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On the Quasi-linear Elliptic PDE $-\nabla \cdot (\nabla u/\sqrt{1-|\nabla u|^2}) = 4\pi \sum_k a_k \delta_{s_k}$ in Physics and Geometry

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Abstract: It is shown that for each finite number N of Dirac measures δ_{s_n} supported at points $s_n \in \mathbb{R}^3$ with given amplitudes $a_n \in \mathbb{R}\backslash\{0\}$ there exists a unique real-valued function $u \in C^{0,1}(\mathbb{R}^3)$, vanishing at infinity, which distributionally solves the quasi-linear elliptic partial differential equation of divergence form $-\nabla \cdot (\nabla u/\sqrt{1-|\nabla u|^2}) = 4\pi \sum_{n=1}^{N} a_n \delta_{s_n}$. Moreover, $u \in C^{\omega}(\mathbb{R}^3\backslash\{s_n\}_{n=1}^{N})$. The result can be interpreted in at least two ways: (a) for any number N of point charges of arbitrary magnitude and sign at prescribed locations s_n in three-dimensional Euclidean space there exists a unique electrostatic field which satisfies the Maxwell-Born-Infeld field equations smoothly away from the point charges and vanishes as $|s| \to \infty$; (b) for any number N of integral mean curvatures assigned to locations $s_n \in \mathbb{R}^3 \subset \mathbb{R}^{1,3}$ there exists a unique asymptotically flat, almost everywhere space-like maximal slice with point defects of Minkowski spacetime $\mathbb{R}^{1,3}$, having lightcone singularities over the s_n but being smooth otherwise, and whose height function vanishes as $|s| \to \infty$. No struts between the point singularities ever occur.

1. Introduction

In this paper we will prove the existence of unique, essentially smooth distributional solutions to the quasi-linear elliptic partial differential problem of divergence form

$$\nabla \cdot \frac{\nabla u(s)}{\sqrt{1-|\nabla u(s)|^2}} + 4\pi \sum_{n=1}^{N} a_n \delta_{s_n}(s) = 0 \qquad \text{for} \qquad s \in \mathbb{R}^3, \tag{1}$$

$$u(s) \to 0 \qquad \text{as} \qquad |s| \to \infty; \tag{2}$$

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here, δ_{s_n} is the unit Dirac measure supported at $s_n \in \mathbb{R}^3$, and the $a_n \in \mathbb{R}\backslash\{0\}$ are amplitudes. More precisely, we will prove the following theorem:

Theorem 1.1. *For any finite sets $\{s_n\}_{n=1}^N \subset \mathbb{R}^3$ and $\{a_n\}_{n=1}^N \subset \mathbb{R}\backslash\{0\}$ there exists a unique real function $u \in C^{0,1}(\mathbb{R}^3)$ which solves (1), (2) in the sense of distributions. Furthermore, $|\nabla u(s)| < 1$ for $s \in \mathbb{R}^3\backslash\{s_n\}_{n=1}^N$, and $\lim_{s\to s_n} |\nabla u(s)| = 1$ for each s_n. Thus, $u \in C^\omega(\mathbb{R}^3\backslash\{s_n\}_{n=1}^N)$.*

Remark 1.2. Evidently our theorem allows that some of the support points for the Dirac measures coincide; however, any such situation is identical to a reformulation of the problem with fewer but distinct points, with a re-assignment of amplitude values. Thus, without loss of generality we will henceforth assume that all the s_n are distinct. The amplitudes may or may not be distinct, though.

Our result has applications in physics and geometry. It governs objects as diverse as, on the one hand, the classical electrostatic fields of the Maxwell-Born-Infeld field theory [BoIn1934, Pry1935b, Gib1998, Kie2004a], and maximal spacelike hypersurfaces with lightcone defects in the Minkowski spacetime [BaSi1982, Eck1986, KlMi1993, Kly1995, Kly2003] on the other. Applications are discussed in Sect. 4.

