Internal and labelled sequent calculi: an equivalence result for conditional logic $\mathbb{V}$

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Conditional logics extend the language of classical propositional logic with the connective $>$, suited to represent conditional sentences that cannot be captured by material implication. Counterfactuals are possibly the most famous example of conditionals: they describe a state of affairs which has not obtained and its consequences. Lewis [4] defined a whole family of conditional logics specifically tailored to capture counterfactual reasoning. Lewis characterized his logics by the so-called *sphere semantics*, a generalization of relational semantics. Sphere models are special types of neighbourhood models, with the property that for any world x, the family of neighbourhoods of x is nested.

There is a link between conditionals logics and substructural logics: the derivability relation defined by the conditional operator does not satisfy the property of *weakening*, called monotonicity or strengthening in the context of conditional logics: if $\Gamma \vdash A$ it does not hold that $\Gamma, \Delta \vdash A$, where $\vdash$ is interpreted as "counterfactually entails". An analysis of the relation between conditional logics, belief revision and substructural logics has been carried on recently in [1], taking non-commutative Lambek calculus as the basic underlying logic.

We here take into account logic $\mathbb{V}$, the basic system of Lewis' family of counterfactual conditional logics. The language of this logic is defined by adding to the language of classical propositional logic the comparative plausibility operator $A \preccurlyeq B$, read "A is at least as plausible as B". Comparative plausibility is equivalent to the counterfactual conditional, but simpler to treat. An internal sequent calculus for $\mathbb{V}$, called $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, was presented in [2]. We introduce **G3V**, a labelled sequent calculus for the system defined on the basis of [5], and prove the equivalence of the two calculi. This correspondence between the internal and labelled calculi may shed light on the relation between syntax and semantics, the latter explicitly encoded in the labelled calculus.

The calculus **G3V** is a **G3**-style labelled calculus based on neighbourhood semantics, similarly to the calculi for conditional doxastic logic **G3CDL** and preferential conditional logic **G3CL** [3, 5]. Unlike **G3CL**, which is based on the conditional operator $>$, the present calculus takes as primitive the comparative plausibility operator $\preccurlyeq$, thus being (to the best of our knowledge) the first labelled system which explicitly accounts for this connective. The semantic condition for $\preccurlyeq$ is $x \Vdash A \preccurlyeq B$ iff $\forall \alpha \in I(x)(\alpha \Vdash^{\exists} B \rightarrow \alpha \Vdash^{\exists} A)$, where $\alpha \Vdash^{\exists} A$ iff $\exists y \in \alpha(y \Vdash A)$. This justifies the following (sound) rules:

$$\frac{a \Vdash^{\exists} B, a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq B} \preccurlyeq R \text{ (a new)}$$

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$$\frac{a \in I(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} B \quad a \Vdash^{\exists} A, a \in I(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta}{a \in I(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta} \preccurlyeq L$$

As for the internal sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ [2], its sequents are enriched with a new syntactic structure, called "block". A block $[S_1, ..., S_k \lhd A]$ is interpreted as a disjunction of comparative plausibility formulas: $(S_1 \preccurlyeq A) \vee ... \vee (S_k \preccurlyeq A)$. Blocks might occur in the consequent of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequents, and the intended interpretation of a sequent is the following:

$$\iota(\Gamma \Rightarrow \Delta', [\Sigma_1 \lhd B_1], \ldots, [\Sigma_n \lhd B_n]) \ := \ \bigwedge \Gamma \rightarrow \bigvee \Delta' \vee \bigvee_{1 \leq i \leq n} \bigvee_{A \in \Sigma_i} (A \preccurlyeq B_i).$$

