkin Methods

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Contents

1 Introduction

The *discontinuous Galerkin* (dG) methods are a class of finite element methods, which use piecewise polynomial but discontinuous approximations. In general, dG methods are **nonconforming**, i.e., the discrete spaces are not subspaces of the continuous ones. The dG methods are very flexible:

- high-order schemes,
- *hp*-variants of adaptive techniques,
- non-matching and non-uniform grids,
- local conservativity (e.g., local conservation of mass),
- use of numerical flux densities to approximate the flux densities of the continuous problem,
- stability for convection-dominated problems (e.g., $-\varepsilon u'' + u' = f$ with small $\varepsilon > 0$),
- parts can easily be parallelised, as the degrees of freedom have a very local character.

Compared with conforming finite element methods, in dG methods,

- the numerical analysis, i.e., the stability and error analysis, is more involved,
- the errors are often measured in discrete norms only,
- parameters have to be chosen, e.g., penalty parameters must be sufficiently large,
- the number of degrees of freedom could be much higher.

In particular, dG methods are widely used for hyperbolic problems or convection-dominated problems since dG methods have the potential for constructing efficient, accurate and robust methods in these fields. For more details, see, e.g., [1, 2, 3, 5, 6, 7, 8, 10] and references therein.

1.1 First Example in 1D

In this subsection, a first example for a dG method in 1D is given. This example is based on [1, Subsection 7.4.2], [2, Subsection 4.2.1], [3, Subsection 2.4] and shows the basic ideas, where notations are kept as simple as possible. For this purpose, consider Poisson's equation

$$-u''(x) = f(x) \quad \text{for } x \in (a,b), \qquad u(a) = u(b) = 0, \tag{1}$$

where $a, b \in \mathbb{R}$, $a < b$ and $f \in L^2(a,b)$ is a given function.

1.1.1 Continuous Setting

First, we recall the Sobolev spaces

$$\begin{aligned} H^1(a,b) :=& \{w \in L^2(a,b) :\ w' \in L^2(a,b)\}, \\ H^1_0(a,b) :=& \{w \in H^1(a,b) :\ w(a) = w(b) = 0\} \subset H^1(a,b) \end{aligned}$$

with the Hilbertian norms

$$\begin{aligned} \|w\|_{H^1(a,b)} &:= \left(\|w\|^2_{L^2(a,b)} + \|w'\|^2_{L^2(a,b)}\right)^{1/2}, \\ \|w\|_{H^1_0(a,b)} &:= |w|_{H^1(a,b)} := \|w'\|_{L^2(a,b)}, \end{aligned}$$

where

$$\begin{aligned} \langle w, z\rangle_{L^2(a,b)} &:= \int_a^b w(x)z(x)\mathrm{d}x \qquad \text{for } w, z \in L^2(a,b), \\ \|w\|_{L^2(a,b)} &:= \left(\langle w, w\rangle_{L^2(a,b)}\right)^{1/2} = \left(\int_a^b |w(x)|^2 \,\mathrm{d}x\right)^{1/2} \qquad \text{for } w \in L^2(a,b). \end{aligned}$$

In $H^1_0(a,b)$, the Poincaré inequality

$$\forall w \in H^1_0(a,b) : \quad \|w\|_{L^2(a,b)} \le \frac{b-a}{\pi}\|w'\|_{L^2(a,b)} \tag{2}$$

holds true, i.e., the norms $\|\cdot\|_{H^1(a,b)}$ and $|\cdot|_{H^1(a,b)}$ are equivalent in $H^1_0(a,b)$. For $k \in \mathbb{N}$, the Sobolev space $H^k(a,b)$ is defined recursively, i.e.,

$$H^k(a,b) := \{v \in H^1(a,b) :\ v' \in H^{k-1}(a,b)\}$$

with the Hilbertian norm

$$\|v\|_{H^k(a,b)} := \left(\sum_{i=0}^{k} \left\|v^{(i)}\right\|^2_{L^2(a,b)}\right)^{1/2},$$

where $v^{(i)}$ denotes the weak i-th derivative of v and $v^{(0)} := v$. Note that functions in $H^1(a,b)$ are continuous, i.e., $H^1(a,b) \subset C[a,b]$, and more general, $H^k(a,b) \subset C^{k-1}[a,b]$ for $k \in \mathbb{N}$ with $C^0[a,b] := C[a,b]$.

Next, the variational formulation of Poisson's equation (1) is motivated. Assume that the solution u of Poisson's equation (1) is smooth. Then, multiply (1) by a sufficiently smooth test function v with $v(a) = v(b) = 0$, integrate via (a,b) and use integration by parts to get

$$\int_a^b f(x)v(x)\mathrm{d}x \stackrel{!}{=} -\int_a^b u''(x)v(x)\mathrm{d}x = \int_a^b u'(x)v'(x)\mathrm{d}x - u'(b)\underbrace{v(b)}_{=0} + u'(a)\underbrace{v(a)}_{=0}.$$

Thus, for a given function $f \in L^2(a,b)$, the variational formulation of Poisson's equation (1) is to find a function $u \in H_0^1(a,b)$ such that

$$\forall v \in H_0^1(a,b): \quad a(u,v) = \langle f, v \rangle_{L^2(a,b)}, \tag{3}$$

where the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(\cdot,\cdot)\colon\ H_0^1(a,b) \times H_0^1(a,b) \to \mathbb{R}, \quad a(w,z) := \int_a^b w'(x) z'(x) \mathrm{d}x.$$

The bilinear form $a(\cdot,\cdot)$ is continuous by the Cauchy–Schwarz inequality, i.e.,

$$\forall w, z \in H_0^1(a,b): \quad |a(w,z)| \leq |w|_{H^1(a,b)} |z|_{H^1(a,b)},$$

and coercive (or elliptic), i.e.,

$$\forall w \in H_0^1(a,b): \quad |a(w,w)| \geq |w|_{H^1(a,b)}^2.$$

Note that the bilinear form $a(w,z) = \int_a^b w'(x)z'(x)\mathrm{d}x$ is also well-defined and continuous for functions $w, z \in H^1(a,b)$, but is **not** coercive for functions $w \in H^1(a,b)$.

The Lax–Milgram lemma [5, Lemma 25.2] yields the unique solvability of the variational formulation (3), i.e., a unique element $u \in H_0^1(a,b)$ exists such that (3) is satisfied and the stability estimate

$$\begin{aligned} |u|_{H^1(a,b)} \leq \|f\|_{[H_0^1(a,b)]'} &= \sup_{0 \neq z \in H_0^1(a,b)} \frac{\left|\langle f, z \rangle_{L^2(a,b)}\right|}{|z|_{H^1(a,b)}} \\ &\leq \sup_{0 \neq z \in H_0^1(a,b)} \frac{\|f\|_{L^2(a,b)} \|z\|_{L^2(a,b)}}{|z|_{H^1(a,b)}} \leq \frac{b-a}{\pi} \|f\|_{L^2(a,b)} \end{aligned}$$

holds true, where the Cauchy–Schwarz inequality and the Poincaré inequality (2) are used.

1.1.2 Conforming Discretisation

For a discretisation parameter $N \in \mathbb{N}, N \geq 3$, we consider decompositions

$$[a,b] = \bigcup_{\ell=1}^{N} K_\ell,$$

where the elements $K_\ell := [x_{\ell-1}, x_\ell] \subset \mathbb{R}$ with mesh sizes $h_\ell = x_\ell - x_{\ell-1}$ are defined via the decomposition

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$$

of the interval (a,b). The number of elements is N and the number of vertices is $N+1$. The maximal and the minimal mesh sizes are denoted by $h := h_{\max} := \max_\ell h_\ell$ and $h_{\min} := \min_\ell h_\ell$, respectively. Furthermore, we introduce the mesh

$$\mathcal{T}_N := \{K_1, K_2, \ldots, K_N\}.$$

Next, for a fixed polynomial degree $p \in \mathbb{N}$,

$$S_h^p(\mathcal{T}_N) := \left\{v_h \in C[a,b] : \forall \ell \in \{1,\dots,N\} \colon v_{h|K_\ell} \in \mathbb{P}_1^p(K_\ell)\right\}$$

denotes the space of piecewise polynomial, continuous functions on intervals, where $\mathbb{P}_1^p(A)$ is the space of polynomials on a subset $A \subset \mathbb{R}$ of global degree at most p. The subspace

$$S_h^p(\mathcal{T}_N) \cap H_0^1(a,b) \subset H_0^1(a,b)$$

of $H_0^1(a,b)$ is **conforming**. Thus, we consider the conforming discretisation of the variational formulation (3) to find $u_h^{\mathrm{c}} \in S_h^p(\mathcal{T}_N) \cap H_0^1(a,b)$ such that

$$\forall v_h \in S_h^p(\mathcal{T}_N) \cap H_0^1(a,b) : \quad a(u_h^{\mathrm{c}}, v_h) = \langle f, v_h \rangle_{L^2(a,b)}.$$

The discrete variational formulation is uniquely solvable with the stability estimate

$$|u_h^{\mathrm{c}}|_{H^1(a,b)} \leq \frac{b-a}{\pi} \|f\|_{L^2(a,b)},$$

due to the Lax–Milgram lemma [5, Lemma 25.2] and the quasi-optimal error estimate

$$|u - u_h^{\mathrm{c}}|_{H^1(a,b)} \leq \inf_{v_h \in S_h^p(\mathcal{T}_N) \cap H_0^1(a,b)} |u - v_h|_{H^1(a,b)}$$

holds true, which follows from Céa's lemma [5, Lemma 26.13]. Standard error estimates yield

$$|u - u_h^{\mathrm{c}}|_{H^1(a,b)} \leq C h^p$$

and by a duality argument (Aubin–Nitsche trick),

$$\|u - u_h^{\mathrm{c}}\|_{L^2(a,b)} \leq C h^{p+1}$$

with a constant $C > 0$, provided that $u \in H^{p+1}(a,b)$. The number of degrees of freedom is

$$\dim S_h^p(\mathcal{T}_N) \cap H_0^1(a,b) = N - 1 + N(p-1) = pN - 1.$$

1.1.3 Nonconforming Discretisation: dG Method

In this subsection, a heuristic derivation of a dG method for Poisson's equation (1) is given. Using the notations of Subsection 1.1.1 and Subsection 1.1.2, for $k \in \mathbb{N}$, we introduce the **broken Sobolev space**

$$H^k(\mathcal{T}_N) := \{v \in L^2(a,b) : \forall \ell \in \{1,\dots,N\} : v_{|\mathring{K}_\ell} \in H^k(\mathring{K}_\ell)\}$$

with norm

$$\|v\|_{H^k(\mathcal{T}_N)} := \left(\sum_{\ell=1}^N \left\|v_{|\mathring{K}_\ell}\right\|^2_{H^k(\mathring{K}_\ell)}\right)^{1/2}.$$

