Dispersive blow up for Schrödinger type equations

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We aim to study the appearance and role of dispersive singularities in Schrödinger type equations and in the linearized water waves equations (possible relevance to rogue waves).

Three different type of blow-up for conservative equations

- 1. "Hyperbolic blow-up" (blow-up of gradients).

Typical example the Burgers equation

$$u_t + uu_x = 0, \quad u(\cdot, 0) = u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

Blow-up of $||u_x(\cdot, t)||_\infty$ at $T^\star = -\frac{1}{\inf u_0'}$, but u remains bounded (shock formation).

- Influence of a "weak" dispersive perturbation ("dispersive Burgers") :

$$u_t + uu_x - D^\alpha u_x = 0, \quad \widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

($\alpha = 2$: KdV, $\alpha = 1$: Benjamin-Ono. Global well-posedness for $\alpha > 1$)

The case $\alpha = -\frac{1}{2}$ is close (for short waves) to the dispersion of gravity waves that have the phase velocity :
$p(\xi) = \left(\frac{\tanh \xi}{\xi}\right)^{1/2}$

Blow-up in the case $\alpha < 1$ is delicate

- $-1 < \alpha < 0$: one still have blow-up of gradient (Castro-Córdoba-Gancedo 2010, after previous works of Naumkin-Shishmarev 1994, Constantin-Escher 1998).
- $0 < \alpha < 1$, Linares-Pilod-S, Klein-S (numerics), in progress. It is claimed by Zakharov 2000 that there is no hyperbolic blow-up.

- 2. "Nonlinear dispersive blow-up".
- L^2 critical and super critical focusing NLS : $p \geq \frac{4}{n}$ in

$$i\psi_t + \Delta\psi + |\psi|^p\psi = 0, \quad \psi = \psi(x,t), x \in \mathbb{R}^n, t \in \mathbb{R}.$$

(Vlasov-Petrishev-Talanov 1971, Zakharov 1972 ; Glassey 1977, Ginibre-Velo 1979 ; Merle-Raphaël 2005).
Blow-up of $|\nabla\psi|_{L^2}$ and of $|\psi|_\infty$.

- Similar result for the L^2 critical and super critical KdV equation

$$u_t + u^p u_x + u_{xxx} = 0, \quad p \geq 4.$$

- Proved for $p = 4$ (Martel-Merle 2002), conjectured for $p > 4$ (numerics by Bona-Dougalis-Karakashian-McKinney 1990).
- For dispersible Burgers, the L^2 critical case corresponds to $\alpha = \frac{1}{2}$.

- 3. "Dispersive blow-up".

Dispersive blow-up is a focussing type of behavior which is due to both the unbounded domain in which the problem is set and the propensity of the dispersion relation to propagate energy at different speeds. These two aspects allow the possibility that widely separated, small disturbances may come together locally in space-time, thereby forming a large deviation from the rest position. Possible relevance for explaining the genesis of rogue waves on the surface of large bodies of water and in electrical networks.

One of the proposed routes to rogue-wave formation is *concurrence*. This is the idea that the ambiant wave motion in a big body of water possesses a large amount of energy which could, in the right circumstances, temporarily coallesce in space, leading to giant waves.

Rogue waves, (freak waves), occur in both deep and shallow water. While the free surface Euler equations could be taken as the overall governing equations in both deep and shallow water, there is much to be learned from approximate models.
These, however, differ in deep and shallow water regimes. Bona-S. 1993 dealt with the shallow water situation, exemplified by the Korteweg-de Vries equation and Boussinesq-type systems of equations.
Interest is focussed here on Schrödinger type equations (occurring for instance as deep water wave models) and on the linearized surface water waves.

Old results on KdV

The starting point is the classical paper of Benjamin, Bona and Mahony (1972) where they pointed out the "bad" behaviour of the "Airy" equation (which was introduced by Stokes !) with respect to high frequencies.

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \\ u(.,0) = \phi \end{cases} \tag{1}$$

Take

$$\phi(x) = \frac{Ai(-x)}{(1+x^2)^m},$$

with

$$\frac{1}{8} < m < \frac{1}{4},$$

where Ai is the Airy function (now the name is correct though Airy did not introduced it in the context of water waves !).

Then $\phi \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$.
The solution $u \in C(\mathbb{R}_+; L^2(\mathbb{R}))$ is given by

$$\frac{c}{t^{\frac{1}{3}}} \int_{\mathbb{R}} Ai(\frac{x-y}{t^{\frac{1}{3}}}) \frac{Ai(-y)}{(1+y^2)^m} dy.$$

When $(x,t) \to (0,1)$, $u(x,t) \to c \int_{\mathbb{R}} \frac{Ai^2(-y)}{(1+y^2)^m} dy = +\infty$.

