Fresnel versus Kummer surfaces: geometrical optics in dispersionless linear (meta)materials and vacuum

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Outline

- Dispersionless linear (meta)materials and vacuum. Find 3 components: principal part, skewon part & axion part.
- Bateman's treatment of dispersionless linear media (1910). Seemingly first to include **non-zero axion** part.
- Geometrical optics results in a quartic **Fresnel surface**.
- Bateman relates geometrical optics and lines in real projective space. For dispersionless linear media with no skewon part, the Fresnel surface is a **Kummer surface**.
- What if the medium has **non-zero skewon** part? Does the Fresnel surface still coincide with Kummer one? If not, is Fresnel surface a K3 or a Calabi-Yau manifold?

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$$\mathcal{H}_a = \beta_a{}^c E_c + \tfrac{1}{2}(\mu^{-1})_a{}^{cd} B_{cd}\,,$$
$$\mathcal{D}_{ab} = \varepsilon'_{ab}{}^c E_c + \tfrac{1}{2}\alpha_{ab}{}^{cd} B_{cd}\,.$$

Latin indices range from 1 to 3. Observe that $\varepsilon'_{ab}{}^c$ is the **permittivity**, $(\mu^{-1})_a{}^{cd}$ is the **inverse permeability**, while $\beta_a{}^c$ and $\alpha_{ab}{}^{cd}$ describe **magneto-electric** effects.

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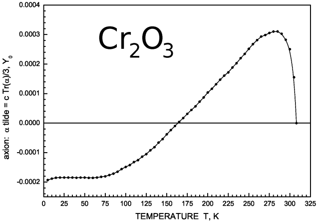
- Using 2-forms $H = d\sigma \wedge \mathcal{H} + \mathcal{D}$ and $F = -d\sigma \wedge E + B$ get

$$H = \kappa(F), \qquad \text{that is,} \qquad H_{\alpha\beta} = \tfrac{1}{2}\kappa_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}.$$

  Greek indices go from 0 to 3. Constitutive law in 4-dim.

Principal-Skewon-Axion split of constitutive law



- To decompose medium response translate ${\kappa_{\alpha\beta}}^{\mu\nu}$ into:

$$\chi^{\alpha\beta\mu\nu} := \frac{1}{2}\epsilon^{\alpha\beta\rho\sigma}{\kappa_{\rho\sigma}}^{\mu\nu},$$

  where $\epsilon^{\alpha\beta\gamma\delta} = \{+1, 0, -1\}$ is the Levi-Civita symbol.
- Split χ in **principal** part, **skewon** part and **axion** part:

$$\chi^{\alpha\beta\mu\nu} = {}^{(1)}\chi^{\alpha\beta\mu\nu} + {}^{(2)}\chi^{\alpha\beta\mu\nu} + {}^{(3)}\chi^{\alpha\beta\mu\nu}.$$

  Note: ${}^{(1)}\chi$ is the symmetric-traceless component, ${}^{(2)}\chi$ is the antisymmetric component and ${}^{(3)}\chi$ is the trace.
- Derive equivalent split for κ. Finite axion part observed in nature (Hehl et al. 2008). Finite skewon part not yet.

H. Bateman, Proc. Lond. Math. Soc., s2–8, 1910

- Harry Bateman (Manchester 1882 – New York 1946). Students: Murnaghan (@Hopkins), Truesdell (@Caltech).
- In a modern notation, Bateman's constitutive law reads

$$\check{F}^{\alpha\beta} = -\tfrac{1}{2}\theta^{\alpha\beta\mu\nu}H_{\mu\nu},$$

with $\check{F}^{\alpha\beta} := \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}$. In terms of 3-dim. fields obtain

$$\check{B}^a = -\theta^{0a0c}\mathcal{H}_c - \tfrac{1}{2}\theta^{0acd}\mathcal{D}_{cd},$$
$$\check{E}^{ab} = +\theta^{ab0c}\mathcal{H}_c + \tfrac{1}{2}\theta^{abcd}\mathcal{D}_{cd}.$$

- Bateman **only** requires (Preview: this means no skewon)

$$\theta^{\alpha\beta\mu\nu} = \theta^{\mu\nu\alpha\beta}.$$

It is **not** demanded that fully antisymmetric part of $\theta^{\alpha\beta\mu\nu}$ is zero. Preview: axion component is allowed.