Curiously enough, given the attention that these areas of research have received in the literature, the existence of solutions to (1), (2) as ascertained in Theorem 1.1 has been an unsettled problem. Of course, there is the explicit solution of (1), (2) for $N = 1$ found by Born [Bor1933] and elaborated on further in [Bor1934, BoIn1934, BaSi1982, Eck1986, Gib1998]; it is well-defined for any value of its amplitude a. There is also a semi-explicit solution of (1) (which violates (2), though) for $N = \infty$ found by Hoppe [Hop1994] and further elaborated on in [Hop1995, Gib1998]; it has positive and negative amplitude Dirac sources of magnitude $|a|$ arranged in a cubic lattice and exists for arbitrary $|a|$. However, to the best of our knowledge, generic existence theorems for solutions to (1) have so far been established only: (a) in [KlMi1993, Kly1995] under smallness conditions[1] for the a_n when (1) is restricted to (bounded or unbounded) domains with boundary with Dirichlet data replacing (2); (b) in [Kly2003] for arbitrary a_n but with (2) replaced by prescribing $u(s_n) = u_n$, restricted by the bounds $|u_n - u_m| < |s_n - s_m|$ for $1 \le n < m \le N$ — in this case (2) is generically violated, and it doesn't follow from the proof in [Kly2003] whether (2) can hold for some particular choices of $\{u_n\}_{n=1}^N \subset \mathbb{R}$ and $\{a_n\}_{n=1}^N \subset \mathbb{R}\backslash\{0\}$, given $\{s_n\}_{n=1}^N \subset \mathbb{R}^3$. Our Theorem 1.1 does not follow from adapting the proofs in [KlMi1993, Kly1995, or Kly2003]. In fact, our arguments also extend to the Dirichlet problem in domains with boundary, as will become clear from our proof.

As do their proofs of their theorems in [KlMi1993, Kly1995, and Kly2003], our proof of Theorem 1.1 makes convenient use of the results by Bartnik and Simon [BaSi1982]. Explicitly, in [BaSi1982] Bartnik and Simon prove a number of results for the Dirichlet problem of (1) in bounded domains (with almost arbitrarily irregular boundary!), and they also outline how to pass to unbounded domains using barrier functions as in [Tre1982]. From their results one can extract the following theorem:

Theorem 1.3 (Essentially Bartnik–Simon). *For any finite set $\{s_n\}_{n=1}^N \subset \mathbb{R}^3$ of distinct points and any finite set $\{u_n\}_{n=1}^N \subset \mathbb{R}$, restricted by the bounds*

$$|u_n - u_m| < |s_n - s_m| \quad \text{for } 1 \le n < m \le N, \tag{3}$$

[1] We note that the review of Klyachin's paper [Kly1995] in Mathematical Reviews incorrectly claims that existence was proved for arbitrary amplitudes.

there exists a unique real function $u \in C^{0,1}(\mathbb{R}^3)$ *which weakly solves*

$$\nabla \cdot \frac{\nabla u(s)}{\sqrt{1-|\nabla u(s)|^2}} = 0 \qquad \textit{for} \qquad s \in \mathbb{R}^3 \backslash \{s_n\}_{n=1}^N, \tag{4}$$

$$u(s) \to u_n \qquad \textit{as} \qquad s \to s_n, \tag{5}$$

$$u(s) \to 0 \qquad \textit{as} \qquad |s| \to \infty. \tag{6}$$

Furthermore, $|\nabla u(s)| < 1$ *for* $s \in \mathbb{R}^3 \backslash \{s_n\}_{n=1}^N$*, and* $u \in C^{\omega}(\mathbb{R}^3 \backslash \{s_n\}_{n=1}^N)$.

Theorem 1.3 basically reduces the proof of our Theorem 1.1 to variational arguments which show that for each set of amplitudes $\{a_n\}_{n=1}^N \subset \mathbb{R}\backslash\{0\}$ associated with the points $\{s_n\}_{n=1}^N \subset \mathbb{R}^3$ there exists a unique distributional $C^{0,1}(\mathbb{R}^3)$ solution $u(s)$ of (1), (2) for which $|u(s_n) - u(s_m)| < |s_n - s_m|$ for $1 \le n < m \le N$. The claim that $\lim_{|s-s_n|\to 0} |\nabla u(s)| = 1$ then follows from:

Theorem 1.4 (Rephrasing of Theorem 1.5 in [Eck1986]). *Let* $u(s)$ *be as in Theorem 1.3. Then either* $u(s)$ *can be analytically continued into* s_n *or*

$$\lim_{|s-s_n|\to 0} |\nabla u(s)| = 1.$$

Thus, genuine singularities of $u(s)$ *are lightcone singularities.*

In Sect. 2 we formulate our variational approach to (1), (2) and prove existence of a unique optimizer $u \in C_0^{0,1}(\mathbb{R}^3)$ with $|\nabla u| \le 1$. In Sect. 3 we show with a dual variational argument that $|u(s_n) - u(s_m)| < |s_n - s_m|$ for $1 \le n < m \le N$. Afterwards, in Sect. 4, we discuss applications to physics and geometry. In Sect. 5 we list a few straightforward extensions of our main theorem, only indicating their proofs. In Sect. 6 we conclude with a list of desirable extensions.