Here, $v_{|\mathring{K}_\ell}$ is the restriction of v to the interior $\mathring{K}_\ell$ of the set K_ℓ. Further, we denote the traces of $v_{|\mathring{K}_\ell}$ by

$$v_{|K_\ell}(x_{\ell-1}) := \lim_{x \searrow x_{\ell-1}} v_{|\mathring{K}_\ell}(x), \quad v_{|K_\ell}(x_\ell) := \lim_{x \nearrow x_\ell} v_{|\mathring{K}_\ell}(x),$$

which exist due to $H^1(\mathring{K}_\ell) \subset C(K_\ell)$. Note that

$$H^1(a,b) \subset H^1(\mathcal{T}_N).$$

Next, we motivate a dG method. For this purpose, assume that the solution u of Poisson's equation (1) is smooth, e.g., $u \in H^2(a,b)$. Then, multiply Poisson's equation (1) by a function $v \in H^1(\mathcal{T}_N)$ and integrate via **one** element K_ℓ, which gives

$$\begin{aligned}\int_{x_{\ell-1}}^{x_\ell} f(x)v(x)\mathrm{d}x &\overset{!}{=} -\int_{x_{\ell-1}}^{x_\ell} u''(x)v(x)\mathrm{d}x \\ &= \int_{x_{\ell-1}}^{x_\ell} u'(x)v'_{|K_\ell}(x)\mathrm{d}x - u'_{|K_\ell}(x_\ell)v_{|K_\ell}(x_\ell) + u'_{|K_\ell}(x_{\ell-1})v_{|K_\ell}(x_{\ell-1}),\end{aligned}$$

where integration by parts is used. Summing via the elements yields

$$\begin{aligned}\int_a^b f(x)v(x)\mathrm{d}x &\overset{!}{=} \sum_{\ell=1}^{N} \int_{x_{\ell-1}}^{x_\ell} u'(x)v'_{|K_\ell}(x)\mathrm{d}x + \sum_{\ell=1}^{N} \left(-u'_{|K_\ell}(x_\ell)v_{|K_\ell}(x_\ell) + u'_{|K_\ell}(x_{\ell-1})v_{|K_\ell}(x_{\ell-1})\right) \\ &= \sum_{\ell=1}^{N} \int_{x_{\ell-1}}^{x_\ell} u'(x)v'_{|K_\ell}(x)\mathrm{d}x + u'_{|K_1}(x_0)v_{|K_1}(x_0) - u'_{|K_N}(x_N)v_{|K_N}(x_N) \\ &\quad + \sum_{\ell=1}^{N-1} \Big(u'_{|K_{\ell+1}}(x_\ell)v_{|K_{\ell+1}}(x_\ell) - u'_{|K_\ell}(x_\ell)v_{|K_\ell}(x_\ell)\Big).\end{aligned}$$

Due to the assumption $u \in H^2(a,b)$, we have

$$\forall \ell \in \{1,\dots,N-1\}: \quad u'_{|K_{\ell+1}}(x_\ell) = u'_{|K_\ell}(x_\ell) = \{u'\}_{x_\ell},$$

where the **average** of a function $w \in H^1(\mathcal{T}_N)$ on x_ℓ, $\ell \in \{0,\dots,N\}$, is defined as

$$\{w\}_{x_\ell} := \begin{cases} w_{|K_1}(x_0), & \ell = 0, \\ \frac{1}{2}w_{|K_{\ell+1}}(x_\ell) + \frac{1}{2}w_{|K_\ell}(x_\ell), & \ell \in \{1,\dots,N-1\}, \\ w_{|K_N}(x_N), & \ell = N. \end{cases}$$

Thus, it follows that

$$\begin{aligned}\int_a^b f(x)v(x)\mathrm{d}x &\overset{!}{=} \sum_{\ell=1}^{N} \int_{x_{\ell-1}}^{x_\ell} u'(x)v'_{|K_\ell}(x)\mathrm{d}x + u'_{|K_1}(x_0)v_{|K_1}(x_0) - u'_{|K_N}(x_N)v_{|K_N}(x_N) \\ &\quad + \sum_{\ell=1}^{N-1} \{u'\}_{x_\ell}\left(v_{|K_{\ell+1}}(x_\ell) - v_{|K_\ell}(x_\ell)\right) \\ &= \sum_{\ell=1}^{N} \int_{x_{\ell-1}}^{x_\ell} u'(x)v'_{|K_\ell}(x)\mathrm{d}x - \sum_{\ell=0}^{N} \{u'\}_{x_\ell} [\![v]\!]_{x_\ell},\end{aligned}$$

where for $\ell \in \{0, \ldots, N\}$,

$$[\![w]\!]_{x_\ell} := \begin{cases} -w_{|K_1}(x_0), & \ell = 0, \\ w_{|K_\ell}(x_\ell) - w_{|K_{\ell+1}}(x_\ell), & \ell \in \{1, \ldots, N-1\}, \\ w_{|K_N}(x_N), & \ell = N, \end{cases} \tag{4}$$

denotes the **jump** of the function $w \in H^1(\mathcal{T}_N)$ across x_ℓ. To obtain a symmetric bilinear form, we use the property

$$\forall w \in H_0^1(a,b) : \forall \ell \in \{0, \ldots, N\} : \quad [\![w]\!]_{x_\ell} = 0$$

for $w = u$ to add the vanishing term

$$-\sum_{\ell=0}^{N} \{v'\}_{x_\ell} [\![u]\!]_{x_\ell}$$

to conclude that

$$\sum_{\ell=1}^{N} \int_{x_{\ell-1}}^{x_\ell} u'(x) v'_{|K_\ell}(x) \mathrm{d}x - \sum_{\ell=0}^{N} \{u'\}_{x_\ell} [\![v]\!]_{x_\ell} - \sum_{\ell=0}^{N} \{v'\}_{x_\ell} [\![u]\!]_{x_\ell} = \int_a^b f(x) v(x) \mathrm{d}x \tag{5}$$

for all $v \in H^2(\mathcal{T}_N)$. Note that we require $v \in H^2(\mathcal{T}_N)$ since traces of v' are not defined for all $v \in H^1(\mathcal{T}_N)$.

Next, we motivate to add additional terms on the left side of (5). Replacing on the left side of (5) the function u by v, we get

$$\sum_{\ell=1}^{N} \left\| v'_{|\mathring{K}_\ell} \right\|^2_{L^2(\mathring{K}_\ell)} - 2 \sum_{\ell=0}^{N} \{v'\}_{x_\ell} [\![v]\!]_{x_\ell}$$

for all $v \in H^2(\mathcal{T}_N)$, where the sign of the second term is not clear, i.e., coercivity of the related bilinear cannot be expected. Hence, this and to mimic the continuity of the approximate solution, the **penalty** term

$$\sum_{\ell=0}^{N} \omega_{x_\ell} [\![u]\!]_{x_\ell} [\![v]\!]_{x_\ell}$$

with **penalty parameters** $\omega_{x_\ell} > 0$, $\ell = 0, \ldots, N$, is added to the left side of (5). Thus, the solution u of the weak formulation (3), when satisfying $u \in H_0^1(a,b) \cap H^2(a,b)$, fulfils

$$\sum_{\ell=1}^{N} \int_{x_{\ell-1}}^{x_\ell} u'(x) v'_{|K_\ell}(x) \mathrm{d}x - \sum_{\ell=0}^{N} \{u'\}_{x_\ell} [\![v]\!]_{x_\ell} - \sum_{\ell=0}^{N} \{v'\}_{x_\ell} [\![u]\!]_{x_\ell} + \sum_{\ell=0}^{N} \omega_{x_\ell} [\![u]\!]_{x_\ell} [\![v]\!]_{x_\ell} = \int_a^b f(x) v(x) \mathrm{d}x \tag{6}$$

for all $v \in H^2(\mathcal{T}_N)$. The dG method is the discretisation of (6) by **discontinuous**, piecewise polynomial functions

$$S_h^{p,\mathrm{dG}}(\mathcal{T}_N) := \Big\{ v_h \in L^2(a,b) : \forall \ell \in \{1,\dots,N\} \colon v_{h|\mathring{K}_\ell} \in \mathbb{P}_1^p(\mathring{K}_\ell) \Big\}$$

for a polynomial degree $p \in \mathbb{N}_0$. Thus, the **symmetric interior penalty discontinuous Galerkin method** (SIP) is to find $u_h \in S_h^{p,\mathrm{dG}}(\mathcal{T}_N)$ such that

$$\sum_{\ell=1}^{N} \int_{x_{\ell-1}}^{x_\ell} u'_{h|K_\ell}(x) v'_{h|K_\ell}(x) \mathrm{d}x - \underbrace{\sum_{\ell=0}^{N} \{u'_h\}_{x_\ell} [\![v_h]\!]_{x_\ell}}_{\text{consistency term}} - \underbrace{\sum_{\ell=0}^{N} \{v'_h\}_{x_\ell} [\![u_h]\!]_{x_\ell}}_{\text{symmetry term}} + \underbrace{\sum_{\ell=0}^{N} \omega_{x_\ell} [\![u_h]\!]_{x_\ell} [\![v_h]\!]_{x_\ell}}_{\text{penalty term}} = \int_a^b f(x) v_h(x) \mathrm{d}x \quad (7)$$

for all $v_h \in S_h^{p,\mathrm{dG}}(\mathcal{T}_N)$ with penalty parameters $\omega_{x_\ell} > 0$, $\ell = 0,\dots,N$, which have to be **chosen sufficiently large**. Note that the solution u_h of the SIP method (7) fulfils the homogeneous Dirichlet conditions only in a **weak sense**. The number of degrees of freedom is

$$\dim S_h^{p,\mathrm{dG}}(\mathcal{T}_N) = N(p+1) = pN + N.$$

In the next sections, the SIP method (7) is generalised to problems in higher dimension and is analysed with the help of an abstract nonconforming error analysis.