Actually one can prove with some extra work using the asymptotics of the Airy function (Bona-S. 1993) that
u is continuous on $\mathbb{R} \times \mathbb{R}_+^*$ except at $(x,t) = (0,1)$.

A similar "dispersive blow-up" holds true for the KdV equation (or any generalized KdV equation). For instance (Bona-S. 1993) :

Theorem

Let $(x*, t*) \in \mathbb{R} \times \mathbb{R}_+^*$. *There exists* $\phi \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *such that the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} + uu_x + \frac{\partial^3 u}{\partial x^3} = 0, \\ u(.,0) = \phi \end{cases} \tag{2}$$

has a unique solution $u \in C([0,\infty); L^2(\mathbb{R}) \cap L^2_{loc}(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))$ *which is continuous on* $(\mathbb{R} \times \mathbb{R}_+^*) \setminus (x*, t*)$ *and satisfies*

$$\lim_{(x,t)\to(x*,t*)} |u(x,t)| = +\infty.$$

Sketch of proof : one can reduce to $(x*, t*) = (0, 1)$. We take ϕ leading to the dispersive blow-up of the linear Airy equation. Write the solution as

$$u(x,t) = S(t)\phi(x) + \int_0^t \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{1}{3}}} Ai\Big(\frac{(x-y)}{(t-s)^{\frac{1}{3}}}\Big) uu_x(y,s)dsdy$$

$$= S(t)\phi(x) + C\int_0^t \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{2}{3}}} Ai'\Big(\frac{(x-y)}{(t-s)^{\frac{1}{3}}}\Big) u^2(y,s)dsdy$$

This seems silly since Ai' grows as $(-x)^{\frac{1}{4}}$ as $x \to -\infty$. The solution is to work in a weighted L^2 space and u^2 will compensate the growth of Ai'.

More precisely one can solves the Cauchy problem in the weighed space $L^2(\mathbb{R}, w)$ where

$$w(x) = w_\sigma(x) = \begin{cases} 1 & \text{for } x < 0 \\ (1+x^2)^\sigma & \text{for } x > 1. \end{cases}$$

Choosing $\frac{3}{16} < m < \frac{1}{4}$, the initial data $\phi \in L^2(\mathbb{R}, w_\sigma)$ where $\sigma \geq \frac{1}{16}$. The linear part still blows up at $(0, 1)$. On the other hand the Duhamel integral is shown to be a continuous function of (x, t). So the nonlinear solution blows up exactly at $(0, 1)$.

- Similar results for any generalized KdV equations and for the C^k norms (for data in H^k).
- Alternative proof for modified KdV (Linares-Scialom 1993).

The same method works for the linear Schrödinger equation, (and more generally for linear dispersive equations with unbounded phase velocity , $\lim_{k\to\infty} c(k) = \frac{\omega(k)}{k} = \infty$) :

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta u = 0, \\ u(.,0) = \phi \end{cases} \tag{3}$$

Theorem

Let $(x^*, t^*) \in \mathbb{R}^d \times (0, +\infty)$ *be given. There exist functions* ϕ *lying in the class* $C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *such that the corresponding solution u of (3) satisfies*

1. $u \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^d))$,
2. *u is a continuous function of* (x, t) *on* $\mathbb{R}^d \times ((0, +\infty) \setminus \{t^*\})$,
3. $u(\cdot, t^*)$ *is a continuous function of x on* $\mathbb{R}^d \setminus \{x^*\}$ *and*
4. $$\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*)\\(x,t)\neq(x^*,t^*)}} |u(x,t)| = +\infty.$$

Take :

$$\phi(y) = \frac{e^{-i|y|^2}}{(1+|y|^2)^m}. \tag{4}$$

with

$$\frac{d}{4} < m \leq \frac{d}{2}. \tag{5}$$

Remark

In particular, one deduces from the previous Theorem that for any fixed $t \in (0, +\infty) \setminus \{t^*\}$, the function $x \mapsto u(x,t)$ is continuous on $\mathbb{R}^d$ (but not necessarily bounded).

Remark

With some modifications, a similar analysis applies to what is sometimes called the "hyperbolic" Schrödinger equation, namely

$$\begin{cases} i\partial_t u + \partial_{xx} u - \partial_{yy} u = 0, \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\ u(x,0) = \phi(x), \quad \text{for} \quad x \in \mathbb{R}^2. \end{cases} \tag{6}$$

This equation is the linearization about the rest state of a model for deep water surface gravity waves (Zakharov 1968).

