Dualities for Płonka sums of algebras

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It is a common trend in mathematics to study dualities for algebraic structures and, in particular, for those arising from mathematical logic. The first step towards this direction traces back to the pioneering work by Stone for Boolean algebras [14]. Later on, Stone duality has been extended to the more general case of distributive lattices by Priestley [10], [11].

The construction that follows has been firstly introduced in the late 1960's by J. Płonka [8, 7]. Roughly speaking, given a system of algebras of a fixed type, indexed by a semilattice, their Płonka sum is a new algebra constructed on the disjoint sum of elements of such system.

Recall that a *direct system of algebras* consists in

1. a semilattice $I = \langle I, \vee \rangle$;
2. a family of algebras $\{\mathbf{A}_i : i \in I\}$ with disjoint universes;
3. a homomorphism $f_{ij}\colon \mathbf{A}_i \to \mathbf{A}_j$, for every $i, j \in I$ such that $i \leq j$ such that f_{ii} is the identity map for every $i \in I$, and if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$.

Given a direct system of algebras as above, the *Płonka sum* is the new algebra $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ defined as follows: the universe of $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ is the union of the various A_i, and for every n-ary basic operation f and $a_1, \dots, a_n \in \bigcup_{i\in I} A_i$, we set

$$f^{\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}}(a_1, \dots, a_n) \coloneqq f^{\mathbf{A}_j}(f_{i_1 j}(a_1), \dots, f_{i_n j}(a_n))$$

where $a_1 \in A_{i_1}, \dots, a_1 \in A_{i_n}$ and $j = i_1 \vee \dots \vee i_n$.

It turns out that Płonka sums are strictly connected with regular varieties. An identity $\varphi \approx \psi$ is said to be *regular* provided that exactly the same variables occur in φ and ψ. A variety $\mathcal{K}$ is called *regular* whenever it satisfies only identities which are regular. Differently, a variety is called irregular. In particular, a variety $\mathcal{K}$ is called *strongly irregular* whenever it satisfies an identity of the kind $f(x, y) \approx x$, where $f(x, y)$ is any term of the language in which x and y really occur. Examples of strongly irregular varieties abound in logic, since every variety with a lattice reduct is irregular as witnessed by the term $f(x, y) \coloneqq x \wedge (x \vee y)$. In his pioneering work [7], Płonka proved that any regular variety $\mathcal{K}$ can be represented as Płonka sum of a suitable strongly irregular variety $\mathcal{V}$. In this case $\mathcal{K}$ is called the *regularization* of $\mathcal{V}$, as $\mathcal{K}$ satisfies exactly the regular identities holding in $\mathcal{V}$.

Although, over the years, (examples of) regular varieties have been studied in depth from a purely algebraic perspective (see for example [1, 4, 5, 6]), the connection with logic is motivated by the recent discovery [2] that the algebraic semantics of the logic PWK, known as Paraconsistent Weak Kleene, is a subquasivariety of the regularization of the variety of Boolean algebras. This showed that the notions of Płonka sums and regularization find interesting applications in logic as well.

In the present contribution, we provide a very general method to construct a duality for algebras admitting a Płonka sum representation in terms of algebras possessing topological duals.

In particular, we will be able to apply to provide a duality for involutive bisemilattices, the algebraic semantics of PWK. In order to provide a concrete description of the dual space, we will rely on the duality established by Gierz and Romanowska [3] between distributive bisemilattices, the regularization of distributive lattices, and compact totally disconnected partially ordered left normal bands with constants, which we refer to as GR spaces. Such duality is constructed using a dualizing object; however, its relevance mainly lies in the use of the technique of Płonka sums [7], [9], as an essential tool [13], [12].

The present work consists of two main results. On one hand, taking advantage of the Płonka sums representation in terms of Boolean algebras and Stone duality, we are able to describe the dual space of an involutive bisemilattice as a semilattice inverse system of Stone spaces. On the other hand, we generalize Gierz and Romanowska duality by considering GR spaces with negation as an additional operation. As a byproduct of our analysis we get a topological description of *semilattice inverse systems* of Stone spaces.

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