2 Abstract Nonconforming Error Analysis

In this section, we present key ingredients for the stability and error analysis of nonconforming discretisation methods, which are investigated in [2, Section 1.3] or [5, Chapter 27]. These ingredients are **coercivity**, **consistency** and **boundedness**. For simplicity, consider the real Hilbert spaces $V \subset L^2(\Omega)$, $W \subset L^2(\Omega)$ with the Hilbertian norms $\|\cdot\|_V$, $\|\cdot\|_W$, where $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is a fixed domain and $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ is the usual inner product in $L^2(\Omega)$. Further, the bilinear form

$$a(\cdot,\cdot) \colon \; V \times W \to \mathbb{R}$$

is assumed to be continuous, i.e., a constant $C_a > 0$ exists such that

$$\forall v \in V : \forall w \in W : \quad |a(v,w)| \le C_a \|v\|_V \|w\|_W.$$

For a given right-hand side $f \in L^2(\Omega)$, we assume that the variational formulation to find $u \in V$ such that

$$\forall w \in W : \quad a(u,w) = \langle f, w \rangle_{L^2(\Omega)} \tag{8}$$

is uniquely solvable with the stability estimate

$$\|u\|_V \le C_{\mathrm{exact}} \|f\|_{L^2(\Omega)},$$

where C_{exact} is a positive constant. In other words, the exact problem (8) is **well-posed**.

Remark 2.1. *Here are some comments on the setting of the exact problem* (8)*:*

1. *Note that the ansatz space V and the test space W could be* ***different****.*

2. *We do* ***not*** *assume that the bilinear form $a(\cdot,\cdot)$ in* (8) *fulfils the inf-sup theorem (Banach–Nečas theorem, see [5, Theorem 25.9]), or the Lax–Milgram lemma [5, Lemma 25.2] in the case $V = W$.*

3. *We restrict the right-hand side to $L^2(\Omega)$. In many applications, more general right-hand sides can be treated, e.g., $f \in W'$, where W' is the dual space of W.*

Next, we introduce a finite-dimensional space $V_h \subset L^2(\Omega)$ with (discrete) Hilbertian norm $\|\cdot\|_{V_h}$, and a discrete bilinear form

$$a_h(\cdot,\cdot)\colon\ V_h \times V_h \to \mathbb{R}.$$

The discrete problem is to find $u_h \in V_h$ such that

$$\forall w_h \in V_h: \quad a_h(u_h, w_h) = \ell_h(w_h), \tag{9}$$

where the given right-hand side $\ell_h\colon\ V_h \to \mathbb{R}$ is linear.

Remark 2.2. *Note that the ansatz and test spaces in the discrete problem* (9) *are* ***equal****, i.e., coercivity is possible.*

The unique solvability and the stability of the discrete problem (9) is stated in the next theorem.

Theorem 2.3 (Discrete Lax–Milgram lemma)**.** *Assume that the bilinear form $a_h(\cdot,\cdot)$ of the discrete problem* (9) *is coercive, i.e., there exists a constant $C_{\text{coe}} > 0$ such that*

$$\forall v_h \in V_h: \quad a_h(v_h, v_h) \geq C_{\text{coe}} \|v_h\|_{V_h}^2. \tag{10}$$

Then, for any linear right-hand side $\ell_h\colon\ V_h \to \mathbb{R}$, a unique solution $u_h \in V_h$ of the discrete problem (9) *exists and the stability estimate*

$$\|u_h\|_{V_h} \leq \frac{1}{C_{\text{coe}}} \|\ell_h\|_{V_h'}$$

holds true with $\|\ell_h\|_{V_h'} = \sup_{0 \neq w_h \in V_h} \frac{|\ell_h(w_h)|}{\|w_h\|_{V_h}}$.

Proof. The proof is based on [3, Corollary 1.7].

The bilinear form $a_h(\cdot,\cdot)$ and the linear form ℓ_h are continuous, as V_h is finite dimensional. Note that all norms on finite dimensional spaces are equivalent. Thus, all assumptions of the Lax–Milgram lemma [5, Lemma 25.2] are satisfied, which states the unique solvability of the discrete problem (9).

Next, we prove the stability estimate. For $u_h = 0$, the assertion is trivial. For $u_h \neq 0$, the stability follows from

$$\|u_h\|_{V_h}^2 \leq \frac{1}{C_{\text{coe}}} a_h(u_h, u_h) = \frac{1}{C_{\text{coe}}} \frac{\ell_h(u_h)}{\|u_h\|_{V_h}} \|u_h\|_{V_h} \leq \frac{1}{C_{\text{coe}}} \sup_{0 \neq w_h \in V_h} \frac{|\ell_h(w_h)|}{\|w_h\|_{V_h}} \|u_h\|_{V_h},$$

i.e., the assertion is proven. □

Next, we address abstract error estimates of the type of Céa's lemma [5, Lemma 26.13]. In dG methods, we have $V_h \not\subset V$ in general. Thus, Céa's lemma [5, Lemma 26.13] is not applicable, i.e., we need a more involved error analysis. We introduce a subspace

$$V_{\text{smo}} \subset V.$$

We assume that the discrete bilinear form $a_h(\cdot,\cdot)$ can be **extended** to $V_{\text{smo}} \times V_h$, and the norm $\|\cdot\|_{V_h}$ can be **extended** to V_{smo}, where the extensions are denoted again by $a_h(\cdot,\cdot)$ and $\|\cdot\|_{V_h}$, respectively. With this notation, the consistency is formulated.

Definition 2.4 (Consistency)**.** *The discrete problem* (9) *is* ***consistent****, if the exact solution u of the variational formulation* (8) *satisfies $u \in V_{\text{smo}}$ such that*

$$\forall w_h \in V_h : \quad a_h(u, w_h) = \ell_h(w_h).$$

In other words, the exact solution u satisfies the discrete problem (9).

Remark 2.5. *For* ***conforming*** *methods with $V_h \subset V = W$, consistency follows from the* ***Galerkin orthogonality***

$$\forall w_h \in V_h : \quad a(u - u_h, w_h) = 0.$$

The last ingredient in the error analysis is the **boundedness**. For this purpose, we introduce the subspace

$$V_{\text{bnd}} := V_{\text{smo}} + V_h = \{v_{\text{s}} + v_h : v_{\text{s}} \in V_{\text{smo}},\, v_h \in V_h\} \subset L^2(\Omega)$$

with a norm $\|\cdot\|_{V_{\text{bnd}}}$. We assume that $u \in V_{\text{smo}}$. Thus, the approximation error $u - u_h$ belongs to the space V_{bnd}, i.e., $u - u_h \in V_{\text{bnd}}$.

Definition 2.6 (Boundedness)**.** *Assume that $a_h(\cdot,\cdot)$ can be extended to $V_{\text{bnd}} \times V_h$ and $\|\cdot\|_{V_h}$ can be extended to V_{bnd}. The discrete bilinear form $a_h(\cdot,\cdot)$ is* ***bounded in*** *$V_{\text{bnd}} \times V_h$ if*

- *the norm $\|\cdot\|_{V_{\text{bnd}}}$ satisfies*

$$\forall v \in V_{\text{bnd}} : \|v\|_{V_h} \leq \|v\|_{V_{\text{bnd}}},$$

- *a constant $C_{\text{bnd}} > 0$ exists such that*

$$\forall v \in V_{\text{bnd}} : \forall w_h \in V_h : \quad |a_h(v, w_h)| \leq C_{\text{bnd}} \|v\|_{V_{\text{bnd}}} \|w_h\|_{V_h}.$$

With these ingredients, the abstract error estimate is stated in the next theorem, see [2, Theorem 1.35].

Theorem 2.7 (Abstract nonconforming error estimate)**.** *Let $u \in V$ be the solution of the exact problem* (8) *for the right-hand side $f \in L^2(\Omega)$, satisfying $u \in V_{\text{smo}}$. Let $u_h \in V_h$ be the solution of the discrete problem* (9) *for the linear right-hand side $\ell_h \colon V_h \to \mathbb{R}$. Assume that the norm $\|\cdot\|_{V_h}$ can be extended to V_{bnd}, and the discrete bilinear form $a_h(\cdot,\cdot)$*

- *is coercive, i.e.,* (10)*,*
- *can be extended to* $V_{\mathrm{bnd}} \times V_h$*,*
- *is bounded, i.e., Definition 2.6,*

and the discrete problem (9) *is consistent, i.e., Definition 2.4.*

Then, the error estimate

$$\|u - u_h\|_{V_h} \leq \left(1 + \frac{C_{\mathrm{bnd}}}{C_{\mathrm{coe}}}\right) \inf_{v_h \in V_h} \|u - v_h\|_{V_{\mathrm{bnd}}}$$

holds true.

Proof. Let $v_h \in V_h$ be an arbitrary element. The triangle inequality yields

$$\|u - u_h\|_{V_h} \leq \|u - v_h\|_{V_h} + \|v_h - u_h\|_{V_h} \leq \|u - v_h\|_{V_{\mathrm{bnd}}} + \frac{C_{\mathrm{bnd}}}{C_{\mathrm{coe}}} \|u - v_h\|_{V_{\mathrm{bnd}}},$$

where the estimate

$$\|u_h - v_h\|_{V_h}^2 \leq \frac{1}{C_{\mathrm{coe}}} \underbrace{a_h(u_h - v_h, u_h - v_h)}_{= a_h(u - v_h, u_h - v_h)} \leq \frac{C_{\mathrm{bnd}}}{C_{\mathrm{coe}}} \|u - v_h\|_{V_{\mathrm{bnd}}} \|u_h - v_h\|_{V_h},$$

i.e., $\|u_h - v_h\|_{V_h} \leq \frac{C_{\mathrm{bnd}}}{C_{\mathrm{coe}}} \|u - v_h\|_{V_{\mathrm{bnd}}}$, is used. □

Remark 2.8. *Note that the norms of the error estimate of Theorem 2.7 are* ***different****.*

3 Finite Element Spaces

In this section, we introduce the geometric setting, e.g., a mesh, its faces, and the discrete spaces, which are used for the dG method. For this purpose, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0, L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$.