2. The Variational Approach

In this section we prove:

Proposition 2.1. *There exists a unique* $u \in C_0^{0,1}(\mathbb{R}^3) \cap \{v : |\nabla v| \le 1\}$ *for which*

$$0 = \int_{\mathbb{R}^3} \Big(\nabla\psi(s) \cdot \frac{\nabla u(s)}{\sqrt{1-|\nabla u(s)|^2}} - 4\pi\,\psi(s) \sum_{1\le n\le N} a_n \delta_{s_n}(s) \Big) \mathrm{d}^3 s \tag{7}$$

holds for all $\psi \in C_0^{\infty}(\mathbb{R}^3)$*, where* ∇u *denotes weak derivative and where* $\mathrm{d}^3 s$ *is three-dimensional Lebesgue measure.*

Thus, $u \in C_0^{0,1}(\mathbb{R}^3) \cap \{v : |\nabla v| \le 1\}$ is a distributional solution of (1), (2).

2.1. Preliminary considerations. Equation (1) is the formal Euler-Lagrange equation for the variational principle

$$\mathcal{F}(v) = \int_{\mathbb{R}^3} \Big(1 - \sqrt{1-|\nabla v(s)|^2} - 4\pi\, v(s) \sum_{1\le n\le N} a_n \delta_{s_n}(s) \Big) \mathrm{d}^3 s \ \to\ \min \tag{8}$$

over a suitable set of functions v, and (7) is its weak version. In particular, if $C_b^0(\mathbb{R}^3) \cap C_b^1(\mathbb{R}^3 \backslash \{s_n\}_{n=1}^N)$ denotes the Banach space of bounded continuous real functions on $\mathbb{R}^3$ which have a bounded continuous derivative on the indicated punctured domain, equipped with their usual norm, then $\mathcal{F}$ is well-defined for those $v \in C_b^0(\mathbb{R}^3) \cap C_b^1(\mathbb{R}^3 \backslash \{s_n\}_{n=1}^N)$ which satisfy $|\nabla v| \leq 1$ on $\mathbb{R}^3 \backslash \{s_n\}_{n=1}^N$ and which vanish sufficiently fast as $|s| \to \infty$; in particular, $|\nabla v(s)| = O(|s|^{-2})$ is fast enough. Eventually, in Sect. 3, we will show that $\mathcal{F}$ does take its minimizer on this set of functions. However, since the indicated Banach spaces are not convenient spaces to work with, here we shall characterize (8) as upper limit of a sequence of variational functionals which are defined on larger, more convenient spaces of functions. The minimizer of $\mathcal{F}$ will be obtained as the limit of a family of minimizers of these approximating variational principles. In particular, we show that the minimizer solves (7).

2.2. A monotone family of variational principles. For $x \geq 0$ we define the extended real-valued function

$$f(x) = \begin{cases} 1 - \sqrt{1-x} & \text{for } x \in [0, 1] \\ \infty & \text{for } x > 1 \end{cases}. \tag{9}$$

The K^{th} Taylor polynomial of f about $x = 0$, given by

$$\mathrm{Tay}_K[f](x|0) = \sum_{k=0}^{K} f^{(k)}(0) x^k, \tag{10}$$

has Taylor coefficients

$$f^{(k)}(0) = \begin{cases} 0 & \text{for } k = 0 \\ \frac{1}{2} & \text{for } k = 1 \\ \frac{(2k-3)!!}{(2k)!!} & \text{for } k > 1 \end{cases}, \tag{11}$$

so that $\mathrm{Tay}[f](x|0) \equiv \left\{\mathrm{Tay}_K[f](x|0)\right\}_{K=1}^{\infty}$, the Maclaurin series of $f(x)$, viz.

$$\mathrm{Tay}[f](x|0) = \frac{1}{2} x + \sum_{k=2}^{\infty} \frac{(2k-3)!!}{(2k)!!} x^k, \tag{12}$$

is a pointwise strictly increasing sequence of strictly convex, strictly increasing functions of $x > 0$ which vanish at $x = 0$. The series (12) converges absolutely to $f(x)$ for $x \in [0, 1]$ but diverges for $x > 1$; since all coefficients are positive, $\mathrm{Tay}_K[f](x|0) \nearrow \infty$ for $x > 1$, so we are entitled to say that $\mathrm{Tay}[f](x|0)$ actually converges to the extended real-valued function $f(x)$ for all $x \geq 0$. In the following, for brevity we shall simply write $f_K(x)$ for $\mathrm{Tay}_K[f](x|0)$.

