Quantum Oscillations Can Prevent the Big Bang Singularity in an Einstein-Dirac Cosmology

Felix Finster, Regensburg

Fakultät für Mathematik
Universität Regensburg

Johannes-Kepler-Forschungszentrum
für Mathematik, Regensburg

Talk at Workshop “Electromagnetic Spacetimes”
Wolfgang-Pauli-Institut Wien, 22.11.2012

joint work with Christian Hainzl (Tübingen)

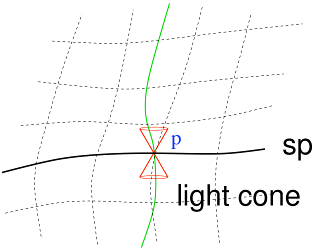
- “Quantum oscillations can prevent the big bang singularity in an Einstein-Dirac cosmology,” arXiv:0809.1693 [gr-qc], *Found. Phys.* **40** (2010) 116-124
- “A spatially homogeneous and isotropic Einstein-Dirac cosmology,” arXiv:1101.1872 [math-ph], *J. Math. Phys* **52**, 042501 (2011)

Our mathematical model of space-time is Minkowski space or, more generally, a Lorentzian manifold (M, g)

- 4-dimensional topological manifold
- metric g of signature $(+ - - -)$

Tangent space T_pM is vector space with indefinite inner product

encodes causal structure : $\begin{cases} g(u,u) > 0 & : \quad u \text{ is timelike} \\ g(u,u) = 0 & : \quad u \text{ is lightlike} \\ g(u,u) < 0 & : \quad u \text{ is spacelike} \end{cases}$



The gravitational field is described by the curvature of M

∇ : covariant derivative, Levi-Civita connection,

$$\nabla_i X = \left(\partial_i X^j + \Gamma^j_{\ ik} X^k \right) \frac{\partial}{\partial x^j}$$

$R^i_{\ jkl}$: Riemann curvature tensor,

$$\nabla_i \nabla_j X - \nabla_j \nabla_i X = R^l_{\ ijk} X^k \frac{\partial}{\partial x^l}$$

$R_{ij} = R^l_{\ ilj}$: Ricci tensor, $R = R^i_i$: scalar curvature

Einstein's equations: $R_{jk} - \frac{1}{2} R\, g_{jk} = 8\pi T_{jk}$

T_{jk} : enery-momentum tensor, describes matter

"matter generates curvature"

vice versa:

"curvature affects the dynamics of matter"

equations of motion, depend on type of matter:

- classical point particles: geodesic equation
- dust: perfect fluid
- quantum mechanical matter:
  equations of wave mechanics
  (Dirac or Klein Gordon equation)
- ......

coupling Einstein equations with equations of motion yields

system of nonlinear hyperbolic PDEs

Dirac spinors in Minkowski space

Relativic wave equation with spin

- $\mathcal{D}$ differential operator of first order with $\mathcal{D}^2 = -\Box$

$$\mathcal{D} = i\gamma^j \partial_j \qquad \mathcal{D}^2 = -\gamma^j \gamma^k \partial_{jk}$$

The Dirac matrices γ^j are (4×4)-matrices with

$$\gamma^j \gamma^k + \gamma^k \gamma^j = 2\, g^{jk}\, \mathbb{1}$$

Dirac representation:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

with the three Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Dirac equation is eigenvalue equation

$$\boxed{\mathcal{D}\Psi = m\Psi}$$

The Zitterbewegung ("trembling motion")

observed by Schrödinger (1930)

- consider only time-dependence

$$i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \partial_t \Psi = m \Psi$$

- can be solved with plane waves:

$$\Psi = \begin{pmatrix} \chi_+ \, e^{-imt} \\ \chi_- \, e^{imt} \end{pmatrix}$$

positive frequency, "large component"
negative frequency, "small component"