3.1 Mesh and its Properties

In this subsection, the notation for the mesh is introduced. The domain Ω is decomposed as

$$\overline{\Omega} = \bigcup_{\ell=1}^{N} K_\ell$$

with N closed, mutually disjoint sets $K_\ell \subset \mathbb{R}^d$ with nonempty interior, i.e.,

$$\mathcal{T}_\nu := \{K_\ell\}_{\ell=1}^{N}$$

is an admissible decomposition or mesh of Ω for an index $\nu \in \mathbb{N}$. Here, the sets K_ℓ are called elements and are intervals for $d = 1$, triangles for $d = 2$ and tetrahedra for $d = 3$. Recall that a decomposition is called *admissible* if two neighbouring elements join either a vertex ($d = 1, 2, 3$), an edge ($d = 2, 3$), or a triangle ($d = 3$). The local mesh sizes are given as the diameter of the element K_ℓ, i.e.,

$$h_\ell := h_{K_\ell} := \sup_{x,y \in K_\ell} |x - y| \quad \text{for } \ell = 1, \ldots, N.$$

In addition,

$$h := h_{\max}(\mathcal{T}_\nu) := \max_{\ell=1,\ldots,N} h_\ell \qquad \text{and} \qquad h_{\min} := h_{\min}(\mathcal{T}_\nu) := \min_{\ell=1,\ldots,N} h_\ell$$

are the global mesh size and minimal local mesh size, respectively. In the following, a sequence

$$(\mathcal{T}_\nu)_{\nu \in \mathbb{N}} := \{\mathcal{T}_\nu \colon\ \nu \in \mathbb{N}\}$$

of decompositions of Ω is considered. The sequence $(\mathcal{T}_\nu)_{\nu \in \mathbb{N}}$ of decompositions of Ω is called *shape-regular*, if a constant $c_{\mathrm{F}} > 0$ exists such that

$$\forall \nu \in \mathbb{N} \colon\ \forall K \in \mathcal{T}_\nu \colon \quad \sup_{x,y \in K} |x - y| \leq c_{\mathrm{F}}\, \mathrm{r}_K, \tag{11}$$

where r_K is the radius of the largest ball that can be inscribed in the element $K \in \mathcal{T}_\nu$. The sequence $(\mathcal{T}_\nu)_{\nu \in \mathbb{N}}$ of decompositions of Ω is called *globally quasi-uniform*, if a constant $c_G \geq 1$ exists such that

$$\forall \nu \in \mathbb{N} \colon \quad \frac{h_{\max}(\mathcal{T}_\nu)}{h_{\min}(\mathcal{T}_\nu)} \leq c_G.$$

Remark 3.1. *In the literature, a shape-regular sequence of decomposition is also called regular or quasi-uniform. In that case, a globally quasi-uniform sequence is called uniform.*

Assumption 3.2. *In the whole work, the sequence $(\mathcal{T}_\nu)_{\nu \in \mathbb{N}}$ of decompositions of Ω is assumed to be admissible and shape-regular.*

3.2 Faces, Broken Sobolev Space, Averages, Jumps

In this subsection, we introduce further notation, which is used for the dG methods. The set $\mathcal{F}_\nu$ is called *faces of* $\mathcal{T}_\nu$ and consists of $(d-1)$-dimensional sides of elements in $\mathcal{T}_\nu$, i.e., endpoints of intervals for $d = 1$, edges of triangles for $d = 2$, or faces of tetrahedra for $d = 3$. In greater detail, for any face $F \in \mathcal{F}_\nu$, one of the two following conditions is satisfied:

1. There exist two distinct elements $K_\ell, K_r \in \mathcal{T}_\nu$ with $K_\ell \neq K_r$ such that $F = \partial K_\ell \cap \partial K_r$. Then, F is called an *interface*. The set of all interfaces, i.e., all inner faces, is denoted by $\mathcal{F}_\nu^{\mathrm{I}}$.

2. There exists an element $K_\ell \in \mathcal{T}_\nu$ such that $F = \partial K_\ell \cap \partial\Omega$. Then, F is called a *boundary face*. The set of all boundary faces is denoted by $\mathcal{F}_\nu^{\mathrm{B}}$.

Thus, we have

$$\mathcal{F}_\nu = \mathcal{F}_\nu^{\mathrm{I}} \cup \mathcal{F}_\nu^{\mathrm{B}}.$$

Additionally, for an element $K \in \mathcal{T}_\nu$, we define the set

$$\mathcal{F}_K := \{F \in \mathcal{F}_\nu : F \subset \partial K\}. \tag{12}$$

Definition 3.3 (Local length scale h_F)**.** *Let $F \in \mathcal{F}_\nu$ be a given face. For $d \in \{2,3\}$, we set*

$$h_F := \sup_{x,y\in F} |x-y| .$$

For $d = 1$, we distinguish two cases:

1. *For an interface $F \in \mathcal{F}_\nu^{\mathrm{I}}$, there exist two distinct elements $K_\ell, K_r \in \mathcal{T}_\nu$ with $K_\ell \neq K_r$ such that $F = \partial K_\ell \cap \partial K_r$. Then, we set*

$$h_F := \min\{h_{K_\ell}, h_{K_r}\}.$$

2. *For a boundary face $F \in \mathcal{F}_\nu^{\mathrm{B}}$, there exists an element $K_\ell \in \mathcal{T}_\nu$ such that $F = \partial K_\ell \cap \partial\Omega$. Then, we set*

$$h_F := h_{K_\ell}.$$

Next, we introduce the normal vectors.

Definition 3.4 (Normals of elements)**.** *The outer unit normal of an element $K_\ell \in \mathcal{T}_\nu$ is denoted by $\underline{n}_{K_\ell}$.*

For every face $F \in \mathcal{F}_\nu$, we **choose** a unit normal $\underline{n}_F$, called *face normal*, such that the chosen normal $\underline{n}_F$ of a boundary face has the same orientation as the outer normal of $\partial\Omega$.

Assumption 3.5. *The choice of the face normals is fixed.*

For example, the chosen face normal points from the element with the higher element number into the one with the lower element number, see [4, Chapter 10] for more details and other choices.

Next, we introduce the average and the jump of functions, which belong to the broken Sobolev space. For this purpose, we use the usual Lebesgue space $L^2(\Omega)$ and the (classical) Sobolev spaces $H^k(\Omega)$ for $k \in \mathbb{N}$ with the inner products $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$, $\langle\cdot,\cdot\rangle_{H^k(\Omega)}$ and the induced norms $\|\cdot\|_{L^2(\Omega)} = \sqrt{\langle\cdot,\cdot\rangle_{L^2(\Omega)}}$, $\|\cdot\|_{H^k(\Omega)} = \sqrt{\langle\cdot,\cdot\rangle_{H^k(\Omega)}}$, respectively. Moreover, the subspace

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : \gamma_0 v = 0\}$$

is endowed with the Hilbertian norm

$$\|v\|_{H_0^1(\Omega)} := |v|_{H^1(\Omega)} := \|\nabla v\|_{L^2(\Omega)}.$$

Here, for a bounded Lipschitz domain $D \subset \mathbb{R}^d$, the linear continuous mapping

$$\gamma_0 \colon \; H^1(D) \to L^2(\partial D)$$

is the usual trace operator, see [9, Theorem 3.37]. For simpler notation, we write $v_{|A} := (\gamma_0 v)_{|A}$ for any set $A \subset \partial D$. To be complete, we recall the formula for integration by parts, i.e., for any bounded Lipschitz domain $D \subset \mathbb{R}^d$, the equation

$$\forall v \in H^2(D) : \forall w \in H^1(D) : \int_D \nabla v \cdot \nabla w \mathrm{d}x = -\int_D \Delta v w \mathrm{d}x + \int_{\partial D} (\nabla v)_{|\partial D} \cdot \underline{n}\, w_{|\partial D} \mathrm{d}s_x \quad (13)$$

holds true, where $\underline{n}$ is the outer unit normal of the domain D, see [9, Lemma 4.1] for a proof.

Definition 3.6 (Broken Sobolev space)**.** *For $k \in \mathbb{N}$,*

$$H^k(\mathcal{T}_\nu) := \{v \in L^2(\Omega) : \; \forall \ell \in \{1, \dots, N\} : \, v_{|\mathring{K}_\ell} \in H^k(\mathring{K}_\ell)\}$$

*is the **broken Sobolev space** with the norm*

$$\|v\|_{H^k(\mathcal{T}_\nu)} := \left(\sum_{\ell=1}^{N} \left\| v_{|\mathring{K}_\ell} \right\|^2_{H^k(\mathring{K}_\ell)} \right)^{1/2}.$$

*The **broken gradient** $\nabla_h \colon \; H^1(\mathcal{T}_\nu) \to [L^2(\Omega)]^d$ is defined by*

$$\nabla_h v(x) := \begin{cases} \nabla (v_{|\mathring{K}_\ell})(x), & x \in \mathring{K}_\ell, \\ \underline{0}, & \textit{otherwise} \end{cases}$$

for any $v \in H^1(\mathcal{T}_\nu)$.

With this notation, we introduce the average and the jump of functions in $H^1(\mathcal{T}_\nu)$.

Definition 3.7 (Average and jump)**.** *For $M_f \in \mathbb{N}$, we consider a (possibly vector-valued) function $w = (w_1, \dots, w_{M_f})^\top \in [H^1(\mathcal{T}_\nu)]^{M_f}$. The average and the jump are defined componentwise. We distinguish between interfaces and boundary faces.*

1. *Interfaces: Consider a face $F \in \mathcal{F}_\nu^{\mathrm{I}}$ and elements $K_\ell, K_r \in \mathcal{T}_\nu$ with $K_\ell \neq K_r$ such that $F = \partial K_\ell \cap \partial K_r$ and the face normal $\underline{n}_F$ points from K_ℓ to K_r, i.e.,*

$$\underline{n}_{K_\ell|F} = \underline{n}_F = -\underline{n}_{K_r|F}.$$

*Then, the **average** of the function w on the interface F is defined as function $\{w\}_F : F \to \mathbb{R}^{M_f}$ by*

$$\{w\}_F := \frac{1}{2}(w_{|\mathring{K}_\ell})_{|F} + \frac{1}{2}(w_{|\mathring{K}_r})_{|F},$$

*and the **jump** of the function w across the interface F is given as function $[w]_F : F \to \mathbb{R}^{M_f}$ by*

$$[w]_F := (w_{|\mathring{K}_\ell})_{|F} - (w_{|\mathring{K}_r})_{|F}.$$

2. *Boundary faces: Consider a face* $F \in \mathcal{F}_\nu^{\mathrm{B}}$ *and an element* $K_\ell \in \mathcal{T}_\nu$ *such that* $F = \partial K_\ell \cap \partial\Omega$*. Then, the* ***average*** *of the function* w *on the boundary face* F *is defined as function* $\{w\}_F : F \to \mathbb{R}^{M_f}$ *by*

$$\{w\}_F := (w_{|\mathring{K}_\ell})_{|F},$$

and the ***jump*** *of the function* w *across the boundary face* F *is given as function* $[w]_F : F \to \mathbb{R}^{M_f}$ *by*

$$[w]_F := (w_{|\mathring{K}_\ell})_{|F}.$$

Note that for any face $F \in \mathcal{F}_\nu$, the average and the jump satisfy $\{w\}_F \in [L^2(F)]^{M_f}$ and $[w]_F \in [L^2(F)]^{M_f}$, where $M_f \in \mathbb{N}$ and $w \in [H^1(\mathcal{T}_\nu)]^{M_f}$.