Bateman: no skewon part, but axion part allowed

- Link Bateman's medium tensor θ to **inverse** of κ and χ:

$$\theta^{\alpha\beta\mu\nu} = -\tfrac{1}{2}\epsilon^{\alpha\beta\rho\sigma}\kappa^{-1}_{\rho\sigma}{}^{\mu\nu} = -\tfrac{1}{4}\epsilon^{\alpha\beta\rho\sigma}\epsilon^{\mu\nu\eta\theta}\chi^{-1}_{\rho\sigma\eta\theta}.$$

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- Inverse of a symmetric "matrix" is symmetric. Thereby,

$$\theta^{\alpha\beta\mu\nu} = \theta^{\mu\nu\alpha\beta} \qquad \text{implies} \qquad \chi^{\alpha\beta\mu\nu} = \chi^{\mu\nu\alpha\beta}.$$

Condition on Bateman's θ imposes that χ is symmetric.
Skewon part vanishes: ${}^{(2)}\chi = 0$, or equivalently ${}^{(2)}\kappa = 0$.

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- "*These conditions* [constitutive laws] *may not correspond to anything occurring in nature; nevertheless, their investigation was thought to be of some mathematical interest on account of the connection which is established between line geometry and the theory of partial differential equations*", ibid. (1910).

Geometrical optics & the Fresnel surface

- 4-dimensional Maxwell's equations with exterior calculus

$$\mathrm{d}H = J, \qquad \mathrm{d}F = 0.$$

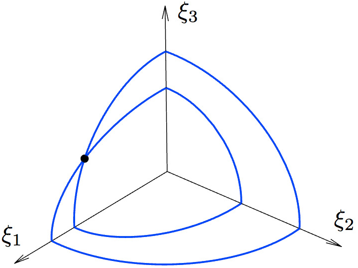
- Below, current density 3-form J is zero. **Geometrical optics** describes the propagation of fast varying fields.
- Have two **equivalent** approaches to geometrical optics,
  - via Hadamard's method: consider discontinuous fields.
  - via characteristic polynomial: approximate plane-waves.

  First approach, see Hehl and Obukhov (2003). Second approach, see Schuller et al. (2010) or Perlick (2011).
- Geometrical optics says amplitude **2-forms** $\{h, f\}$ obey

$$q \wedge f = 0, \qquad q \wedge h = 0.$$

Here, $q = -\omega \mathrm{d}\sigma + k$ is **wave-covector**. Replacement $\mathrm{d} \to q$ similar to $\partial_t \to -\mathrm{i}\omega$ and $\vec{\nabla} \to \mathrm{i}\vec{k}$ with Fourier tr.

Geometrical optics & the Fresnel surface (contd.)



- Geom. optics laws $q \wedge f = 0$ and $q \wedge h = 0$ **equivalent** to

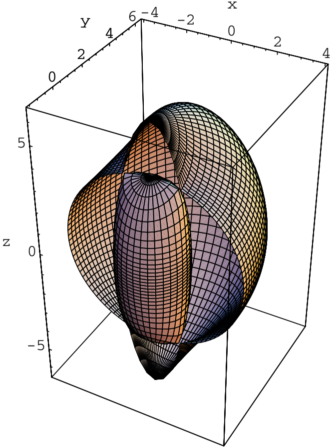
$$f \wedge f = 0, \qquad h \wedge f = 0, \qquad h \wedge h = 0.$$