With the help of the Taylor polynomials we now define the family of functionals

$$\mathcal{F}_K(v) = \int_{\mathbb{R}^3} \Big(f_K\left(|\nabla v|^2\right) - 4\pi v(s) \sum_{1 \leq n \leq N} a_n \delta_{s_n}(s) \Big) \mathrm{d}^3 s, \tag{13}$$

which for $K \geq 2$ are well-defined[2] on $\bigcap_{1 \leq k \leq K} \dot{W}_0^{1,2k}(\mathbb{R}^3)$. For the $|\nabla v|$ term this is seen right away from the definition of $\dot{W}_0^{1,2k}(\mathbb{R}^3)$ as the closure of the compactly supported C^∞ functions on $\mathbb{R}^3$ w.r.t. $\|v\|^{2k}_{\dot{W}_0^{1,2k}(\mathbb{R}^3)} \equiv \int_{\mathbb{R}^3} |\nabla v(s)|^{2k} \mathrm{d}^3 s$, so that for $v \in \bigcap_{1 \leq k \leq K} \dot{W}_0^{1,2k}(\mathbb{R}^3)$ we have $f_K(|\nabla v|^2) \in L^1(\mathbb{R}^3)$. To see that also the source term in (13) is well-defined we note that $\dot{W}_0^{1,2k}(\mathbb{R}^3) \subset W^{1,2k}_{\mathrm{loc}}(\mathbb{R}^3)$ so that we can apply Sobolev's original embedding theorem, according to which for any ball $B \subset \mathbb{R}^3$ we have[3] $W^{1,2k}(B) \hookrightarrow C_b^0(B)$ whenever $k \geq 2$, and conclude that $\delta_{s_n} \in \dot{W}_0^{-1,(2k)'}(\mathbb{R}^3)$ for all $k \geq 2$. This establishes that $\mathcal{F}_K$ is well-defined on $\bigcap_{1 \leq k \leq K} \dot{W}_0^{1,2k}(\mathbb{R}^3)$ for $K \geq 2$, and any finite N. Incidentally, since elements of $\dot{W}_0^{1,2}(\mathbb{R}^3)$ tend to zero at spatial infinity[4] a.e., we see that the $v \in \bigcap_{1 \leq k \leq K} \dot{W}_0^{1,2k}(\mathbb{R}^3)$ with $K \geq 2$ are in fact equivalent to a subset of the bounded continuous functions on $\mathbb{R}^3$ which vanish at spatial infinity.

Let $\mathscr{A} \equiv \bigcap_{1 \leq k \leq \infty} \dot{W}_0^{1,2k}(\mathbb{R}^3) \cap \{v : |\nabla v| \leq 1\}$ be the set of admissible (equivalence classes of) functions for $\mathcal{F}$. Since $\mathscr{A} \subset \bigcap_{1 \leq k \leq K} \dot{W}_0^{1,2k}(\mathbb{R}^3)$ for all $K \in \mathbb{N}$, and since $f_K(x) \nearrow f(x)$ for all $x \geq 0$, monotone convergence now yields that $\mathcal{F}(v) = \lim_{K \to \infty} \mathcal{F}_K(v)$ for all $v \in \mathscr{A}$.

2.3. Existence of a family of unique minimizers. For each $K \geq 2$, the functional $\mathcal{F}_K$ is clearly convex over $\bigcap_{1 \leq k \leq K} \dot{W}_0^{1,2k}(\mathbb{R}^3)$. Moreover, $\mathcal{F}_K$ is bounded below and coercive w.r.t. the topology of $\bigcap_{1 \leq k \leq K} \dot{W}_0^{1,2k}(\mathbb{R}^3)$ whenever $K \geq 2$. To see this we have to estimate the source term in $\mathcal{F}_K$.

We rewrite $\mathcal{F}_K$ as

$$\mathcal{F}_K(v) = \sum_{k=1}^{K} c_k \|v\|^{2k}_{\dot{W}_0^{1,2k}(\mathbb{R}^3)} - 4\pi \sum_{1 \leq n \leq N} a_n v(s_n), \tag{14}$$