- By linearity, the dispersive blow-up is stable to smooth (and localized) perturbations of the initial data leading to DBU.
- Can be used (by truncation) to construct smooth initial data which are arbitrary small in $L^\infty(\mathbb{R}^d)$ and lead to solutions which have very large values at a specified, dispersive blow-up point :

Let ϕ the initial data leading to DBU at $(x,t)=(0,\frac{1}{4})$. Let $\phi_R(y)=\rho\left(\frac{|y|}{R}\right)\phi(y)$, where $\rho\in\mathcal{C}_0^\infty(\mathbb{R}),\ \rho\geq 0, \rho\equiv 0$ on $[0,1]$, $\rho\equiv 0$ for $|y|\geq 2$.
Let $\delta>0$ small and $m>0$ large. Take as initial data $u_0(y)=\delta\phi_R(y)$.
Then $u_0\in H^k(\mathbb{R}^d)\ \forall k$ and $||u_0||_\infty=O(\delta)$ as $\delta\to 0$.
The solution corresponding to u_0 satisfies

$$u\left(0,\frac{1}{4}\right)=\delta\int_{\mathbb{R}^d}\frac{\rho\left(\frac{|y|}{R}\right)}{(1+|y|^2)^m}dy\geq\delta\int_{|y|\leq R}\frac{1}{(1+|y|^2)^m}dy$$

The last integral is bounded below by $\delta O(R^{d-2m})$ when $m<\frac{d}{2}$ and $\delta O(\log R)$ when $m=\frac{d}{2}$.

Same result for the NLS in one dimension :

$$iu_t + u_{xx} \pm |u|^p u = 0$$

- By using Duhamel formula and Strichartz to control the integral term one obtains DBU when $1 \leq p < 3$ (Bona-S. 2010).
- This proof does not seem to work straightforwardly in dimension $d > 1$ by lack of integrability of $t \to \frac{1}{t^{d/2}}$ at 0.

- An alternate proof that works for any p's (Bona-S-Sparber 2013)

Use the H^s theory of Cazenave-Weissler (1990). When particularized to $n = 1$, their results yield local well-posedness for the Cauchy problem for the 1D NLS provided the following conditions are satisfied :

(i) $0 < s < \frac{1}{2}$.
(ii) $0 < p \leq \frac{4}{1-2s}$.

In particular the solution u to the Cauchy problem lies in $C([0, T_0]; H^s(\mathbb{R}))$ for some $T_0 > 0$.

We want furthermore $H^s(\mathbb{R}) \subset L^{p+1}(\mathbb{R})$, (so that $|u|^p u \in C([0, T_0]; L^1(\mathbb{R}))$), implying that

(iii) $\frac{1}{p+1} \geq \frac{1}{2} - s$.

Take $s = \frac{1}{2} - \epsilon$, where $\epsilon > 0$ is (arbitrarily) small.
Then the conditions (i), (ii) and (iii) are fulfilled provided $0 < p < \min(\frac{2}{\epsilon}, \frac{1-\epsilon}{\epsilon}) = \frac{1-\epsilon}{\epsilon}$.

- This will allow us to prove dispersive blow-up in the 1D case with p arbitrarily large.

Admit for the moment :

- Let f be defined by $f(x)=\frac{\exp(-ix^2)}{(1+x^2)^{\frac{1}{2}}}$. Then $f\in H^s(\mathbb{R})$ for any $0\leq s<\frac{1}{2}$.

We take $u_0=f$ as initial data. By the results in Bona-S 2010, the free part $\exp(it\partial_x^2)u_0$ exhibits a dispersive blow-up point at $(x,t)=(0,1)$.
On the other hand, the Duhamel part has the representation (in the usual sense of Lebesgue integrals) for $0<t\leq T_0$,

$$F(x,t)=\pm\int_0^t\int_{\mathbb{R}}\frac{1}{(t-s)^{\frac{1}{2}}}\exp\left(i\frac{|x-y|^2}{4(t-s)}\right)|u|^p u(y,s)dyds \quad (7)$$

Since F is clearly a continuous function on $\mathbb{R}\times[0,T_0]$, it is locally bounded and there is has a dispersive blow-up at $(0,1)$.

- Proof of the assertion.

Recording that $\frac{1}{(1+x^2)^{\frac{1}{2}}}$ is the inverse Fourier transform of the Bessel function K_0, one can write f as a convolution

$$f(x) = Ce^{i\alpha x^2} \star K_0, \tag{8}$$

where C and α are real constants. This expresses f as the value at a certain time of the Schrödinger group acting on K_0. Since the Schrödinger group is unitary on all Sobolev spaces, the proof is reduced to proving that $K_0 \in H^s(\mathbb{R})$ for any $0 \leq s < \frac{1}{2}$. Since $K_0 \in C^\infty(\mathbb{R} \setminus \{0\})$, this results from the fast decay of K_0 at infinity and from its logarithmic singularity at the origin, $K_0(x) \sim -\log x$.