- phases drop out of absolute value
- phases do **not** drop out of off-diagonal expectation values:

$$\left\langle \Psi, \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \Psi \right\rangle_{\mathbb{C}^4} \sim \sin(2mt), \cos(2mt)$$

for velocity operator: Zitterbewegung

more generally: quantum oscillations in observables

Dirac spinors on a Lorentzian manifold

Let (M, g) be a Lorentzian manifold,

$$\text{Dirac operator} \qquad \mathcal{D} = i\gamma^j \nabla_j$$

where Dirac matrices again satisfy the anti-commutation relations

$$\gamma^j \gamma^k + \gamma^k \gamma^j \;=\; 2g^{jk}$$

and ∇ is the metric connection on the spinor bundle

- compatible with inner product $\prec.|.\succ$ on spinors,

$$\partial \prec\Psi|\Phi\succ = \prec(\nabla\Psi)|\Phi\succ + \prec\Psi|(\nabla\Phi)\succ$$

- curvature related to Riemann tensor by

$$[\nabla_j, \nabla_k] \;=\; \frac{1}{8}\, R_{jklm} \gamma^l \gamma^m$$

$$\boxed{\text{Dirac equation} \qquad \mathcal{D}\Psi = m\Psi}$$

The spherically symmetric, static ED equations

F-Smoller-Yau (1998-2000)

- spherically symmetric, static metric:

$$ds^2 = \frac{1}{T(r)^2}\, dt^2 - \frac{1}{A(r)}\, dr^2 - r^2\, d\Omega^2$$

- consider a singlet of two Dirac particles:

$$\Psi_a(t,r) = e^{-i\omega t}\, \frac{\sqrt{T}}{r} \begin{pmatrix} \alpha(r)\, e_a \\ i\sigma^r\, \beta(r)\, e_a \end{pmatrix} \qquad (a = 1, 2)$$

frequency ω is the energy of the Dirac particle

The spherically symmetric, static ED equations

Dirac equation:

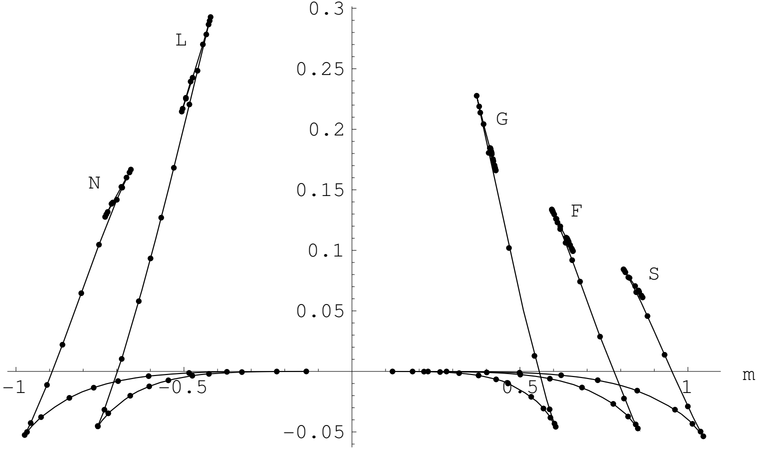
$$\begin{aligned}\sqrt{A}\,\alpha' &= \frac{1}{r}\,\alpha \;-\; (\omega T + m)\,\beta \\ \sqrt{A}\,\beta' &= (\omega T - m)\,\alpha \;-\; \frac{1}{r}\,\beta\end{aligned}$$

Einstein equations:

$$\begin{aligned}r\,A' &= 1 - A \;-\; 16\pi\omega T^2\,(\alpha^2 + \beta^2) \\ 2rA\,\frac{T'}{T} &= A - 1 \;-\; 16\pi\omega T^2\,(\alpha^2 + \beta^2) \\ &\quad + 32\pi\,\frac{1}{r}\,T\,\alpha\beta \;+\; 16\pi\,mT\,(\alpha^2 - \beta^2)\end{aligned}$$

The spherically symmetric, static ED equations

Particlelike solutions



Total Binding Energy $\rho - 2|m|$ (where ρ is the ADM-mass)

The spherically symmetric, static ED equations

Nonexistence of black hole solutions

- spherical symmetry + horizon $\Longrightarrow$ no flux of Dirac current across horizon
- current conservation $\Longrightarrow$ no Dirac current outside horizon

As a consequence, Ψ vanishes identically outside the horizon.