where $c_k = f^{(k)}(0) > 0$. Now $\{s_n\}_{n=1}^N$ is given, so there exists an open ball $B \subset \mathbb{R}^3$ such that $\{s_n\}_{n=1}^N \subset B$. Since the restriction of any $v \in \dot{W}_0^{1,4}(\mathbb{R}^3)$ to B is in $\dot{W}^{1,4}(B)$, and since $\dot{W}^{1,4}(B) \subset W^{1,4}(B)$ (though no embedding, clearly), we can apply the Sobolev embedding theorem in the form $W^{1,4}(B) \hookrightarrow C_b^0(B)$, and using $|a_n| \leq \max_{1 \leq n \leq N} |a_n| < \infty$, for all $K \geq 2$ we obtain the estimate

$$\mathcal{F}_K(v) \geq \sum_{k=1}^{K} c_k \|v\|^{2k}_{\dot{W}_0^{1,2k}(\mathbb{R}^3)} - 4\pi N A \|v\|_{W^{1,4}(B)}, \tag{15}$$

where A is a positive constant. Now $\|v\|^4_{W^{1,4}(B)} = \|v\|^4_{L^4(B)} + \|\nabla v\|^4_{L^4(B)}$, so that $\|v\|_{W^{1,4}(B)} \leq \|v\|_{L^4(B)} + \|\nabla v\|_{L^4(B)}$. Next, since $v \in \dot{W}_0^{1,4}(\mathbb{R}^3)$ for $K \geq 2$, we have the nontrivial estimate $\|\nabla v\|_{L^4(B)} \leq \|v\|_{\dot{W}_0^{1,4}(\mathbb{R}^3)} < \infty$. Furthermore, by Hölder's inequality, $\|v\|_{L^4(B)} \leq |B|^{1/12} \|v\|_{L^6(B)}$, and the special case $\dot{W}_0^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ of Sobolev's

[2] The functional $\mathcal{F}_1$ is not well-defined on the "canonical" domain of the Dirichlet integral, which is $\dot{W}_0^{1,2}(\mathbb{R}^3)$, for which reason we don't have any use for $\mathcal{F}_K$ when $K = 1$.

[3] By the Sobolev-Morrey embedding theorem we even have $W^{1,2k}(B) \hookrightarrow C^{0,1-3/2k}(B)$ for $k \geq 2$ and any $B \subset \mathbb{R}^3$.

[4] This is not true for all elements of $\dot{W}_0^{1,2k}(\mathbb{R}^3)$ when $k \geq 2$.

embedding theorem then yields the estimate $\|v\|_{L^4(B)} \leq |B|^{1/12} S \|v\|_{\dot{W}_0^{1,2}(\mathbb{R}^3)} < \infty$, where $S > 0$ is the sharp Sobolev constant. And so, for $K \geq 2$ we find

$$\mathcal{F}_K(v) \geq \sum_{k=1}^{K} c_k \|v\|^{2k}_{\dot{W}_0^{1,2k}(\mathbb{R}^3)} - 4\pi N \left(A' \|v\|_{\dot{W}_0^{1,2}(\mathbb{R}^3)} + A \|v\|_{\dot{W}_0^{1,4}(\mathbb{R}^3)} \right), \tag{16}$$

where A' is another positive constant. This lower estimate for $\mathcal{F}_K$ is manifestly bounded below and coercive on $\bigcap_{1 \leq k \leq K} \dot{W}_0^{1,2k}(\mathbb{R}^3)$ whenever $K \geq 2$. Therefore, for each $K \geq 2$ the functional $\mathcal{F}_K$ takes on a unique minimum for some $v_K \in \bigcap_{1 \leq k \leq K} \dot{W}_0^{1,2k}(\mathbb{R}^3)$. We set $\mathcal{F}_K(v_K) \equiv F_K$.

2.4. Weak convergence of the family of minimizers. Since $F_K = \mathcal{F}_K(v_K) > \mathcal{F}_{K'}(v_K) \geq \mathcal{F}_{K'}(v_{K'}) = F_{K'}$ whenever $K > K'$, the minimum values F_K of the family of variational functionals form a strictly monotonic increasing sequence. And since $f_K(x) < f(x)$ for all K when $x > 0$, this sequence $\{F_K\}_{K=2}^{\infty}$ is bounded above by $\mathcal{F}(\hat{v})$, where $\hat{v} \in C^{0,1}(\mathbb{R}^3)$ is the following convenient trial function: let $2r := \min\{|s_k - s_l|\}_{1 \leq k < l \leq N}$, then $\hat{v}$ is defined by

$$\hat{v}(s) = \begin{cases} \operatorname{sign}(a_n)\,(r - |s_n - s|) & \text{for } \; s \in B_r(s_n) \\ 0 & \text{for } \; s \in \mathbb{R}^3 \setminus \bigcup_{1 \leq n \leq N} B_r(s_n) \end{cases}. \tag{17}$$

One readily calculates that

$$\mathcal{F}(\hat{v}) = N\Big(|B_r| - 4\pi r\, \overline{|a|} \Big), \tag{18}$$

where

$$\overline{|a|} \equiv \frac{1}{N} \sum_{1 \leq n \leq N} |a_n|. \tag{19}$$

Thus, $\lim_{K \to \infty} F_K =: F \leq \inf_v \mathcal{F}(v) \leq \mathcal{F}(\hat{v})$ exists, and $F_K < F$ for all $K \geq 2$.