Extensions (Bona-S-Sparber 2013)

- Fourth order nonlinear Schrödinger equations

$$iu_t + \alpha\Delta u + \beta\Delta^2 u + \lambda|u|^p u = 0, \quad \text{in } \mathbb{R}^n \times \mathbb{R} \tag{9}$$

  where α, β, λ are real constants, $\beta \neq 0$, together with the linear Schrödinger equations

$$iu_t - \epsilon\Delta u + \Delta^2 u = 0, \quad \epsilon \in \{0, -1, +1\}. \tag{10}$$

  (nonlinear case via Strichartz or a H^s theory *à la Cazenave-Weissler* for 4th order NLS. Works in $\mathbb{R}^n, n \leq 3$).

- NLS with anisotropic dispersion

$$iu_t + \alpha\Delta u + i\beta\partial^3_{x_1x_1x_1} u + \gamma\partial^4_{x_1x_1x_1x_1} u + |u|^p u = 0. \tag{11}$$

- A natural question is to ask for which class of potentials V the *linear* Schrödinger equation

$$iu_t + \frac{1}{2}\Delta u + V(x,t)u = 0 \tag{12}$$

displays the dispersive blow-up phenomena.
An easy case is that of harmonic potentials, $V(x,t) = \pm\frac{1}{2}\omega|x|^2$, the $+$ sign corresponding to the repulsive case and the $-$ sign to the attractive case (see Carles 2003).
In both cases the corresponding group has an explicit representation. In the attractive case, the solution is given by (Mehler's formula)

$$u(x,t) = e^{-in\frac{\pi}{4}\mathrm{sgn}\ t}\left|\frac{\omega}{2\pi\sin\omega t}\right|^{\frac{n}{2}}\int_{\mathbb{R}^n} e^{\frac{i\omega}{\sin\omega t}\left(\frac{|x|^2+|y|^2}{2}\cos\omega t - x\cdot y\right)}u_0(y)dy, \tag{13}$$

while in the repulsive case,

$$u(x,t) = e^{-in\frac{\pi}{4}\text{sgn } t}\left|\frac{\omega}{2\pi \sinh \omega t}\right|^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i\omega}{\sinh \omega t}\left(\frac{|x|^2+|y|^2}{2}\cosh \omega t - x\cdot y\right)} u_0(y)dy, \tag{14}$$

Mehler's formula is valid for $|t| < \frac{\pi}{2\omega}$, while that in the repulsive case makes sense for $t > 0$.

- In both cases one can prove the DBU property.

Another simple case is that of a linear potential, for instance $V(x,t) = \alpha x_1$ where α is a non zero real constant. The corresponding Schrödinger equation is thus

$$iu_t + \Delta u + \alpha x_1 u = 0,$$

and the evolution is given by the Avron-Herbst formula :

$$u(x,t) = \frac{e^{-it\alpha x_1} e^{-i\alpha^2 t^3/3}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-i\frac{|x-y|^2}{4t}} u_0(y_1 - \alpha t^2, y^\perp) dy \qquad (15)$$

The dispersive blow-up at $(x,t) = (0, \frac{1}{4})$ is obtained by choosing the initial data

$$u_0(y) = \frac{e^{i(y_1 + \frac{\alpha}{16})^2 + |y^\perp|^2)}}{1 + |y|^2)^m}$$

with

$$\frac{n}{4} < m \leq \frac{n}{2}.$$

Gross-Pitaevskii equation

$$i\psi_t + \Delta\psi + (1 - |\psi|^2)\psi = 0, \quad \psi = \psi(x,t),\ (x,t) \in \mathbb{R}^n \times \mathbb{R}, \quad (16)$$

Ginzburg-Landau energy :

$$E(\Psi) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\Psi|^2 + \frac{1}{4}\int_{\mathbb{R}^d} (1 - |\Psi|^2)^2 \equiv \int_{\mathbb{R}^d} e(\Psi) < \infty. \quad (17)$$

In some sense, $|\psi(x,t)|$ has to tend to one at infinity and our proof for the classical NLS has to be a bit modified.

More precisely, by writing $\psi = 1 + u$, $u \in H^1(\mathbb{R}^N)$, one obtains the following equation for u :

$$iu_t + \Delta u - 2\,\mathrm{Re}\,u = u^2 + 2|u|^2 + |u|^2 u. \tag{18}$$

Following Gustafson-Nakanishi-Tsaï (2006) one writes GP as

$$iv_t - Hv = U(3u_1^2 + u_2^2 + |u|^2 u_1) + iu_2(2u_1 + |u|^2) \tag{19}$$

where

$$u = u_1 + iu_2, \quad v = u_1 + iUu_2, \quad U = \sqrt{-\Delta(2-\Delta)^{-1}}, \quad H = \sqrt{-\Delta(2-\Delta)}.$$