"The Dirac particle must fall into the black hole"

The spherically symmetric, static ED equations

Existence results for ED solutions for small m:

- Eric Bird ($\approx$ 2005 at UMich): Schauder's fixed point theorem
- Simona Rota Nodari (Paris): Relate to the Choquard equation
- John Stuart (Cambridge): Variational methods

Numerics for time-dependent ED system:

- Jason Ventrella ($\approx$ 1998, student of Matt Choptuik)
- Benedikt Zeller (PhD thesis ETH 2009)

Non-Existence of EDYM black hole solutions:

- Yann Bernard ($\approx$ 2005 now Freiburg)

Cosmological solutions

Consider homogeneous and isotropic space-times:

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

- time coordinate t, spatial coordinates r and $\Omega \in S^2$)
- $R(t)$ is so-called scale function
- k determines spatial geometry:

$$\begin{cases} \text{k=0} & \text{flat universe, } r > 0, & M \simeq \mathbb{R} \times \mathbb{R}^3 \\ \text{k=1} & \text{closed universe, } r \in (0,1), & M \simeq \mathbb{R} \times S^3 \\ \text{k=-1} & \text{open universe, } r > 0 & M \simeq \mathbb{R} \times H^3 \end{cases}$$

Ansatz for matter must be chosen consistently.

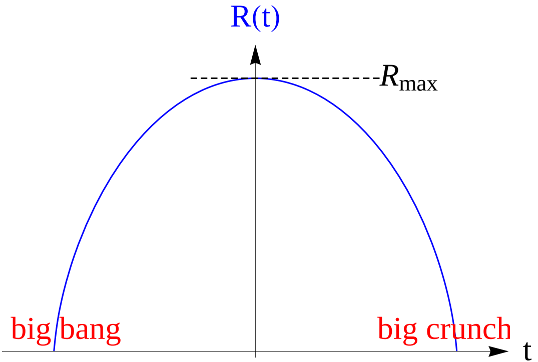
- ► gives a system of nonlinear ODEs

The Friedmann solution

Simplest example:

- closed case (other cases similar)
- choose matter as dust

one gets a single ODE: $$\dot{R}^2 + 1 = \frac{R_{\max}}{R}$$



Do quantum effects prevent singularities?

Different effects discussed in physics literature:

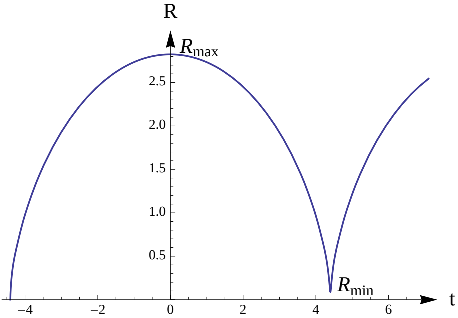
- Wheeler-DeWitt (1967): first ideas in this direction
- Padmanabhan, Narlikar (1982):
  Quantum conformal fluctuations
- Turok, Perry, Steinhardt (2004): String theory, *M*-Theory
- Bojowald (2008): Loop quantum gravity,
  reduction to finite number of degrees of freedom,
  "Once Before Time: A Whole Story of the Universe" (2010)

In all these cases, quantum effects of gravity are essential.

An Einstein-Dirac cosmology

Here we consider a system with classical gravity,
but with quantum mechanical matter (Dirac equation)

- “quantum oscillations” can prevent singularities



- time-periodic solutions with infinite number of expansion and contraction cycles
- simple equations, can be analyzed rigorously

A homogeneous and isotropic Einstein-Dirac system

Back to homogeneous and isotropic space-time,
here only closed case:

$$ds^2 = dt^2 - R^2(t)\, d\sigma^2_{S^3}\ ,$$

where $d\sigma^2$ is the line element on the unit S^3.
The Dirac operator becomes

$$\mathcal{D} = i\gamma^0 \left(\partial_t + \frac{3\dot{R}(t)}{2R(t)} \right) + \frac{1}{R(t)} \begin{pmatrix} 0 & \mathcal{D}_{S^3} \\ -\mathcal{D}_{S^3} & 0 \end{pmatrix} ,$$

where $\mathcal{D}_{S^3}$ is the Dirac operator on S^3.