Remark 3.8. *We give some comments on the jumps.*

- *For* $d = 1$ *and a function* $w \in H^1(\mathcal{T}_\nu)$*, the jumps of Definition 3.7 and the jumps given in* (4) *fulfil*
$$\forall \ell \in \{0, \dots, N\} : \quad [w]_{x_\ell} \underline{n}_{x_\ell} = [\![w]\!]_{x_\ell} \tag{14}$$
with faces $\mathcal{F}_\nu = \{x_0, \dots, x_N\}$ *and chosen face normals*
$$\underline{n}_{x_\ell} = \begin{cases} -1, & \ell = 0, \\ 1, & \textit{otherwise}. \end{cases}$$

- *In the literature, for* $d \in \{1, 2, 3\}$ *and a function* $w \in H^1(\mathcal{T}_\nu)$*, the vector-valued quantity* $[w]_F \underline{n}_F$ *with* $F \in \mathcal{F}_\nu$ *is also commonly used as an alternative definition of the jumps.*

The next lemma states properties of $H^1(\Omega)$ in connection with jumps and the broken gradient ∇_h.

Lemma 3.9 (Properties of $H^1(\Omega)$)**.** *The following statements are valid:*

1. *For* $w \in H^1(\mathcal{T}_\nu)$*, the equivalence*
$$w \in H^1(\Omega) \quad \Longleftrightarrow \quad \forall F \in \mathcal{F}_\nu^{\mathrm{I}} : [w]_F = 0 \textit{ almost everywhere on } F$$
holds true.

2. *For* $w \in H^1(\Omega)$*, we have that*
$$\nabla_h w = \nabla w \quad \textit{in } L^2(\Omega).$$

Proof. The proof of the first statement is given in [4, Theorem 18.8], whereas the second statement is proven in [4, Lemma 18.9]. □

3.3 Polynomial Spaces

In this subsection, the discrete spaces of the dG methods are stated. For this purpose, we fix a polynomial degree $p \in \mathbb{N}_0$. For a subset $\emptyset \neq A \subset \mathbb{R}^d$, the space $\mathbb{P}_d^p(A)$ is the space of all polynomials on A of global degree at most p. The dimension of the vector space $\mathbb{P}_d^p(A)$ is

$$\dim \mathbb{P}_d^p(A) = \binom{p+d}{p} = \frac{(p+d)!}{p!d!},$$

see, e.g., [4, Section 7.3].

Definition 3.10. *The* ***broken polynomial space*** *is given by*

$$S_h^{p,\mathrm{dG}}(\mathcal{T}_\nu) := \left\{ v_h \in L^2(\Omega) : \forall \ell \in \{1, \dots, N\} \colon v_{h|\mathring{K}_\ell} \in \mathbb{P}_d^p(\mathring{K}_\ell) \right\}.$$

Note that $S_h^{p,\mathrm{dG}}(\mathcal{T}_\nu) \subset H^k(\mathcal{T}_\nu)$ for any $k \in \mathbb{N}$. The dimension of the space $S_h^{p,\mathrm{dG}}(\mathcal{T}_\nu)$ is

$$\dim S_h^{p,\mathrm{dG}}(\mathcal{T}_\nu) = N \frac{(p+d)!}{p!d!}.$$

Next, we state the discrete trace inequality.

Lemma 3.11 (Discrete trace inequality)**.** *Let the mesh sequence $(\mathcal{T}_\nu)_{\nu\in\mathbb{N}}$ be shape-regular with constant $c_{\mathrm{F}} > 0$, see* (11)*, and let $p \in \mathbb{N}_0$ be the polynomial degree. Then, a constant $C_{\mathrm{tr}} > 0$, only depending on c_{F}, p, d, exists such that*

$$\forall \nu \in \mathbb{N} : \forall K \in \mathcal{T}_\nu : \forall F \in \mathcal{F}_K : \forall q \in \mathbb{P}_d^p(K) : \quad h_K^{1/2} \|q\|_{L^2(F)} \leq C_{\mathrm{tr}} \|q\|_{L^2(K)}$$

and

$$\forall \nu \in \mathbb{N} : \forall K \in \mathcal{T}_\nu : \forall q \in \mathbb{P}_d^p(K) : \quad h_K^{1/2} \|q\|_{L^2(\partial K)} \leq C_{\mathrm{tr}} \cdot (d+1)^{1/2} \|q\|_{L^2(K)}$$

hold true.

Proof. For the first inequality, see the proof of Lemma 1.46 in [2] or Subsection 12.2 in [4].

The second inequality follows by

$$h_K \|q\|_{L^2(\partial K)}^2 = \sum_{F \in \mathcal{F}_K} h_K \|q\|_{L^2(F)}^2 \leq \sum_{F \in \mathcal{F}_K} C_{\mathrm{tr}}^2 \|q\|_{L^2(K)}^2 = C_{\mathrm{tr}}^2 \cdot (d+1) \|q\|_{L^2(K)}^2,$$

where the frist inequality is used. □

Remark 3.12 (p-dependency of C_{tr})**.** *The constant C_{tr} scales like $\sqrt{p(p+d)}$ for $p \to \infty$, see Remark 1.48 in [2] and references there.*

As last ingredient, we need the discrete Poincaré inequality.

Lemma 3.13 (Discrete Poincaré inequality). *Let the mesh sequence $(\mathcal{T}_\nu)_{\nu\in\mathbb{N}}$ be shape-regular with constant $c_\mathrm{F} > 0$, see* (11)*, and let $p \in \mathbb{N}_0$ be the polynomial degree. Then, a constant $C_\mathrm{dP} > 0$, only depending on c_F, p, Ω, exists such that*

$$\forall \nu \in \mathbb{N} : \forall w_h \in S_h^{p,\mathrm{dG}}(\mathcal{T}_\nu) : \quad \|w_h\|_{L^2(\Omega)} \le C_\mathrm{dP} \left(\|\nabla_h w_h\|^2_{L^2(\Omega)} + \sum_{F\in\mathcal{F}_\nu} \frac{1}{h_F} \|[w_h]_F\|^2_{L^2(F)} \right)^{1/2}$$

holds true.

Proof. See the proof of Corollary 5.4 in [2] or Lemma 2.45 in [3] and references there. □

4 Poisson's Equation

This section is based on [2, Subsection 4.2] and [5, Chapter 38], where the notation and the assumptions of Section 2 and Section 3 are used.

4.1 Model Problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, which is an interval for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. Further, let $f \in L^2(\Omega)$ be given. Poisson's equation is to find a function u such that

$$-\Delta u = f \text{ in } \Omega, \quad u_{|\partial\Omega} = 0.$$

We set

$$V := H^1_0(\Omega)$$

with the Hilbertian norm

$$\|v\|_V := |v|_{H^1(\Omega)} := \|\nabla v\|_{L^2(\Omega)}, \quad v \in V.$$

The variational formulation to find $u \in V$ such that

$$\forall w \in V : \quad \int_\Omega \nabla u(x) \cdot \nabla w(x) \mathrm{d}x = \int_\Omega f(x) w(x) \mathrm{d}x \tag{15}$$

is uniquely solvable with the stability estimate

$$\|u\|_V \le C_\mathrm{exact} \|f\|_{L^2(\Omega)},$$

where $C_\mathrm{exact} > 0$ is a constant, see [5, Chapter 31] for more details.

4.2 Symmetric Interior Penalty Discontinuous Galerkin Method (SIP)

Consider a shape-regular mesh sequence $(\mathcal{T}_\nu)_{\nu\in\mathbb{N}}$ of admissible meshes as considered in Section 3 and a fixed polynomial degree $p \in \mathbb{N}_0$. With the notation of Section 3, we set

$$V_h := S_h^{p,\mathrm{dG}}(\mathcal{T}_\nu),$$

see Definition 3.10, with Hilbertian norm

$$\|v_h\|_{V_h} := \sqrt{\|\nabla_h v_h\|_{L^2(\Omega)}^2 + |v_h|_{\mathrm{J}}^2}, \quad v_h \in V_h, \tag{16}$$

where

$$|v_h|_{\mathrm{J}} := \sqrt{\sum_{F\in\mathcal{F}_\nu} \frac{1}{h_F} \|[v_h]_F\|_{L^2(F)}^2}, \quad v_h \in V_h,$$

is the so-called **jump seminorm**. Here, the local length scale h_F is defined in Definition 3.3. Note that $\|\cdot\|_{V_h}$ in (16) is actually a norm. To prove this, assume that $\|v_h\|_{V_h} = 0$ for a function $v_h \in V_h$. Then, v_h is piecewise constant in Ω and $[v_h]_F = 0$ for all $F \in \mathcal{F}_\nu$, due the definition of $\|\cdot\|_{V_h}$. Hence, zero jumps across interfaces yield that v_h is constant in Ω and zero on the boundary $\partial\Omega$, since the jumps across boundary faces are also zero. Thus, $v_h = 0$ and $\|\cdot\|_{V_h}$ is a norm.

Next, we define the **discrete bilinear form**

$$a_h(\cdot,\cdot)\colon\ V_h \times V_h \to \mathbb{R},$$

by

$$\begin{aligned} a_h(v_h, w_h) := \int_\Omega \nabla_h v_h \cdot \nabla_h w_h \mathrm{d}x - \underbrace{\sum_{F\in\mathcal{F}_\nu} \int_F \{\nabla_h v_h\}_F \cdot \underline{n}_F [w_h]_F \mathrm{d}s_x}_{\text{consistency term}} \\ - \underbrace{\sum_{F\in\mathcal{F}_\nu} \int_F [v_h]_F \{\nabla_h w_h\}_F \cdot \underline{n}_F \mathrm{d}s_x}_{\text{symmetry term}} + \underbrace{\sum_{F\in\mathcal{F}_\nu} \omega_F \int_F [v_h]_F [w_h]_F \mathrm{d}s_x}_{\text{penalty term}} \end{aligned} \tag{17}$$

for $v_h, w_h \in V_h$, where $\omega_F > 0$ are the **penalty parameters**, which have to be chosen. Here, the jumps and averages for vector-valued functions are defined componentwise, see Definition 3.7.