The free evolution is, given by

$$v = e^{-itH} v_0,$$

or

$$v = G \star_x v_0$$

where

$$G(x,t) = \mathcal{F}^{-1} e^{it([(|\xi|^2(|\xi|^2+2)]^{\frac{1}{2}})}$$

Observe that

$$[(|\xi|^2(|\xi|^2+2)]^{\frac{1}{2}} = |\xi|^2 + a(\xi),$$

where $a(\xi) = \frac{2|\xi|^2}{(|\xi|^2(|\xi|^2+2))^{\frac{1}{2}}+|\xi|^2} = 1 - \frac{2|\xi|^2}{\left([|\xi|^2(|\xi|^2+2)]^{\frac{1}{2}}+|\xi|^2\right)^2} = 1 + r(\xi).$

Thus,

$$e^{it([(|\xi|^2(|\xi|^2+2)]^{\frac{1}{2}})} = e^{it|\xi|^2} e^{it} e^{itr(\xi)} = e^{it} e^{it|\xi|^2}(1 + f_t(|\xi|)$$

where

$$f_t(|\xi|) = 2it \sin(\frac{t}{2} r(\xi)) e^{it\frac{r(\xi)}{2}},$$

is continuous, smooth on $\mathbb{R}^d$ and decays to 0 like $|\xi|^{-2}$ as $|\xi| \to \infty$, uniformly on compact temporal intervals in $(0 + \infty)$, since $r(|\xi|)$ does so. This decomposition leads to an associated splitting of G

$$G(x,t) = e^{it} \int_{\mathbb{R}^d} e^{it|\xi|^2} e^{ix\cdot\xi} d\xi + e^{it} \int_{\mathbb{R}^d} f_t(|\xi|) e^{it|\xi|^2} e^{ix\cdot\xi} d\xi = I_t^1(x) + I_t^2(x). \tag{20}$$

Obviously,

$$I_t^1(x) = \frac{e^{it}}{(4\pi i t)^{d/2}} e^{\frac{i|x|^2}{4t}}.$$

On the other hand, for any fixed t, $f_t \in H^k(\mathbb{R}^d)$ for any $k \in \mathbb{N}$ and so is I_t^2 by the H^k unitarity of the Schrödinger group. In particular, I_t^2 is a bounded function of x uniformly on compact temporal intervals.

Dispersive blow-up at $(x,t) = (0, \frac{1}{4})$ is thus obtained by taking $v_0(x) = \frac{e^{-i|x|^2}}{(1+|x|^2)^m}$ with $\frac{d}{4} < m \leq \frac{d}{2}$, thus $u_0 = v_{01} + iU^{-1}v_{02}$.

Blowing-up solutions can be constructed for the one-dimensional nonlinear equation following the usual NLS case.

Possible connexion with optical rogue waves

The analysis of optical rogue-wave formation in Dudley-Genty-Eggleton 2008, Dudley 2008 is based on the generalized nonlinear Schrödinger equation

$$\begin{aligned}\frac{\partial A}{\partial t} + \frac{\alpha}{2}A \quad &- \quad \sum_{k\geq 2} \frac{i^{k+1}}{k!}\beta_k \frac{\partial^k A}{\partial z^k} \\ &= \quad i\gamma\left(1 + i\tau_{shock}\frac{\partial}{\partial z}\right)\left(A(z,t)\int_{-\infty}^{+\infty} R(z')|A(z',t)|^2 dz'\right).\end{aligned}$$

The variable t in fact connotes distance along the fiber, whereas z is in reality the temporal variable. The physical problem is in fact a boundary-value problem, but this is normally converted to an initial-value problem by viewing the independent variables as indicated in the present notation.

In this generalized Schrödinger equation, the dispersion is represented by its Taylor series and the nonlinearity features what is usually called a response function of the form $R(z) = (1 - f_R)\delta + f_R h_R(z)$, where δ is the Dirac mass. Thus the nonlinearity generally includes both instantaneous electronic and delayed Raman contributions.

We sketch below a proof of dispersive blow-up also for this model, thus providing a rigorous account of a possible explanation of the formation of optical rogue wave formation.
Consider first the linear part and for convenience, truncate the Taylor expansion of the dispersion so the linear model becomes

$$\begin{cases} \frac{\partial A}{\partial t} + \frac{\alpha}{2}A - \sum_{2\leq k\leq K} i^{k+1}\gamma_k \frac{\partial^k A}{\partial z^k} = 0, \\ A(x,0) = A_0(x) \end{cases} \tag{21}$$

where $\gamma_K \neq 0$. By changing the independent variable from A to $B = e^{-\alpha t}A$, one may take it that the damping coefficient α is zero.