A homogeneous and isotropic Einstein-Dirac system

- Employ the separation ansatz

$$\Psi^\ell_\lambda = R(t)^{-\frac{3}{2}} \begin{pmatrix} \alpha(t)\, \psi^\ell_\lambda(r, \theta, \varphi) \\ \beta(t)\, \psi^\ell_\lambda(r, \theta, \varphi) \end{pmatrix},$$

where ψ^ℓ_λ are eigenfunctions of the spatial Dirac operator,

$$\mathcal{D}_{S^3}\, \psi^\ell_\lambda = \lambda\, \psi^\ell_\lambda\,, \qquad \ell = 1, \ldots, \lambda^2 - \frac{1}{4}$$

- Then the Dirac equation reduces to an ODE,

$$i \frac{d}{dt} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} m & -\lambda/R \\ -\lambda/R & -m \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

A homogeneous and isotropic Einstein-Dirac system

- The energy-momentum tensor:

$$T_{kl} = \text{Re}\left(i \prec\Psi|\gamma_{(k}\nabla_{l)}\Psi\succ\right)$$

In order to get a homogeneous and isotropic system, occupy a whole eigenspace

$$T_{kl} = \sum_{\ell=1}^{\lambda^2-1/4} \text{Re}\left(i \prec\Psi_\lambda^\ell|\gamma_{(k}\nabla_{l)}\Psi_\lambda^\ell\succ\right)$$

- thus $\lambda^2 - \frac{1}{4}$ particles
- all wave functions have the same time dependence

in physical terms: a coherent many-particle quantum state

A homogeneous and isotropic Einstein-Dirac system

- Putting it all together gives the Einstein-Dirac equations

$$
\begin{aligned}
i \frac{d}{dt} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \begin{pmatrix} m & -\lambda/R \\ -\lambda/R & -m \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
\dot{R}^2 + 1 &= \frac{m}{R} \left(|\alpha|^2 - |\beta|^2 \right) - \frac{\lambda}{R^2} \left(\overline{\beta} \alpha + \overline{\alpha} \beta \right) .
\end{aligned}
$$

system of nonlinear ODEs

The Bloch representation

To simplify the equations:

introduce Bloch vector $\quad \vec{v} = \left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \vec{\sigma} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\rangle_{\mathbb{C}^2}$

rotate Bloch vector $\quad \vec{w} = Uv$ with $U = U(R) \in \mathrm{SO}(3)$

Einstein-Dirac equations in the Bloch representation:

$$\dot{\vec{w}} = \vec{d} \wedge \vec{w}\,, \qquad \dot{R}^2 + 1 = -\frac{1}{R^2}\sqrt{\lambda^2 + m^2 R^2}\, w_1\,,$$

where

$$\vec{d} := \frac{2}{R}\sqrt{\lambda^2 + m^2 R^2}\, e_1 - \frac{\lambda m R}{\lambda^2 + m^2 R^2}\,\frac{\dot{R}}{R}\, e_2\,.$$

- Similarity to movement of a spinning top:
  Bloch vector $\vec{w}$ precedes around “moving rotation axis” $\vec{d}$

Simple limiting cases

- Limit $\lambda \to 0$:

$$\dot{\vec{w}} = 2m\, e_1 \wedge \vec{w}\,, \qquad \dot{R}^2 + 1 = -\frac{m}{R}\, w_1$$

  w_1 is constant, get back to Friedmann equation for dust

- Limit $m \to 0$:

$$\dot{\vec{w}} = \frac{2|\lambda|}{R}\, e_1 \wedge \vec{w}\,, \qquad \dot{R}^2 + 1 = -\frac{|\lambda|}{R^2}\, w_1$$