We start by providing a translation from $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ to **G3V**. The result relies on the fact that each block can be interpreted in the language of the labelled system as expressing the semantic condition which corresponds to a block. We define a translation t_x such that to each $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequent $\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A_1], ..., [\Sigma_n \lhd A_n]$ there corresponds a **G3V** sequent

$$a_1 \in I(x), \ldots, a_n \in I(x), a_1 \Vdash^{\exists} A_1, \ldots, a_n \Vdash^{\exists} A_n, \Gamma^t \Rightarrow \Delta^t, a_1 \Vdash^{\exists} \Sigma_1, \ldots, a_n \Vdash^{\exists} \Sigma_n$$

with $a_1, \ldots, a_n$ distinct labels and for $\Sigma_i = S_i^1, \ldots, S_i^k$, $a_i \Vdash^{\exists} \Sigma_i = a_i \Vdash^{\exists} S_i^1, \ldots, a_i \Vdash^{\exists} S_i^k$. Intuitively, we introduce a neighbourhood label for each block occurring in the derivation, and introduce in the derivation the semantic condition corresponding to (a disjunction of) $\preccurlyeq$ formulas. Then, we show that the rules of the sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ can be simulated in **G3V**: thus, the translation is sound, i.e. if a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, then its translation $(\Gamma \Rightarrow \Delta)^{t_x}$ is derivable in **G3V**.

Proving the other direction of the equivalence requires more work, mainly due to the fact that there are **G3V**-derivable sequents that cannot be translated into $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequents or, in other worlds, that **G3V**-derivable sequents are *more* than $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivable sequents.

To face this problem we devise a different, and more complex, proof strategy, composed of the following steps. First, we define an *inverse* translation, int_x, which transforms **G3V**-sequents into $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$-sequents. Thanks to int_x, we are able to treat a wider class of sequents: it allows for a translation of sequents containing inclusions. Let $\mathcal{R}^{\subseteq}$ contains zero or more inclusions $a_i \subseteq a_j$ for $1 \leqslant i \leqslant j \leqslant n$. Here is the definition of the inverse translation int_x:

$$\begin{gathered}(\mathcal{R}^{\subseteq}, a_1 \in I(x), ..., a_n \in I(x), a_1 \Vdash^{\exists} A_1, ..., a_n \Vdash^{\exists} A_n, x : \Gamma \Rightarrow \\ \Rightarrow x : \Delta, a_1 \Vdash^{\exists} \Sigma_1, ..., a_n \Vdash^{\exists} \Sigma_n)^{int_x} \\ := \Gamma \Rightarrow \Delta, \Pi\end{gathered}$$

where Γ is obtained from $x : \Gamma$ by removing the label x, Δ is obtained from $x : \Delta$ by removing the label x, and Π contains n blocks $[\Lambda_1 \lhd A_1], ..., [\Lambda_n \lhd A_n]$ and $\Lambda_i = \Sigma_i \cup \bigcup \{\Sigma_j \mid a_i \subseteq a_j \textit{ occurs in the antecedent}\}$. Intuitively, for each inclusion $a_i \subseteq a_j$, we add to the consequent of the sequent $a_i \Vdash^{\exists} \Sigma_j$ (this passage is implicit). Consequently we add to the left-hand side of the block corresponding to formulas labelled with a_i also the formulas labelled with a_j (this is the explicit operation described in the definition).

Second, we define the notion of normal form derivations in **G3V**: the idea is that we cannot translate *any* derivation, but only those constructed following a certain order of application of the rules. We prove that any derivation in **G3V** can be transformed into a normal form derivation. Finally, we prove that if $\Gamma \Rightarrow \Delta$ is derivable in **G3V**, then its inverse translation $(\Gamma \Rightarrow \Delta)^{int_x}$ is derivable in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$. The theorem is proved by simulating the rules of **G3V** within the internal sequent calculus; furthermore, it makes an essential use of the *Jump lemma*. This lemma states that, given a sequent, in some cases we can restrict our attention to formulas labelled with the same world label, since these formulas might be enough to derive the whole sequent. Thanks to this lemma we can simulate some multiple and different occurrences of **G3V** rule with just one rule (the Jump rule) of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$.

The two translations provide us with effective methods to construct derivations: from a $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivation we can build a **G3V** derivation, and from a **G3V** derivation we can build a $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivation (this latter by translating only some **G3V** sequents). Soundness of the translation t_x can be used to establish completeness of **G3V**, from completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$. Moreover, the full equivalence result is particularly interesting since it allows us to detail the mutual correspondences between a labelled and an internal system. The translation t_x makes explicit the semantic intuition "hidden" in the rules of the internal sequent calculus; the translation int_x suggests that only a part of the information contained in a labelled sequent is relevant to derive the conclusion. Generally speaking, we achieve a better understanding of both calculi, and a deeper insight on the different methods of derivation construction they employ.

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