As a corollary, since $\mathcal{F}_{K'}(v_K) < \mathcal{F}_K(v_K)$ when $K > K'$, we have that $\mathcal{F}_{K'}(v_K) < F$ whenever $K > K'$. By coercivity, for any fixed $K' \geq 2$ there now exists a positive constant C such that $\|v_K\|_{\dot{W}_0^{1,2K'}(\mathbb{R}^3)} < CF$ for all $K > K'$. Now, since $\dot{W}_0^{1,2K'}(\mathbb{R}^3)$ is a separable, reflexive Banach space for all $1 \leq K' < \infty$, the closed ball $\big\{ v : \|v\|_{\dot{W}_0^{1,2K'}(\mathbb{R}^3)} < CF \big\}$ is weakly compact. Therefore the sequence $\{v_K\}_{K=2}^{\infty}$ contains a weakly convergent subsequence in $(\dot{W}_0^{1,2} \cap \dot{W}_0^{1,2K'})(\mathbb{R}^3)$ for each $K' \geq 1$. By a diagonal argument we can pick the subsequence so that its weak limit in each $(\dot{W}_0^{1,2} \cap \dot{W}_0^{1,2K'})(\mathbb{R}^3)$ is one and the same $v_\infty \in \bigcap_{1 \leq k \leq K'} \dot{W}_0^{1,2k}(\mathbb{R}^3)$ for all $1 \leq K' < \infty$, hence $v_\infty \in \bigcap_{1 \leq k < \infty} \dot{W}_0^{1,2k}(\mathbb{R}^3)$.

Now, functions in $\bigcap_{1 \leq k < \infty} \dot{W}_0^{1,2k}(\mathbb{R}^3)$ are not necessarily in $\dot{W}_0^{1,\infty}$, but the uniform (in K) upper bound F on the $\mathcal{F}_K(v_K)$ guarantees that the weak limit v_∞ of the v_K is actually in $\dot{W}_0^{1,\infty}$; indeed, we even have $|\nabla v_\infty| \leq 1$ a.e. For assume to the contrary that $|\nabla v_\infty| \not\leq 1$ a.e. Then there exists an $\Omega \subset \mathbb{R}^3$ with $|\Omega| > 0$ such that $|\nabla v_\infty| \geq 1 + 2\epsilon$ a.e. in Ω. But then there exists a $\widetilde{K}$ such that $|\nabla v_K| \geq 1 + \epsilon$ a.e. in Ω whenever $K > \widetilde{K}$. And then we have $\mathcal{F}_K(v_K) \geq |\Omega| \sum_{k=\widetilde{K}+1}^{K} \frac{(2k-3)!!}{(2k)!!} (1+\epsilon)^{2k} \nearrow \infty$ as $K \to \infty$, which is a contradiction to $\mathcal{F}_K(v_K) < F$ for all K. Thus, not only is $v_\infty \in \dot{W}_0^{1,\infty}$, also $|\nabla v_\infty| \leq 1$ a.e. in $\mathbb{R}^3$, as claimed. So $v_\infty \in \bigcap_{1 \leq k \leq \infty} \dot{W}_0^{1,2k}(\mathbb{R}^3) \cap \{v : |\nabla v| \leq 1\} = \mathscr{A}$.

2.5. The limit of the minimizers of the $\mathcal{F}_K$ minimizes $\mathcal{F}$. We have just proved that $v_\infty \in \mathscr{A}$. We now show that $\mathcal{F}(v_\infty) = \min_{v\in\mathscr{A}} \mathcal{F}(v)$.

Since $\mathscr{A} \subset (W^{1,2} \cap W^{1,2K})(\mathbb{R}^3)$ for each $K \geq 1$, $\mathcal{F}_K(v)$ is well-defined for each $v \in \mathscr{A}$ and each $K \in \mathbb{N}$; in particular, $\mathcal{F}_K(v_\infty)$ is well-defined for each $K \in \mathbb{N}$. Moreover, $\lim_{K\to\infty} \mathcal{F}_K(v) = \mathcal{F}(v)$ for each $v \in \mathscr{A}$, by monotone convergence.