  again w_1 is constant, gives Friedmann equation in radiation dominated universe

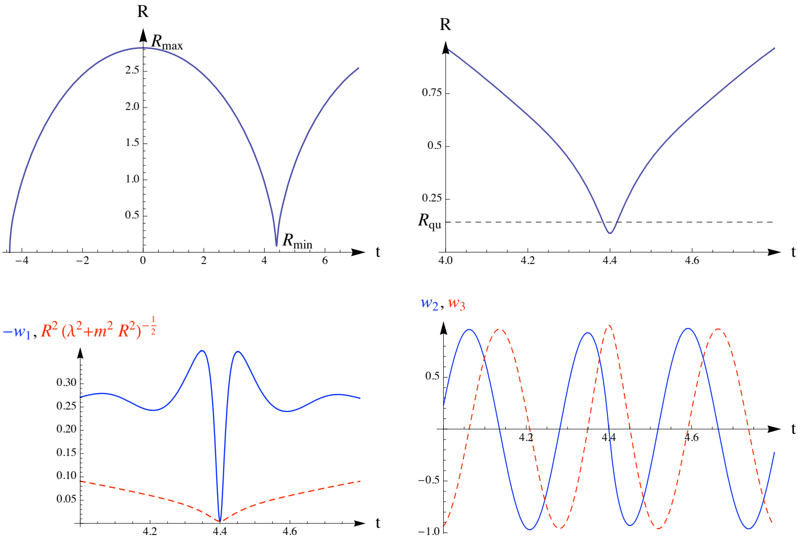
The intermediate region is characterized by

$$R_{\text{qu}} = \lambda/m\,.$$

Here w_1 and thus energy-momentum tensor is oscillatory

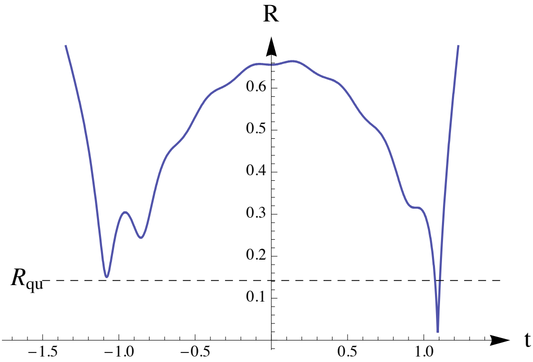
Quantum oscillations of the energy-momentum tensor

Numerical results



Example for $\lambda = \frac{3}{2}$, $m = 10.5448$ and $w_1(R_{max}) = -0.2675$.

Numerical results



Example for $\lambda = \frac{3}{2}$, $m = 10.5448$ and $w_1(R_{\max}) = -0.0608$.

The “approximation of instantaneous tilt”

$$\dot{\vec{w}} = \vec{d} \wedge \vec{w}\,, \qquad \vec{d} := \frac{2}{R}\,\sqrt{\lambda^2 + m^2 R^2}\; e_1 \;-\; \frac{\lambda m R}{\lambda^2 + m^2 R^2}\;\frac{\dot{R}}{R}\; e_2$$

- Begin at $t = 0$, omit the second summand

$$\vec{d} = \frac{2}{R}\,\sqrt{\lambda^2 + m^2 R^2}\; e_1$$

  gives dust approximation, solvable in closed form

- At a radius

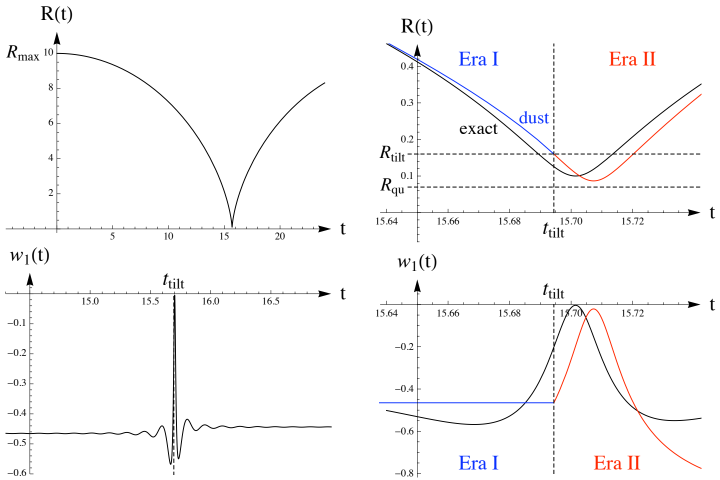
$$R_{\text{tilt}} = \frac{\lambda^{\frac{2}{5}}\; R_{\text{qu}}^{\frac{1}{5}}}{m^{\frac{4}{5}}}$$