Demonstrating dispersive blow-up for the linear equation (21) can be reduced (by perturbation arguments very similar to those used below for the linearized water-wave equation) to showing dispersive blow-up for the linear equation with *homogeneous* dispersion, *viz.*

$$\begin{cases} \dfrac{\partial A}{\partial t} - i^{K+1}\dfrac{\partial^K A}{\partial z^K} = 0, \\ \\ A(x,0) = A_0(x), \end{cases} \tag{22}$$

where γ_K is set equal to 1 without loss of generality. Equation (22) specializes to the Airy and the linear Schrödinger equations as particular cases when $K = 3$ and 2, respectively. When $K \geq 4$ one can use Sidi-Sulem-Sulem 1986 or Ben Artzi-S. 1999 to evaluate the corresponding fundamental solution and then construct suitable smooth initial data (a weighted version of the fundamental solution) which leads to dispersive blow-up.

This linear theory may then be extended to the nonlinear case. When the coefficient $\tau_{shock} = 0$, the equation is "semilinear" and the result follows for instance by using Strichartz estimates as for the one-dimensional nonlinear Schrödinger equation. (This is especially transparent when the instantaneous electronic contribution vanishes, that is when $f_R = 1$, but it holds without this restriction.)

When $\tau_{shock} \neq 0$, the nonlinear term involves a derivative with respect to z. Assume now that $K \geq 3$. The crux of the matter is to analyze the double integral term in the Duhamel representation of the solution and to show that it defines a continuous function of space and time. When $K = 3$ we are reduced to the Korteweg de Vries case which was dealt with already in Bona-S 1993 by using a theory of the Cauchy problem in suitable weighted L^2-spaces. This analysis was also extended to a class of fifth order Korteweg–de Vries equations. This extension is easily made for any odd value of K greater than 7. When $K \geq 4$ is even, the equation is of Schrödinger type and the weighted space theory (which uses in a crucial way that the phase velocity of the linear equation has a definite sign) does not appear to work. One has to rely instead on the higher-order smoothing properties of the linear group that appertain to the higher-order dispersion.

As an example, we focus on a simpler model (Taki *et al* 2010) that can be easily reduced to the KdV case, namely we consider the Cauchy problem

$$u_t + i\alpha u_{xx} + \beta u_{xxx} + i\gamma |u|^2 u = 0 \quad u(\cdot, 0) = u_0, \tag{23}$$

where α, β, γ are non zero real constants.
By using the factorization

$$\left(\xi + \frac{\alpha}{3\beta}\right)^3 = \xi^3 + \frac{\alpha}{\beta}\xi^2 + \frac{\alpha^2\xi}{3\beta^2} + \frac{\alpha^3}{27\beta^3},$$

the fundamental solution of the linearized equation (23) can be expressed as

$$\mathfrak{A}(x,t) = \frac{1}{(t\beta)^{1/3}} \exp\left(\frac{4it\alpha^3}{27\beta^2}\right) \exp\left(\frac{-i\alpha x}{2\beta^2}\right) \mathrm{Ai}\left(\frac{1}{t^{1/3}\beta^{1/3}}(x - \frac{\alpha^2}{3\beta}t)\right),$$

where the Airy function is defined by

$$\mathrm{Ai}(z) = \int_{-\infty}^{\infty} e^{i(\xi^3 + iz\xi)} d\xi.$$

Proceedings as in Bona -S 1993, the dispersive blow-up for (23) is then obtained at $(x, t) = (0, 1)$ by taking

$$u_0(x) = \frac{A(-x)}{(1+x^2)^m}$$

with

$$\frac{1}{8} < m \leq \frac{1}{4},$$

and

$$A(x) = \mathrm{Ai}\left(\frac{\alpha^2 + x}{3\beta^{4/3}}\right).$$

Actually the proof of dispersive blow-up for the nonlinear case is easier since, contrary to the KdV equation, the Duhamel term does not involve any x derivative and can be shown to be bounded by using only Strichartz estimates.

Linearized water waves

Motivated" by the freak (rogue) wave question we consider here the linearized (at the trivial state) water wave system in one or two spatial dimension :

$$\begin{cases} \eta_{tt} + \omega^2(|D|)\eta = 0 \quad in \quad \mathbb{R}^d \times \mathbb{R}_+, \\ \eta(.,0) = \eta_0, \quad \eta_t(.,0) = \eta_1. \end{cases} \tag{24}$$

Here $d = 1, 2$ and $\omega^2(|D|) = g|D|\tanh(h_0|D|), \quad |D| = (-\Delta)^{1/2}$.
We consider the finite depth case and will scale the equations so that the gravity constant g and the mean depth h_0 are equal to 1.
The solution is explicit via Fourier transform ($\mathbf{k} = (k_1, k_2)$) :

$$\hat{\eta}(\mathbf{k}, t) = \hat{\eta}_0(\mathbf{k}) \cos[t(|\mathbf{k}| \tanh|\mathbf{k}|)^{\frac{1}{2}}] + \frac{\sin[t(|\mathbf{k}| \tanh|\mathbf{k}|)^{\frac{1}{2}}]}{(|\mathbf{k}| \tanh|\mathbf{k}|)^{\frac{1}{2}}} \hat{\eta}_1(\mathbf{k}).$$

- Obviously well-posed in L^2.