Additionally, we set the **discrete linear form** $\ell_h\colon\ V_h \to \mathbb{R}$,

$$\ell_h(w_h) := \int_\Omega f(x) w_h(x) \mathrm{d}x, \quad w_h \in V_h. \tag{18}$$

The **SIP method** is to find $u_h \in V_h$ such that

$$\forall w_h \in V_h: \quad a_h(u_h, w_h) = \ell_h(w_h). \tag{19}$$

Note that the solution u_h of the SIP method (19) fulfils the homogeneous Dirichlet conditions only in a **weak sense**. Additionally, for $d = 1$, the SIP method (19) coincides with the SIP method (7) when using the relation (14). In the following, we analyse the SIP method (19) with the help of the abstract results in Section 2. For this purpose, properties of the discrete bilinear form $a_h(\cdot, \cdot)$ in (17) and the discrete linear form ℓ_h in (18) are shown.

4.3 Coercivity and Well-Posedness

In this subsection, the coercivity of the discrete bilinear form $a_h(\cdot, \cdot)$ in (17) and thus, the well-posedness of the SIP method (19) are shown. The following lemma is used to prove coercivity and later, boundedness of the discrete bilinear form $a_h(\cdot, \cdot)$ in (17).

Lemma 4.1 (Bound on the consistency term). *For all $v_h \in V_h + H^2(\Omega)$, $w_h \in V_h + H^2(\Omega)$, the estimate*

$$\left| \sum_{F \in \mathcal{F}_\nu} \int_F \{\nabla_h v_h\}_F \cdot \underline{n}_F [w_h]_F \mathrm{d}s_x \right| \leq \left(\sum_{K \in \mathcal{T}_\nu} \sum_{F \in \mathcal{F}_K} h_F \left\| \nabla(v_{h|\mathring{K}}) \cdot \underline{n}_F \right\|^2_{L^2(F)} \right)^{1/2} |w_h|_{\mathrm{J}}$$

holds true, where the set $\mathcal{F}_K$ is defined in (12) *and*

$$V_h + H^2(\Omega) = \{v_h + v_{\mathrm{s}} \in L^2(\Omega) : \, v_h \in V_h, \, v_{\mathrm{s}} \in H^2(\Omega)\}.$$

Proof. The proof is based on the proof of [5, Lemma 38.5].

Note that for $v \in H^2(\Omega)$, we have $\nabla v \in [H^1(\Omega)]^d$ and thus, traces $(\nabla v)_{|F}$ for $F \in \mathcal{F}_\nu$ are well-defined as functions in $L^2(F)$. Hence, this and the properties of $H^1(\Omega)$ (Lemma 3.9) ensure that all integrals in the asserted estimate exist.

First, for an interface $F \in \mathcal{F}_\nu^{\mathrm{I}}$ with elements $K_{F,\ell}, K_{F,r} \in \mathcal{T}_\nu$ such that $F = \partial K_{F,\ell} \cap \partial K_{F,r}$, we have

$$\begin{aligned}
\|\{\nabla_h v_h\}_F \cdot \underline{n}_F\|^2_{L^2(F)} &= \frac{1}{4} \int_F \Big(\underbrace{\nabla(v_{h|\mathring{K}_{F,\ell}}) \cdot \underline{n}_F}_{=a} + \underbrace{\nabla(v_{h|\mathring{K}_{F,r}}) \cdot \underline{n}_F}_{=b} \Big)^2 \mathrm{d}s_x \\
&\leq \frac{1}{2} \int_F (a^2 + b^2) \mathrm{d}s_x \\
&= \frac{1}{2} \left\| \nabla(v_{h|\mathring{K}_{F,\ell}}) \cdot \underline{n}_F \right\|^2_{L^2(F)} + \frac{1}{2} \left\| \nabla(v_{h|\mathring{K}_{F,r}}) \cdot \underline{n}_F \right\|^2_{L^2(F)},
\end{aligned}$$

where we used the inequality

$$\forall a, b \in \mathbb{R} : \quad (a+b)^2 \leq 2a^2 + 2b^2.$$

Second, for a boundary face $F \in \mathcal{F}_\nu^{\mathrm{B}}$ with element $K_F \in \mathcal{T}_\nu$ such that $F = \partial K_F \cap \partial\Omega$, the equality

$$\|\{\nabla_h v_h\}_F \cdot \underline{n}_F\|_{L^2(F)} = \left\| \nabla(v_{h|\mathring{K}_F}) \cdot \underline{n}_F \right\|_{L^2(F)}$$

holds true.

With these relations and the Cauchy–Schwarz inequality, we conclude that

$$\begin{aligned}
\left|\sum_{F\in\mathcal{F}_\nu}\int_F \{\nabla_h v_h\}_F\cdot \underline{n}_F[w_h]_F \mathrm{d}s_x\right| &\le \sum_{F\in\mathcal{F}_\nu} \|\{\nabla_h v_h\}_F\cdot\underline{n}_F\|_{L^2(F)} h_F^{1/2} h_F^{-1/2} \|[w_h]_F\|_{L^2(F)} \\
&\le \left(\sum_{F\in\mathcal{F}_\nu} \|\{\nabla_h v_h\}_F\cdot\underline{n}_F\|^2_{L^2(F)} h_F\right)^{1/2} \underbrace{\left(\sum_{F\in\mathcal{F}_\nu} \|[w_h]_F\|^2_{L^2(F)} \frac{1}{h_F}\right)^{1/2}}_{=|w_h|_{\mathrm{J}}} \\
&\le \left(\sum_{F\in\mathcal{F}_\nu^{\mathrm{I}}} \frac{h_F}{2}\left(\left\|\nabla(v_{h|\mathring{K}_{F,\ell}})\cdot\underline{n}_F\right\|^2_{L^2(F)} + \left\|\nabla(v_{h|\mathring{K}_{F,r}})\cdot\underline{n}_F\right\|^2_{L^2(F)}\right)\right. \\
&\qquad \left. + \sum_{F\in\mathcal{F}_\nu^{\mathrm{B}}} h_F \left\|\nabla(v_{h|\mathring{K}_F})\cdot\underline{n}_F\right\|^2_{L^2(F)}\right)^{1/2} |w_h|_{\mathrm{J}} \\
&\le \left(\sum_{K\in\mathcal{T}_\nu}\sum_{F\in\mathcal{F}_K} h_F \left\|\nabla(v_{h|\mathring{K}})\cdot\underline{n}_F\right\|^2_{L^2(F)}\right)^{1/2} |w_h|_{\mathrm{J}}\,,
\end{aligned}$$

where in the last inequality, the sums are rewritten in the following way: Running via the interfaces and summing up the two contributions of the related elements identically equals to running via all elements and summing up the contribution on their interior faces. A similar argument holds true for the boundary faces. □

The next lemma is also needed to prove coercivity of the discrete bilinear form $a_h(\cdot,\cdot)$ in (17).

Lemma 4.2. *For all $v_h \in V_h$, we have that*

$$\sum_{K\in\mathcal{T}_\nu}\sum_{F\in\mathcal{F}_K} h_F \left\|\nabla(v_{h|\mathring{K}})\cdot\underline{n}_F\right\|^2_{L^2(F)} \le C_{\mathrm{tr}}^2\cdot(d+1)\|\nabla_h v_h\|^2_{L^2(\Omega)},$$

where the set $\mathcal{F}_K$ is defined in (12) *and $C_{\mathrm{tr}} > 0$ is the constant of the discrete trace inequality (Lemma 3.11).*

Proof. For $v_h \in V_h$, the Cauchy–Schwarz inequality and the discrete trace inequality

(Lemma 3.11) yield that

$$\begin{aligned}
\sum_{K\in\mathcal{T}_\nu}\sum_{F\in\mathcal{F}_K}\underbrace{h_F}_{\le h_K}\Big\|\nabla(v_{h|\mathring{K}})\cdot\underline{n}_F\Big\|^2_{L^2(F)} &\le \sum_{K\in\mathcal{T}_\nu} h_K\sum_{F\in\mathcal{F}_K}\int_F\Big(\nabla(v_{h|\mathring{K}})(x)\cdot\underline{n}_F(x)\Big)^2\mathrm{d}s_x\\
&\le \sum_{K\in\mathcal{T}_\nu} h_K\sum_{F\in\mathcal{F}_K}\int_F\sum_{i=1}^d\Big(\partial_{x_i}(v_{h|\mathring{K}})(x)\Big)^2\underbrace{\sum_{i=1}^d\Big(\underline{n}_F[i](x)\Big)^2}_{=1}\mathrm{d}s_x\\
&= \sum_{K\in\mathcal{T}_\nu} h_K\sum_{i=1}^d\Big\|\partial_{x_i}(v_{h|\mathring{K}})\Big\|^2_{L^2(\partial K)}\\
&\le \sum_{K\in\mathcal{T}_\nu}\sum_{i=1}^d C_{\mathrm{tr}}^2\cdot(d+1)\Big\|\partial_{x_i}(v_{h|\mathring{K}})\Big\|^2_{L^2(K)}\\
&= C_{\mathrm{tr}}^2\cdot(d+1)\|\nabla_h v_h\|^2_{L^2(\Omega)},
\end{aligned}$$

i.e., the assertion. □

The next theorem is the main result of this subsection.

Theorem 4.3. *Let the mesh sequence $(\mathcal{T}_\nu)_{\nu\in\mathbb{N}}$ be shape-regular with constant $c_{\mathrm{F}} > 0$, see* (11)*, and let $p \in \mathbb{N}_0$ be the polynomial degree. Further, let the penalty parameters be such that*

$$\forall\nu\in\mathbb{N}:\ \forall F\in\mathcal{F}_\nu:\quad \omega_F=\frac{\omega_0}{h_F}$$

with a fixed $w_0 > C_{\mathrm{tr}}^2\cdot(d+1)$, where $C_{\mathrm{tr}} > 0$ is the constant of the discrete trace inequality (Lemma 3.11), which only depends on c_{F}, p, d. Then,

1. *the coercivity estimate*

$$\forall v_h\in V_h:\quad a_h(v_h,v_h)\ge C_{\mathrm{coe}}\|v_h\|^2_{V_h}$$

   holds true with the coercivity constant

$$C_{\mathrm{coe}}:=\frac{\omega_0-C_{\mathrm{tr}}^2\cdot(d+1)}{1+\omega_0}>0,$$

2. *the SIP method* (19) *is uniquely solvable with the stability estimate*

$$\|u_h\|_{V_h}\le\frac{C_{\mathrm{dP}}}{C_{\mathrm{coe}}}\|f\|_{L^2(\Omega)},$$

where $C_{\mathrm{dP}} > 0$ is the constant of the discrete Poincaré inequality (Lemma 3.13), which only depends on c_{F}, p, Ω.