  the second summand becomes dominant.
  From then on neglect first summand

$$\vec{d} := -\frac{\lambda m R}{\lambda^2 + m^2 R^2}\;\frac{\dot{R}}{R}\; e_2$$

  again solvable in closed form

The "approximation of instantaneous tilt"



catches "bouncing effect" quite well

- allows to analyze the probability of bouncing, is about 50%

Analytic construction of bouncing solutions

Now to rigorous analysis.

- Introduce the scaling

$$m \to m/\varepsilon, \qquad t \to t/\varepsilon^2,$$
$$R(t) \to \varepsilon R(t/\varepsilon^2), \qquad \lambda \to \lambda, \quad \vec{w}(t) \to \vec{w}(t/\varepsilon^2).$$

- The rescaled equations are

$$\dot{R}^2 + \varepsilon^2 = -\frac{1}{R^2}\sqrt{\lambda^2 + m^2 R^2}\, w_1,$$
$$\dot{\vec{w}} = \left(\varepsilon\, \frac{2}{R}\sqrt{\lambda^2 + m^2 R^2}\, e_1 - \frac{\lambda m \dot{R}}{\lambda^2 + m^2 R^2}\, e_2\right) \wedge \vec{w}.$$

Coefficients are continuous in ε.

Analytic construction of bouncing solutions

- In the limit $\varepsilon \searrow 0$ one gets the microscopic limit equations

$$\dot{R}^2 = -\frac{1}{R^2}\,\sqrt{\lambda^2 + m^2 R^2}\; w_1$$

$$\dot{\vec{w}} = -\frac{\lambda m \dot{R}}{\lambda^2 + m^2 R^2}\; e_2 \wedge \vec{w}$$

- Can be solved by integration:
  Let θ be the angle between $\vec{w}$ and e_1. Then

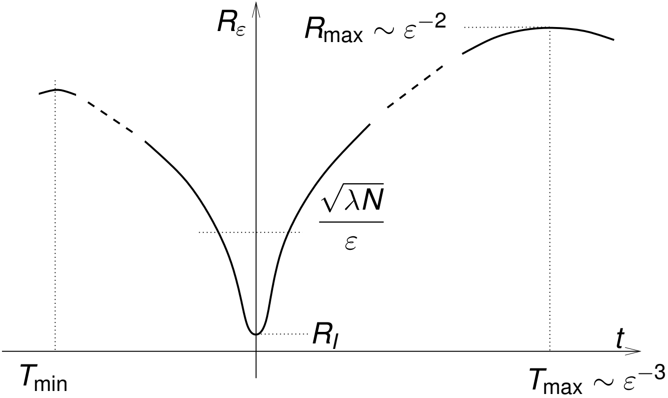
$$\dot{\theta} = -\frac{d}{dt}\,\mathrm{arctan}[R\frac{m}{\lambda}]$$

$$\dot{R}^2 = \frac{1}{R^2}\,\sqrt{\lambda^2 + m^2 R^2}\,\sin\Big(\mathrm{arctan}[R(t)\frac{m}{\lambda}] - \mathrm{arctan}[R_l\frac{m}{\lambda}]\Big)\,.$$

- Use continuous dependence of solutions on ε,
  gives solution near the lower turning point.

Analytic construction of bouncing solutions

- Prove that this solution enters classical dust regime. Gives classical turning point at $T_{\max}$.