Look for well/ill-posedeness in L^∞. Taking $\eta_1 \equiv 0$ we are reduced to proving that, for a fixed $t > 0$, $m_t(\mathbf{k}) = e^{it(\mathbf{k}\tanh(k))^{\frac{1}{2}}}$ is/(is not) a Fourier multiplier in L^∞. Recall that $\{Fourier \quad multipliers \quad in \quad L^\infty\} =$ $\{Fourier \quad transforms \quad of \quad bounded \quad measures\}$.

Noticing that $(\mathbf{k}\tanh(\mathbf{k}))^{\frac{1}{2}} = |\mathbf{k}|^{\frac{1}{2}}(1 - \frac{2}{1+e^{2|\mathbf{k}|}})^{\frac{1}{2}} \equiv |\mathbf{k}|^{\frac{1}{2}} + r(\mathbf{k})$, we have

$$e^{it(\mathbf{k}\tanh(\mathbf{k}))^{\frac{1}{2}}} = (1 + f_t(\mathbf{k}))e^{it|\mathbf{k}|^{\frac{1}{2}}},$$

where

$$f_t(\mathbf{k}) = -2\sin\frac{tr(\mathbf{k})}{2}[\sin\frac{tr(\mathbf{k})}{2} - i\cos\frac{tr(\mathbf{k})}{2}]$$

is continuous, smooth on $\mathbb{R}^d \setminus \{0\}$, and decays exponentially to 0 as $|\mathbf{k}| \to \infty$, uniformly on bounded time intervals

Theorem

Let $(x^*, t^*) \in \mathbb{R}^d \times (0, +\infty)$, $d = 1, 2$ be given. There exists $\eta_0 \in C^\infty(\mathbb{R}^d \setminus \{0\}) \cap C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the solution $\eta \in C_b(\mathbb{R}; L^2(\mathbb{R}^d))$ of (LWW) with $\eta_1 \equiv 0$ is such that

(i) η is a continuous function of x and t on $\mathbb{R}^d \times \big((0, +\infty) \setminus \{t^*\}\big)$,

(ii) $\eta(\cdot, t^*)$ is continuous in x on $\mathbb{R}^d \setminus \{x*\}$,

(iii) $\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*)\\(x,t)\neq(x^*,t^*)}} |\eta(x, t)| = +\infty.$

One may assume that $(x^*, t^*) = (0, 1)$. The idea is to take $\eta_1 = 0$ and $\eta_0(x) = |x|^\lambda \bar{K}(x)$, where $\frac{3d}{2} \leq \lambda \leq 2d$, and

$$K = \mathcal{F}^{-1}\left(\psi(|\mathbf{k}|)e^{i|\mathbf{k}|^{\frac{1}{2}}}\right),$$

where $\psi \in C^\infty(\mathbb{R}), 0 \leq \psi \leq 1, \psi \equiv 0$ on $[0, 1], \psi \equiv 1$ on $[2, +\infty)$.

We use a classical result (Wainger 1965, Miyachi 1981, and Hardy 1913 for $n=1$) on the precise behavior of $\mathfrak{F}(\psi(|\mathbf{k}|)e^{i|\mathbf{k}|^a})(x)$ at 0 which we state in $\mathbb{R}^n$:

Theorem

(Wainger, Miyachi)
Let $0 < a < 1$, $b \in \mathbb{R}$ *and define*
$F^{\epsilon}_{a,b}(\mathbf{x}) =: \mathcal{F}\left(\psi(|\mathbf{k}|)|\mathbf{k}|^{-b}\exp(-\epsilon|\mathbf{k}| + i|\mathbf{k}|^{a})\right)(\mathbf{x})$ *for* $\epsilon > 0$ *and* $\mathbf{x} \in \mathbb{R}^d$. *The following is true of the function* $F^{\epsilon}_{a,b}$.
(i) $F^{\epsilon}_{a,b}(\mathbf{x})$ *depends only on* $|\mathbf{x}|$.
(ii) $F_{a,b}(\mathbf{x}) = \lim_{\epsilon\to 0^+} F^{\epsilon}_{a,b}(\mathbf{x})$ *exists pointwise for* $\mathbf{x} \neq 0$ *and* $F_{a,b}$ *is smooth on* $\mathbb{R}^d \setminus \{0\}$.
(iii) For all $N \in \mathbb{N}$, *and* $\mu \in \mathbb{N}^d$, $|\left(\frac{\partial}{\partial x}\right)^{\mu} F_{a,b}(\mathbf{x})| = \mathcal{O}(|\mathbf{x}|^{-N})$ *as* $|\mathbf{x}| \to +\infty$.
(iv) If $b > d(1-\frac{1}{2})$, $F_{a,b}$ *is continuous on* $\mathbb{R}^d$.
(v) If $b \leq d(1-\frac{1}{2})$, *then for any* $m_0 \in \mathbb{N}$, *the function* $F_{a,b}$ *has the asymptotic expansion*