Now, by the monotonicity of the sequence of Taylor polynomials $\left\{f_K(|\nabla v|)\right\}_{K=1}^{\infty}$, we have $\mathcal{F}(v_\infty) > \mathcal{F}_K(v_\infty) \geq \mathcal{F}_K(v_K) = F_K$ for all $K > 1$. By taking the limit as $K \to \infty$, we obtain $\mathcal{F}(v_\infty) \geq \lim_{K\to\infty} \mathcal{F}_K(v_K) = F$.

On the other hand, recalling (14), we see that each $\mathcal{F}_K$ is obviously weakly lower semi-continuous, so we have $\mathcal{F}_{K'}(v_\infty) \leq \lim_{K\to\infty} \mathcal{F}_{K'}(v_K)$. Recalling now that $\mathcal{F}_{K'}(v_K) < F$ whenever $K > K'$, we obtain $\mathcal{F}_{K'}(v_\infty) \leq F$ for all $K' > 1$. Taking the limit $K' \to \infty$ and recalling that $\lim_{K'\to\infty} \mathcal{F}_{K'}(v) = \mathcal{F}(v)$ for each $v \in \mathscr{A}$, we obtain that $\mathcal{F}(v_\infty) \leq F$. In total, we have shown that $\mathcal{F}(v_\infty) = F$.

It remains to show that there is no $\tilde{v} \in \mathscr{A}$ for which $\mathcal{F}(\tilde{v}) < F$. But this is really easy. For assume to the contrary that there were such a $\tilde{v}$ with $\mathcal{F}(\tilde{v}) = F - \epsilon$. Then $\mathcal{F}_K(\tilde{v}) < F - \epsilon$ for all $K > 1$, which contradicts the fact that for any ϵ we can find a $\widetilde{K}(\epsilon)$ such that $\min_{v\in(W^{1,2}\cap W^{1,2K})(\mathbb{R}^3)} \mathcal{F}_K(v) = \mathcal{F}_K(v_K) > F - \epsilon$ whenever $K > \widetilde{K}(\epsilon)$.

This proves that $\mathcal{F}$ takes its minimum at $v_\infty \in \mathscr{A}$, and the minimum equals F. Moreover, by convexity, the minimizer is unique.

2.6. The minimizer of $\mathcal{F}$ weakly satisfies the Euler-Lagrange equation. We cannot yet conclude that the minimizer v_∞ of $\mathcal{F}(v)$ weakly satisfies the formal Euler-Lagrange equation (1) because for this conclusion we need to know that $|\nabla v_\infty| < 1$ a.e. and so far we only know that $|\nabla v_\infty| \leq 1$ a.e. We now show that $|\nabla v_\infty| < 1$ a.e., which implies that v_∞ weakly satisfies (1), i.e. (7).

Let $\Omega_{\text{crit}} = \bigcap_{\epsilon>0} \overline{\{s : |\nabla v_\infty| > 1 - \epsilon\}}$. Note that Ω_{crit} contains all points s_* at which $|\nabla v_\infty(s_*)| = 1$ as well as all points s_* for which $\text{ess-lim}_{s\to s_*} |\nabla v_\infty(s)| = 1$ without necessarily having $\nabla v_\infty(s)$ itself defined at $s = s_*$. Clearly, Ω_{crit} has finite Lebesgue measure, $|\Omega_{\text{crit}}| < \infty$, for $v_\infty \in \mathscr{A}$ implies that $|\nabla v_\infty(s)| \to 0$ as $|s| \to \infty$. We now show that $|\Omega_{\text{crit}}| = 0$.

For this purpose we assume to the contrary that $|\Omega_{\text{crit}}| > 0$. Then the variation of $\mathcal{F}(v)$ about v_∞ gives to the leading order (i.e. power 1/2) in ψ,

$$\mathcal{F}^{(1/2)}[v_\infty](\psi) := -\int_{\Omega_{\text{crit}}} \sqrt{-2\nabla v_\infty(s) \cdot \nabla\psi(s)}\, \mathrm{d}^3 s, \tag{20}$$

where $\psi \in C_0^\infty(\mathbb{R}^3)$ is any test function satisfying $\nabla v_\infty(s) \cdot \nabla\psi(s) \leq 0$ a.e. on Ω_{crit}. Note that $\mathcal{F}^{(1/2)}[v_\infty](\psi)$ is homogeneous of fractional degree 1/2 in ψ; hence, this *nonlinear* term — if nonzero — will in general dominate the usual linear terms in ψ, indeed. Moreover, whenever $\mathcal{F}^{(1/2)}[v_\infty](\psi) \neq 0$, we manifestly have

$$-\int_{\Omega_{\text{crit}}} \sqrt{-\nabla v_\infty(s) \cdot \nabla\psi(s)}\, \mathrm{d}^3 s < 0. \tag{21}$$