Proof. First, for proving the coercivity, let $v_h \in V_h$ be fixed. We have that

$$\begin{aligned}
a_h(v_h, v_h) &= \|\nabla_h v_h\|_{L^2(\Omega)}^2 - \sum_{F \in \mathcal{F}_\nu} 2 \int_F \{\nabla_h v_h\}_F \cdot \underline{n}_F [v_h]_F \mathrm{d}s_x + \sum_{F \in \mathcal{F}_\nu} \underbrace{\omega_F}_{=\omega_0/h_F} \|[v_h]_F\|_{L^2(F)}^2 \\
&= \|\nabla_h v_h\|_{L^2(\Omega)}^2 - \sum_{F \in \mathcal{F}_\nu} 2 \int_F \{\nabla_h v_h\}_F \cdot \underline{n}_F [v_h]_F \mathrm{d}s_x + \omega_0 \left|v_h\right|_{\mathrm{J}}^2 . \qquad (20)
\end{aligned}$$

For the second term, Lemma 4.1 and Lemma 4.2 give

$$\begin{aligned}
\sum_{F \in \mathcal{F}_\nu} 2 \int_F \{\nabla_h v_h\}_F \cdot \underline{n}_F [v_h]_F \mathrm{d}s_x &\leq 2 \left(\sum_{K \in \mathcal{T}_\nu} \sum_{F \in \mathcal{F}_K} h_F \left\| \nabla (v_{h|\mathring{K}}) \cdot \underline{n}_F \right\|_{L^2(F)}^2 \right)^{1/2} \left|v_h\right|_{\mathrm{J}} \\
&\leq 2 C_{\mathrm{tr}} \cdot \sqrt{d+1} \|\nabla_h v_h\|_{L^2(\Omega)} \left|v_h\right|_{\mathrm{J}} .
\end{aligned}$$

Using the last estimate in (20) yields

$$\begin{aligned}
a_h(v_h, v_h) &\geq \|\nabla_h v_h\|_{L^2(\Omega)}^2 - 2 C_{\mathrm{tr}} \cdot \sqrt{d+1} \|\nabla_h v_h\|_{L^2(\Omega)} \left|v_h\right|_{\mathrm{J}} + \omega_0 \left|v_h\right|_{\mathrm{J}}^2 \\
&\geq \underbrace{\frac{\omega_0 - C_{\mathrm{tr}}^2 \cdot (d+1)}{1 + \omega_0}}_{=C_{\mathrm{coe}}} \underbrace{\left(\|\nabla_h v_h\|_{L^2(\Omega)}^2 + \left|v_h\right|_{\mathrm{J}}^2 \right)}_{=\|v_h\|_{V_h}^2} .
\end{aligned}$$

In the last step, we use the quadratic inequality

$$\forall x, y \in \mathbb{R} : \quad x^2 - 2\beta x y + \eta y^2 \geq \frac{\eta - \beta^2}{1 + \eta} (x^2 + y^2) \qquad (21)$$

for any $\eta, \beta \in \mathbb{R}$ with $\eta > -1$, which can be proven as follows: Let $x, y \in \mathbb{R}$ be fixed. The inequality (21) is equivalent to

$$\frac{1 + \beta^2}{1 + \eta} x^2 - 2\beta x y + \frac{\eta^2 + \beta^2}{1 + \eta} y^2 \geq 0,$$

where the left side defines a binary quadratic form. To verify the last inequality, we show that the binary quadratic form is positive semidefinite, which is the case iff $\eta > -1$ and

$$4 \frac{1 + \beta^2}{1 + \eta} \cdot \frac{\eta^2 + \beta^2}{1 + \eta} - 4\beta^2 \geq 0.$$

Rearranging the last inequality gives $2\eta\beta^2 \leq \eta^2 + \beta^4$, which is true due to the trivial relation $2ab \leq a^2 + b^2$ for any $a, b \in \mathbb{R}$. Thus, the quadratic inequality (21) and hence, the coercivity are proven.

Second, the discrete Lax–Milgram lemma (Theorem 2.3) yields the unique solvability of the SIP method (19) and the stability estimate

$$\begin{aligned}\|u_h\|_{V_h} &\leq \frac{1}{C_{\text{coe}}} \sup_{0 \neq w_h \in V_h} \frac{\left|\int_\Omega f(x) w_h(x) \mathrm{d}x\right|}{\|w_h\|_{V_h}} \\ &\leq \frac{1}{C_{\text{coe}}} \sup_{0 \neq w_h \in V_h} \frac{\|f\|_{L^2(\Omega)} \|w_h\|_{L^2(\Omega)}}{\|w_h\|_{V_h}} \\ &\leq \frac{1}{C_{\text{coe}}} \sup_{0 \neq w_h \in V_h} \frac{\|f\|_{L^2(\Omega)} C_{\text{dP}} \|w_h\|_{V_h}}{\|w_h\|_{V_h}} = \frac{C_{\text{dP}}}{C_{\text{coe}}} \|f\|_{L^2(\Omega)},\end{aligned}$$

where the Cauchy–Schwarz inequality and the discrete Poincaré inequality (Lemma 3.13) are used. □

4.4 Error Analysis

In this subsection, we prove the consistency of the SIP method (19) and state an error estimate in the norm $\|\cdot\|_{V_h}$, where the abstract error analysis of Section 2 is used.

First, we set

$$V_{\text{smo}} := V \cap H^2(\Omega) = H_0^1(\Omega) \cap H^2(\Omega) \subset V.$$

For $v \in V_{\text{smo}}$, we have $\nabla v \in [H^1(\Omega)]^d$ and thus, traces $(\nabla v)_{|F}$ for $F \in \mathcal{F}_\nu$ are well-defined as functions in $L^2(F)$. Hence, this and the properties of $H^1(\Omega)$ (Lemma 3.9) ensure that the discrete bilinear form $a_h(\cdot, \cdot)$ in (17) can be extended to $V_{\text{smo}} \times V_h$ and the norm $\|\cdot\|_{V_h}$ in (16) can be extended to V_{smo}. As in Section 2, we keep the same notations $a_h(\cdot, \cdot)$ and $\|\cdot\|_{V_h}$ also for the extensions.

To prove consistency, we assume that the exact solution u of the variational formulation (15) satisfies the regularity

$$u \in V_{\text{smo}}.$$

This regularity assumption is satisfied under conditions on the domain Ω, e.g., convexity, see [5, Section 31.4] for more details.

Lemma 4.4 (Consistency)**.** *Assume that the exact solution u of the variational formulation* (15) *satisfies the regularity $u \in V_{\text{smo}}$. Then, the SIP method* (19) *is consistent, i.e.,*

$$\forall w_h \in V_h : \quad a_h(u, w_h) = \ell_h(w_h).$$

Proof. For $w_h \in V_h$, properties of $H^1(\Omega)$ (Lemma 3.9) yield

$$\begin{aligned}a_h(u, w_h) - \ell_h(w_h) &= \int_\Omega \underbrace{\nabla_h u}_{=\nabla u} \cdot \nabla_h w_h \mathrm{d}x - \sum_{F \in \mathcal{F}_\nu} \int_F \underbrace{\{\nabla_h u\}_F}_{=(\nabla u)_{|F}} \cdot \underline{n}_F [w_h]_F \mathrm{d}s_x \\ &\quad - \sum_{F \in \mathcal{F}_\nu} \int_F \underbrace{[u]_F}_{=0} \{\nabla_h w_h\}_F \cdot \underline{n}_F \mathrm{d}s_x + \sum_{F \in \mathcal{F}_\nu} \omega_F \int_F \underbrace{[u]_F}_{=0} [w_h]_F \mathrm{d}s_x - \ell_h(w_h) \\ &= \sum_{K \in \mathcal{T}_\nu} \int_K \nabla u \cdot \nabla (w_{h|\mathring{K}}) \mathrm{d}x - \sum_{F \in \mathcal{F}_\nu} \int_F (\nabla u)_{|F} \cdot \underline{n}_F [w_h]_F \mathrm{d}s_x - \ell_h(w_h).\end{aligned}$$

Integration by parts (13), applied elementwise for K, gives

$$
\begin{aligned}
a_h(u, w_h) - \ell_h(w_h) =& \sum_{K \in \mathcal{T}_\nu} \Big(-\int_K \Delta u w_h \mathrm{d}x + \int_{\partial K} (\nabla u)_{|\partial K} \cdot \underline{n}_K (w_{h|\mathring{K}})_{|\partial K} \mathrm{d}s_x \Big) \\
& - \sum_{F \in \mathcal{F}_\nu} \int_F (\nabla u)_{|F} \cdot \underline{n}_F [w_h]_F \mathrm{d}s_x - \ell_h(w_h) \\
=& \int_\Omega \underbrace{-\Delta u}_{=f} w_h \mathrm{d}x - \int_\Omega f w_h \mathrm{d}x \\
& + \sum_{F \in \mathcal{F}_\nu} \int_F (\nabla u)_{|F} \cdot \underline{n}_F [w_h]_F \mathrm{d}s_x - \sum_{F \in \mathcal{F}_\nu} \int_F (\nabla u)_{|F} \cdot \underline{n}_F [w_h]_F \mathrm{d}s_x \\
=& 0.
\end{aligned}
$$

Here, we use the equality

$$
\begin{aligned}
\sum_{F \in \mathcal{F}_\nu} \int_F (\nabla u)_{|F} \cdot \underline{n}_F [w_h]_F \mathrm{d}s_x &= \sum_{K \in \mathcal{T}_\nu} \sum_{F \in \mathcal{F}_K} \int_F (\nabla u)_{|F} \cdot \underline{n}_K (w_{h|\mathring{K}})_{|F} \mathrm{d}s_x \\
&= \sum_{K \in \mathcal{T}_\nu} \int_{\partial K} (\nabla u)_{|\partial K} \cdot \underline{n}_K (w_{h|\mathring{K}})_{|\partial K} \mathrm{d}s_x,
\end{aligned}
$$

which follows from the following arguments: Running via the interfaces and summing up the two contributions of the related elements identically equals to running via all elements and summing up the contribution on their interior faces. A similar argument holds true for the boundary faces. Additionally, recall the properties of $H^1(\Omega)$ (Lemma 3.9), Definition 3.7 and that for all $F \in \mathcal{F}_\nu^{\mathrm{I}}$ with elements $K_\ell, K_r \in \mathcal{T}_\nu$ such that $F = \partial K_\ell \cap \partial K_r$, we have $\underline{n}_F = \underline{n}_{K_\ell|F} = -\underline{n}_{K_r|F}$.

