# Measurable Kleene algebras and structural control*

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Kleene algebras are the mathematical structures modelling the behaviour of so called *regular expressions* in automata theory, introduced by S.C. Kleene [16] with the same syntactic laws that have been then used to define Kleene algebras. Since then, Kleene algebras have established themselves as one of the most important and best known models of computation, and have been applied to interpret actions in dynamic logic [24, 17], to prove the equivalence of regular expressions and finite automata [4, 1], to give fast algorithms for transitive closure and shortest paths in directed graphs [1], and axiomatize algebras of relations [22, 25]. The proof theory of the logic of Kleene algebras is challenging because the axioms and defining rules of the Kleene star $()^*$ cannot be reduced to an analytic presentation [23, 3, 15, 26].

Structural control, a theory of formal linguistics started in [20], aims at establishing systematic forms of communication between different grammatical regimes in formal linguistics. In [20], certain well known extensions of the Lambek calculus are studied as logics for reasoning about the grammatical structure of linguistic resources, in such a way that the requirement of grammatical correctness on the linguistic side is matched by the requirement of derivability on the logical side. In this context, the basic Lambek calculus incarnates the most general grammatical regime, and the 'special' behaviour of its extensions is captured by additional analytic structural rules. A systematic two-way communication between these grammatical regimes is captured by introducing extra pairs of adjoint modal operators (the *structural control* operators), which make it possible to import a degree of flexibility from the special regime into the general regime, and conversely, to endow the special regime with enhanced 'structural discrimination' coming from the general regime. The control operators are normal modal operators inspired by the exponentials of linear logic [9], in which the S4-type axioms guarantee that the 'of course' exponential ! is an *interior operator* and the 'why not' exponential ? is a *closure operator*, and hence each of them can be reobtained as the composition of adjoint pairs of maps between terms of the general regime and terms of the special regime. But in general, the control operators are not assumed to satisfy the modal S4-type conditions that are satisfied by the linear logic exponentials.

In this talk, we explore a general pattern which is common to Kleene algebras, structural control and linear logic. The connection between linear logic and structural control has been identified in [13], and can now be extended to Kleene algebras. The general intuition (cf. [18, 19]) is that, while general programs are encoded as arbitrary elements of a Kleene algebra, the Kleene star makes it possible to access the special regime of reflexive and transitive programs and to import it in a controlled way within the general regime. Hence, the role played by the Kleene star is similar to the one played by exponentials ! and ? in linear logic, which make it possible to access the special regime, captured proof-theoretically by the analytic structural rules of weakening and contraction, and to import it, in a controlled way, into the environment of the general regime. The axiomatization of the Kleene star is similar to ? in the sense that their algebraic interpretations are closure operators, and hence can be obtained as the composition of adjoint maps, which justifies the soundness of the controlled application of the structural rules capturing the special behaviour. However, the embedding from the special regime into the general regime has not as many order-theoretic properties as in linear logic.

In order to solve this problem, we introduce a subclass of Kleene algebras, referred to as *measurable* Kleene algebras,[1] which are Kleene algebras endowed with a *dual* Kleene star operation $()^\star$, associating any element with its reflexive transitive *interior*. In measurable Kleene algebras, the defining properties of the dual Kleene star are those of an *interior* operator, which plays a role similar to ! in linear logic. In this talk, this pattern is used as the semantic support of a proper display calculus for the logic of measurable Kleene algebras, and for establishing a conceptual and technical connection between Kleene algebras and structural control which is potentially beneficial for both areas.

In a similar way to [13], we introduce a *multi-type proper display calculus* for measurable Kleene algebras which is sound, complete, conservative, and have standard subormula property and cut-elimination.

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[1]The name is chosen by analogy with measurable sets in analysis, which we will discuss about in the talk.

The design of this calculus follows the principles of the *multi-type* methodology, introduced in [10, 7, 5, 6], with the aim of displaying dynamic epistemic logic and propositional dynamic logic, and subsequently applied to several other logics (e.g. linear logic with exponentials [13], inquisitive logic [8], semi-De Morgan logic [11], lattice logic [12]) which are not properly displayable in their single-type presentation, and also used as a platform for introducing novel logics [2].

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