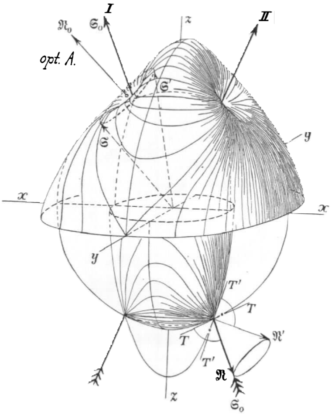
- Assume dispersionless linear (meta)material or vacuum. **Fresnel surface** governing light propagation is given by

$$\hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3}\hat{\epsilon}_{\beta\beta_1\beta_2\beta_3}\chi^{\alpha\alpha_1\beta\beta_1}\chi^{\alpha_2\rho\beta_2\sigma}\chi^{\alpha_3\tau\beta_3\upsilon}q_\rho q_\sigma q_\tau q_\upsilon = 0\,,$$

that is, by a **quartic** equation in $q_\alpha = (-\omega, k_i)$, see Rubilar (2002). Usually plot the inverse phase velocity k_i/ω. Above, Fresnel s. of biaxial crystal (Dahl 2012).

Fresnel surfaces of two other biaxial materials





Left: generated by Tertychniy, 2004. **Right**: Schaefer, 1932.

More on geometric optics: Tamm-Rubilar tensor

- From quartic generating Fresnel surface extract tensor:

$$\mathcal{G}(q) := \mathcal{G}^{\rho\sigma\tau\upsilon} q_\rho q_\sigma q_\tau q_\upsilon = 0\,,$$

$$\mathcal{G}^{\rho\sigma\tau\upsilon} := \frac{1}{4!}\hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3}\hat{\epsilon}_{\beta\beta_1\beta_2\beta_3}\chi^{\alpha\alpha_1\beta\beta_1}\chi^{\alpha_2(\rho\sigma|\beta_2}\chi^{\alpha_3|\tau\upsilon)\beta_3}\,.$$

Note $(\dots)$ is index mixing: $\mathcal{G}^{\rho\sigma\tau\upsilon}$ is **symmetric** under every index swap and has 35 independent components.

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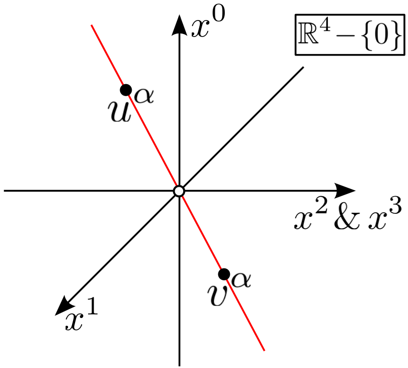
- Axion part does **not** enter geometrical optics (except at interfaces). Zero ${}^{(1)}\chi$ implies zero Tamm-Rubilar tensor.

Bateman's insight on geometrical optics



- Bateman: for media with ${}^{(2)}\kappa = 0$, the Fresnel surface coincides exactly with a **Kummer surface**. Just relate geometrical optics and lines in the real projective space.
- To define the **real projective space** $\mathbb{R}P^3$ consider as identical every two points u^α and v^α in $\mathbb{R}^4 - \{0\}$ that are located on same line: $u^\alpha = \lambda v^\alpha$ for non-zero $\lambda \in \mathbb{R}$.

Bateman's insight on geometrical optics (contd.)

- First step to prove that Fresnel=Kummer is to examine geometric optics eqs. for skewon-free medium $h = \kappa(f)$:

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$$f \wedge \kappa(f) = 0, \qquad \Leftrightarrow \qquad \tfrac{1}{4}\chi^{\alpha\beta\mu\nu} f_{\alpha\beta} f_{\mu\nu} = 0.$$

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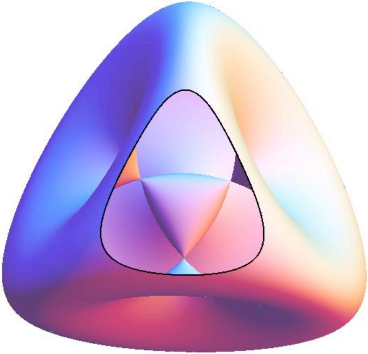
An example of Kummer (Fresnel) surface. . .