Further, the relation $-\Delta u = f$ in $L^2(\Omega)$ is proven by plugging a test function $\varphi \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$ into the variational formulation (15), which leads with integration by parts (13) to

$$
\int_\Omega f(x)\varphi(x)\mathrm{d}x = \int_\Omega \nabla u(x) \cdot \nabla \varphi(x) \mathrm{d}x = -\int_\Omega u(x) \Delta \varphi(x) \mathrm{d}x,
$$

i.e., $-\Delta u = f$ in $L^2(\Omega)$ in the sense of distributions. □

Next, we prove the boundedness of the discrete bilinear form $a_h(\cdot,\cdot)$ in (17). For this purpose, we set

$$
V_{\mathrm{bnd}} := V_{\mathrm{smo}} + V_h = \{ v_{\mathrm{s}} + v_h : v_{\mathrm{s}} \in V_{\mathrm{smo}},\, v_h \in V_h \} \subset L^2(\Omega)
$$

with the norm

$$
\|v\|_{V_{\mathrm{bnd}}} := \Bigg(\|v\|_{V_h}^2 + \sum_{K \in \mathcal{T}_\nu} h_K \Big\| \nabla (v_{|\mathring{K}}) \cdot \underline{n}_K \Big\|_{L^2(\partial K)}^2 \Bigg)^{1/2}, \quad v \in V_{\mathrm{bnd}}.
$$

Note that $\|\cdot\|_{V_{\text{bnd}}}$ is actually a norm, due to $\|\cdot\|_{V_h}$ is a already a norm on V_{bnd}. Further, the discrete bilinear form $a_h(\cdot,\cdot)$ in (17) can be extended to $V_{\text{bnd}} \times V_h$ and the norm $\|\cdot\|_{V_h}$ in (16) can be extended to V_{bnd}. As in Section 2, we keep the same notations $a_h(\cdot,\cdot)$ and $\|\cdot\|_{V_h}$ also for the extensions.

Lemma 4.5 (Boundedness)**.** *Let the assumptions of Theorem 4.3 be satisfied. The discrete bilinear form $a_h(\cdot,\cdot)$ in* (17) *is bounded in $V_{\text{bnd}} \times V_h$ with constant*

$$C_{\text{bnd}} := 2 + C_{\text{tr}}\sqrt{d+1} + \omega_0 > 0$$

independent of h, i.e.,

1. $\forall v \in V_{\text{bnd}}: \quad \|v\|_{V_h} \le \|v\|_{V_{\text{bnd}}}$,
2. $\forall v \in V_{\text{bnd}}: \forall w_h \in V_h: \quad |a_h(v,w_h)| \le C_{\text{bnd}} \|v\|_{V_{\text{bnd}}} \|w_h\|_{V_h}$.

Proof. The first statement is trivial.

For the second inequality, let $v \in V_{\text{bnd}}$ and $w_h \in V_h$ be fixed. We have

$$\begin{aligned} a_h(v,w_h) = &\underbrace{\int_\Omega \nabla_h v \cdot \nabla_h w_h \mathrm{d}x}_{=:I_1} - \underbrace{\sum_{F\in\mathcal{F}_\nu} \int_F \{\nabla_h v\}_F \cdot \underline{n}_F [w_h]_F \mathrm{d}s_x}_{=:I_2} \\ &- \underbrace{\sum_{F\in\mathcal{F}_\nu} \int_F [v]_F \{\nabla_h w_h\}_F \cdot \underline{n}_F \mathrm{d}s_x}_{=:I_3} + \underbrace{\sum_{F\in\mathcal{F}_\nu} \frac{\omega_0}{h_F} \int_F [v]_F [w_h]_F \mathrm{d}s_x}_{=:I_4} . \end{aligned}$$

For I_1, the Cauchy–Schwarz inequality and the definitions of the norms $\|\cdot\|_{V_h}$, $\|\cdot\|_{V_{\text{bnd}}}$ give

$$|I_1| \le \|\nabla_h v\|_{L^2(\Omega)} \|\nabla_h w_h\|_{L^2(\Omega)} \le \|v\|_{V_{\text{bnd}}} \|w_h\|_{V_h}.$$

For I_2, Lemma 4.1 and the definitions of the norms $\|\cdot\|_{V_h}$, $\|\cdot\|_{V_{\text{bnd}}}$ state

$$\begin{aligned} |I_2| &\le \Big(\sum_{K\in\mathcal{T}_\nu} \sum_{F\in\mathcal{F}_K} \underbrace{h_F}_{\le h_K} \Big\| \nabla(v_{|\mathring{K}}) \cdot \underline{n}_F \Big\|_{L^2(F)}^2 \Big)^{1/2} |w_h|_{\mathrm{J}} \\ &\le \Big(\sum_{K\in\mathcal{T}_\nu} h_K \Big\| \nabla(v_{|\mathring{K}}) \cdot \underline{n}_K \Big\|_{L^2(\partial K)}^2 \Big)^{1/2} |w_h|_{\mathrm{J}} \\ &\le \|v\|_{V_{\text{bnd}}} \|w_h\|_{V_h}. \end{aligned}$$

For I_3, Lemma 4.1, Lemma 4.2 and the definitions of the norms $\|\cdot\|_{V_h}$, $\|\cdot\|_{V_{\text{bnd}}}$ yield

$$\begin{aligned} |I_3| &\le \Big(\sum_{K\in\mathcal{T}_\nu} \sum_{F\in\mathcal{F}_K} h_F \Big\| \nabla(w_{h|\mathring{K}}) \cdot \underline{n}_F \Big\|_{L^2(F)}^2 \Big)^{1/2} |v|_{\mathrm{J}} \\ &\le C_{\text{tr}} \sqrt{d+1} \|\nabla_h w_h\|_{L^2(\Omega)} |v|_{\mathrm{J}} \\ &\le C_{\text{tr}} \sqrt{d+1} \|v\|_{V_{\text{bnd}}} \|w_h\|_{V_h}. \end{aligned}$$

For I_4, the Cauchy–Schwarz inequality and the definitions of the norms $\|\cdot\|_{V_h}$, $\|\cdot\|_{V_{\text{bnd}}}$ give

$$|I_4| \le \omega_0 \sum_{F\in\mathcal{F}_\nu} \frac{1}{h_F^{1/2}} \|[v]_F\|_{L^2(F)} \frac{1}{h_F^{1/2}} \|[w_h]_F\|_{L^2(F)} \le \omega_0 \, |v|_{\text{J}} \, |w_h|_{\text{J}} \le \omega_0 \|v\|_{V_{\text{bnd}}} \|w_h\|_{V_h}.$$

To sum up, we conclude that

$$|a_h(v,w_h)| \le \underbrace{(1 + 1 + C_{\text{tr}}\sqrt{d+1} + \omega_0)}_{=C_{\text{bnd}}} \|v\|_{V_{\text{bnd}}} \|w_h\|_{V_h},$$

i.e., the assertion. □

With the last result, all ingredients are given to state an error estimate in $\|\cdot\|_{V_h}$.

Theorem 4.6 (Error estimate in the norm $\|\cdot\|_{V_h}$)**.** *Let the assumptions of Theorem 4.3 and Lemma 4.4 be satisfied, i.e., we assume the following assumptions:*

- *The mesh sequence $(\mathcal{T}_\nu)_{\nu\in\mathbb{N}}$ is shape-regular with constant $c_{\text{F}} > 0$, see* (11)*, and let $p \in \mathbb{N}_0$ be the polynomial degree.*
- *The penalty parameters ω_F are such that*

$$\forall \nu \in \mathbb{N} : \forall F \in \mathcal{F}_\nu : \quad \omega_F = \frac{\omega_0}{h_F}$$

  with a fixed $w_0 > C_{\text{tr}}^2 \cdot (d+1)$, where $C_{\text{tr}} > 0$ is the constant of the discrete trace inequality (Lemma 3.11), which only depends on c_{F}, p, d.
- *The exact solution u of the variational formulation* (15) *satisfies the regularity $u \in V_{\text{smo}} = H_0^1(\Omega) \cap H^2(\Omega)$.*

Then, the discrete solution $u_h \in V_h$ fulfils the error estimate

$$\|u - u_h\|_{V_h} \le \left(1 + \frac{C_{\text{bnd}}}{C_{\text{coe}}}\right) \inf_{v_h\in V_h} \|u - v_h\|_{V_{\text{bnd}}}, \tag{22}$$

where the constant C_{bnd} comes from Lemma 4.5 and the constant C_{coe} comes from Theorem 4.3.

Proof. Theorem 4.3 (coercivity), Lemma 4.4 (consistency) and Lemma 4.5 (boundedness) ensure that the assumptions of Theorem 2.7 (abstract nonconforming error estimate) are satisfied. Thus, the assertion is proven. □

Note that the constant $1 + \frac{C_{\text{bnd}}}{C_{\text{coe}}}$ of the error estimate (22) is **independent** of h.

Next, we state a convergence result, which is optimal for the broken gradient and jump seminorm.

Corollary 4.7. *Let the assumptions of Theorem 4.6 be satisfied. Additional, the polynomial degree p fulfils $p \geq 1$, and assume that the exact solution u belongs to $H^{p+1}(\Omega)$. Then, the error estimate*

$$\|u - u_h\|_{V_h} \leq C h^p \|u\|_{H^{p+1}(\Omega)}$$

holds true with a constant $C > 0$ independent of h.

Proof. First, for an element $K \in \mathcal{T}_\nu$, we define the elementwise $L^2(\mathring{K})$-projection

$$Q_{\mathring{K}} \colon\ L^2(\mathring{K}) \to \mathbb{P}_d^p(\mathring{K})$$

by

$$\forall w_h \in \mathbb{P}_d^p(\mathring{K}) : \quad \int_{\mathring{K}} (Q_{\mathring{K}} z)(x) w_h(x) \mathrm{d}x = \int_{\mathring{K}} z(x) w_h(x) \mathrm{d}x$$

for a given function $z \in L^2(\mathring{K})$.

Next, in the inequality (22), we use the elementwise $L^2(\mathring{K})$-projection $Q_{\mathring{K}}$ on $\mathbb{P}_d^p(\mathring{K})$ as v_h, i.e.,

$$v_{h|\mathring{K}} = Q_{\mathring{K}}(u_{|\mathring{K}})$$

for $K \in \mathcal{T}_\nu$, see [2, Corollary 4.18] for more details. □

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