$$F_{a,b}(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|^{\frac{1}{1-a}(d-b-\frac{ad}{2})}} \exp\left(\frac{i\xi_a}{|\mathbf{x}|^{\frac{a}{1-a}}}\right) \sum_{m=0}^{m_0} \alpha_m |\mathbf{x}|^{\frac{ma}{1-a}} + 0(|\mathbf{x}|)^{\frac{(m_0+1)a}{(1-a)}} + g(\mathbf{x}) \quad (25)$$

as $\mathbf{x} \to 0$, *where* $\xi_a \in \mathbb{R}$, $\xi_a \neq 0$, *and* g *is a continuous function.*

Use this result with $a = 1/2$ and $b = 0$. The choice of λ ensures that $(0, 1)$ is a blow-up point and that the other values of (x, t) are under control.

Remark

The phase velocity $g^{\frac{1}{2}}(\frac{\tanh(|\mathbf{k}|h_0)}{|\mathbf{k}|})^{\frac{1}{2}}\hat{\mathbf{k}}$ is a bounded function of $\mathbf{k}$. This is contrary to the case of the linear KdV-equations (Airy-equation) and the linear Schrödinger equation, where both the phase velocity and the group velocity become unbounded in the short wave limit.
The dispersive blow-up phenomenon observed here is thus not linked to the unboundedness of the phase velocity, but simply to the fact that monochromatic waves (simple waves) propagate at speeds that vary substantially with their wavelength. Indeed, what appears to be important is that the ratio of the phase speeds at different wavenumbers is not suitably bounded.

Consider now the case of the linearized gravity-capillary waves. Again taking all the physical constants equal to 1 to simplify the discussion, the solution of (LWW) becomes

$$\hat{\eta}(\mathbf{k},t) = \hat{\eta}_0(\mathbf{k}) \cos\left[t(|\mathbf{k}| \tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}\right] + \hat{\eta}_1(\mathbf{k}) \frac{\sin\left[t(|\mathbf{k}| \tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}\right]}{(|\mathbf{k}| \tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}}. \quad (26)$$

¿From this formula, it is readily discerned that for $(\eta_0, \eta_1) \in H^k(\mathbb{R}^d) \times H^{k-\frac{3}{2}}(\mathbb{R}^d)$, $k \in \mathbb{N}$, (LWW) has a unique solution $\eta \in C_b(\mathbb{R}; H^k(\mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}; H^{k+\frac{1}{4}}_{loc}(\mathbb{R}^d))$.

Theorem

For $d = 1$ or 2, let $(x^*, t^*) \in \mathbb{R}^d \times (0, +\infty)$, be given. There exists $\eta_0 \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the solution $\eta \in C_b(\mathbb{R}; L^2(\mathbb{R}^d))$ of (LWW) with $\eta_1 \equiv 0$ satisfies

(i) η is a continuous function of x and t on $\mathbb{R} \times \big((0, +\infty) \setminus \{t^*\}\big)$,

(ii) $\eta(\cdot, t^*)$ is continuous in x on $\mathbb{R} \setminus \{x*\}$, and

(iii)

$$\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*)\\(x,t)\neq(x^*,t^*)}} |\eta(x,t)| = +\infty.$$

- Possible link with rogue waves formation.

In all the previous situations one can use the DBU construction to find open sets $\mathcal{U}$ in $H^k(\mathbb{R}^d), k \geq 3$, such that if initial data u_0 is taken from $\mathcal{U}$, then $|u_0|_\infty \leq \epsilon$, but the corresponding solution u with u_0 as initial data has the property that $|u(\cdot, t^*)|_\infty \geq M$, where the positive values of ϵ, M and t^* are specified.

Similar results for Fractional Schrödinger equations.

$$\begin{cases} iu_t + (-\Delta)^{\frac{a}{2}} u = 0, \quad 0 < a < 1, \\ u(\cdot, t) = u_0(\cdot). \end{cases} \tag{27}$$

for $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$. These equations occur in particular as the linearization of some weak turbulence models (Zakharov *et al*) and are reminiscent of the linearized water waves equations.

Open questions

- Extension to NLS in higher dimensions.
- Extensions to more general potentials in the linear case.
- Systems (Davey-Stewartson,..).
- Non constant coefficients.
- Relevance to rogue waves formation ?