First-order logic properly displayed

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The existing sequent calculi for first-order logic [19] contain special rules for the introduction of quantification and for substitution. The application of these rules depends on the free and bounded variables occurring in formulas. For example, in the standard Gentzen calculus for first-order logic the rules

$$\frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x A} \qquad \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}$$

are sound only when x does not occur free anywhere in the conclusions.

A proposal for a display calculus for fragments of first-order logic was first presented in [22, 21]. The key idea of this approach is that existential quantification can be viewed as a diamond-like operator of modal logic and universal quantification can be seen as a box-like operator as discussed in [15, 20]. These similarities, which have been observed and exploited in [15, 20, 22, 21], can be explained in terms of the order-theoretic notion of adjunction: indeed the set-theoretic interpretations of the existential and universal quantification are the left and right adjoint respectively of the inverse projection map and more generally, in categorical semantics, the left and right adjoint of the pullbacks along projections [16],[7, Chapter 15]. However, the display calculus of [22] contains rules with side conditions on the free and bound variables of formulas similar to the ones presented above. This implies that the rules are not closed under uniform substitution, i.e. the display calculus is not *proper* [21, Section 4.1].

We present results based on ongoing work in [11] on a proper display calculus for first-order logic. The design of our calculus is based on the multi-type methodology motivated by considerations discussed in [8, 4], first presented in [5, 2] for DEL and PDL and further developed in [3, 6, 12, 14, 13, 1, 9] in synergy with algebraic techniques [10]. The multi-type approach allows for the co-existence of terms of different types bridged by heterogeneous connectives. The main bookkeeping requirement in multi-type calculi is that in a derivable sequent $x \vdash y$ the structures x and y must be of the same type. Analogously to the notion of proper display calculi (cf. [21]), proper multi-type calculi are those in which parametric parts of rules are closed under uniform substitution within each type.

Using insights from [16, 17, 18], we introduce a proper display calculus for first-order logic. The side conditions on rules are internalised in the calculus by the use of appropriate types. We expand the language by adding a unary heterogeneous connective that serves as both the right adjoint of the existential quantifier and the left adjoint of the universal quantifier. In the context of the calculus, this connective signifies the introduction of a fresh variable to a formula.

In the context of categorical logic [16], the aforementioned adjunction is an instance of a more general construction which also encompasses variable substitution. Observing this, we further expand the language with connectives corresponding to instances of this construction, i.e. connectives to explicitly denote variable substitution. This finer analysis allows the calculus to be modular not only with respect to the logic but also to different appropriate weaker notions of terms. This, in turn, paves the way for a comprehensive study of different notions of quantication.

In our talk we will present the proper display calculus for first-order logic. Combining results from [14] we will focus on presenting a first-order calculus for lattice-based logics and discuss its completeness, soundness, cut-elimination and conservativity.

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