Kummer discovered his surfaces by considering ray tracing in optical instruments (1864). Note: two sheets=birefringence.

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- Singularities of the Kummer surfaces are well studied (Hudson 1905). Singularities of the Fresnel surface (**optical axes**) usually examined in simple cases only.

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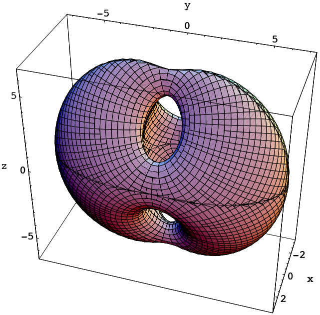
Algebraic methods are still remarkable...

In the literature on Kummer surface apparently no sign of the algebraic compact formula known for the Fresnel surface:

$$\hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3}\hat{\epsilon}_{\beta\beta_1\beta_2\beta_3}\chi^{\alpha\alpha_1\beta\beta_1}\chi^{\alpha_2\rho\beta_2\sigma}\chi^{\alpha_3\tau\beta_3\upsilon}q_\rho q_\sigma q_\tau q_\upsilon = 0\,.$$

Media with skewon: Fresnel=Kummer still true?

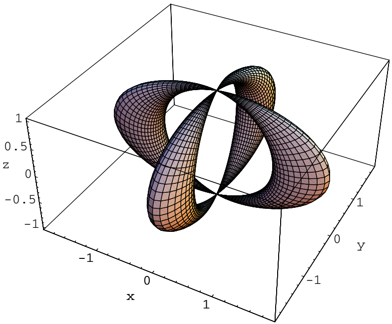


- Bateman's proof that Fresnel=Kummer assumes zero skewon part. If medium has **finite** skewon contribution:

$$\mathcal{G}^{\rho\sigma\tau\upsilon} = \mathcal{G}\left[{}^{(1)}\chi\right]^{\rho\sigma\tau\upsilon} + {}^{(1)}\chi^{\mu(\rho|\nu|\sigma}\,\$_{\mu}{}^{\tau}\,\$_{\nu}{}^{\upsilon)}.$$

- Effect of skewon (2nd term) appears simpler than that of principal (1st term). But can yield **holes** in Fresnel surf!
- **Above**: biaxial medium with $\varepsilon^{ab} = \mathrm{diag}(2.4, 14.8, 54)\varepsilon_0$ and **skewon** $\$_1{}^1 = \$_2{}^2 = \$_3{}^3 = -\frac{1}{3}\$_0{}^0 = 0.25(\varepsilon_0/\mu_0)^{\frac{1}{2}}$.

Skewon: Fresnel=Kummer still true? (contd.)



- Is the Fresnel surface of a medium with finite skewon part **still** a Kummer surface? **If not**, it is more general.
- Look at surfaces types of which Kummer is a subcase:
  - **K3** surfaces: named after Kummer, Kähler and Kodaira.
  - **Calabi-Yau** manifolds: used in superstring theory to compactify 6 spatial dimensions and retrieve $10 - 6 = 4$.
- Plot above and in the previous slide: Tertychniy (2004).

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- Bateman: if medium has **zero** skewon, Fresnel surface is a **Kummer** surface. The natural electromagnetic space of geometrical optics is the **real projective space** $\mathbb{R}P^3$.

Conclusions (Batemania!)

- Dispersionless linear (meta)materials & vacuum: $\kappa_{\alpha\beta}{}^{\mu\nu}$. Decompose this tensor in principal+skewon+axion part.
- Already in 1910, constitutive law of similar kind found in Bateman. Seemingly first to have **non-zero axion** part.
- **Fresnel** surface describes light propagation in geometric optics. Generated by quartic equation (Tamm-Rubilar).
- Bateman: if medium has **zero** skewon, Fresnel surface is a **Kummer** surface. The natural electromagnetic space of geometrical optics is the **real projective space** $\mathbb{R}P^3$.
- What if medium has **finite** skewon? Is Fresnel surface a **K3** surface? or is it more general **Calabi-Yau** manifold?

Thank-you!