Analytic construction of bouncing solutions

THEOREM (Existence of bouncing solutions)

Given $\lambda \in \{\pm\frac{3}{2}, \pm\frac{5}{2}, \ldots\}$ *and* $\delta > 0$ *as well as any radius* $R_>$ *and time* $T_>$, *there is a continuous three-parameter family of solutions* $(R(t), \vec{w}(t))$ *defined on a time interval* $[0, T]$ *with* $T > T_>$ *having the following properties:*

(a) *At* $t = 0$ *and* $t = T$, *the scale function has a local maximum larger than* $R_>$,

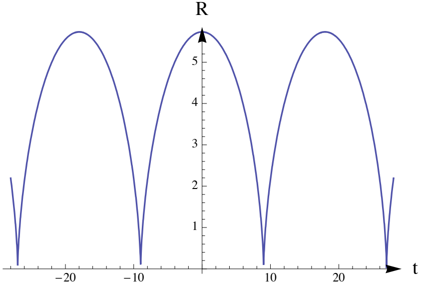
$$R(t) > R_> , \qquad \dot{R}(t) = 0 , \qquad \ddot{R}(t) < 0 .$$

(b) *There is a time* $t_{bounce} \in (0, T)$ *such that* R *is strictly monotone on the intervals* $[0, t_{bounce}]$ *and* $[t_{bounce}, T]$. *Moreover, the scale function becomes small in the sense that*

$$R(t_{bounce}) < \delta \, R_> .$$

Analytic construction of time-periodic solutions

- Choose $\vec{w}(0)$ such that the solution is symmetric under time reversals (in particular $T_{\min} = -T_{\max}$ and $R(T_{\min}) = R(T_{\max})$)
- Task: arrange that $\vec{w}(T_{\min}) = \vec{w}(T_{\max})$.
- This involves only one phase ϕ and the condition $\phi \in 2\pi\mathbb{Z}$. Prove that
$$\phi \sim \varepsilon^{-2}$$
and use continuity.



Analytic construction of time-periodic solutions

THEOREM (Existence of time-periodic solutions)

Given $\lambda \in \{\pm\frac{3}{2}, \pm\frac{5}{2}, \ldots\}$ *and* $\delta > 0$ *as well as any radius* $R_>$ *and time* $T_>$*, there is a one-parameter family of solutions* $(R(t), \vec{w}(t))$ *defined for all* $t \in \mathbb{R}$ *with the following properties:*

(A) *The solution is periodic, i.e.*

$$R(t+T) = R(t)\,, \qquad \vec{w}(t+T) = \vec{w}(t) \qquad \text{for all } t \in \mathbb{R}\,,$$

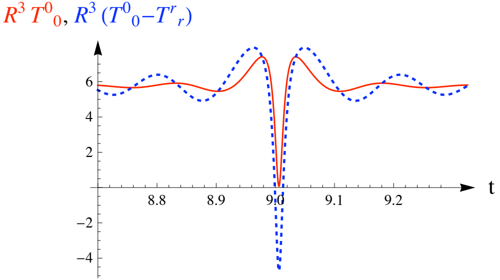
and every $T > 0$ *with this property is larger than* $T_>$*.*

(B) $\inf_{\mathbb{R}} R(t) < \delta\, R_>\,, \qquad R_> < \sup_{\mathbb{R}} R(t)\,.$

The energy conditions

$$\begin{cases} \text{weak energy condition:} & T^0_0 \geq \max\left(0, T^r_r\right) \\ \text{dominant energy condition:} & T^0_0 \geq |T^r_r| \\ \text{strong energy condition:} & T^0_0 \geq \max\left(3\, T^r_r, T^r_r\right). \end{cases}$$

- The strong energy condition is always violated at a bounce.
- All energy conditions are violated close to a bounce for ε small enough.



This gives consistency with Hawking-Penrose singularity theorems.

Physical discussion and summary

- The "bouncing effect" relies on the fact that all particles have the same momentum λ.
- All particle wave functions are coherent ("in phase")

spin condensation

If before the big crunch all fermions of the universe form a coherent many-particle state (spin condensate), then quantum oscillations can prevent the big crunch singularity.

- work with classical gravity and quantum mechanics
- simple ODE system, rigorous results

Thank you for your attention!