But for v_∞ to minimize $\mathcal{F}$ over $\mathscr{A}$ we must have $\mathcal{F}^{(1/2)}[v_\infty](\psi) \geq 0$ for all $\psi \in C_0^\infty(\mathbb{R}^3)$ satisfying $\nabla v_\infty(s) \cdot \nabla\psi(s) \leq 0$ on Ω_{crit} a.e. This together with (21) implies that $\mathcal{F}^{(1/2)}[v_\infty](\psi) \equiv 0$ for all $\psi \in C_0^\infty(\mathbb{R}^3)$ satisfying $\nabla v_\infty(s) \cdot \nabla\psi(s) \leq 0$ a.e. on Ω_{crit}. But this is only possible if $|\Omega_{\text{crit}}| = 0$, as claimed.

Remark 2.2. Our argument above does not show that $\Omega_{\text{crit}} = \{s_n\}_{n=1}^N$.

The result $|\Omega_{\text{crit}}| = 0$ means that $|\nabla v_\infty| < 1$ a.e., and this already implies that the variation of $\mathcal{F}(v)$ about v_∞ to leading order (i.e. power 1) in ψ now reads

$$\mathcal{F}^{(1)}[v_\infty](\psi) = \int_{\mathbb{R}^3} \Big(\nabla\psi(s) \cdot \frac{\nabla v_\infty(s)}{\sqrt{1 - |\nabla v_\infty(s)|^2}} - 4\pi\,\psi(s) \sum_{1\leq n\leq N} a_n \delta_{s_n}(s) \Big) \mathrm{d}^3 s. \tag{22}$$

Since $\mathcal{F}^{(1)}[v_\infty](\psi)$ is linear in ψ, v_∞ can minimize $\mathcal{F}$ over $\mathscr{A}$ only if $\mathcal{F}^{(1)}[v_\infty](\psi) = 0$ for all ψ, which is precisely (7). Thus the Euler-Lagrange equation (1) is satisfied by v_∞ in the weak sense, as claimed. The proof of Proposition 2.1 is complete.

Remark 2.3. We close this section with the remark that alternate, nonvariational routes to Proposition 2.1 are conceivable. In particular, the Dirac sources can be mollified with compactly supported C^∞ functions, and the asymptotic vanishing of $u(s)$ as $|s| \to \infty$ replaced by 0-Dirichlet conditions on ∂B_R, where R is a large ball containing the supports of all mollifiers of the Dirac sources. For this situation the theorems in [BaSi1982] guarantee a classical solution to the so mollified (1). As pointed out by one of the referees, Lemma 2.1 in [BaSi1982] and elliptic regularity theory should now yield uniform Lipschitz bounds on the solution u away from the eventual locations of the Dirac sources when the mollifiers are removed, and the proof of their Lemma 3.1 shows that the limit function solves (7) restricted to B_R. Subsequently one can let $R \to \infty$ by invoking Treiberg's barrier function arguments.

3. Bootstrapping Regularity

In this section we bootstrap the regularity of the minimizer $v_\infty \equiv u$ of $\mathcal{F}(v)$ to the level which guarantees satisfaction of Theorem 1.1.

3.1. Bootstrapping the regularity of u away from Ω_{crit}. By our Proposition 2.1, the unique distributional solution to (1), (2) obtained by minimizing $\mathcal{F}$ in $\mathscr{A}$ takes values $u_n = u(s_n)$ at the s_n which satisfy the inequalities $|u_n - u_m| \leq |s_n - s_m|$ for all $1 \leq n < m \leq N$. Hence we can invoke Corollary 4.2 to Theorem 4.1 of [BaSi1982] to extract the following proposition for our setting.

Proposition 3.1. *Let $u(s) \in C^{0,1}(\mathbb{R}^3)$ be the unique distributional solution to (1), (2) which minimizes $\mathcal{F}$ in $\mathscr{A}$. Then $u \in C^\omega(\mathbb{R}^3 \backslash \Omega_{\text{crit}})$. Moreover, Ω_{crit} (the singular set K for u in [BaSi1982]) is a subgraph of $\mathcal{K}_N \equiv \mathcal{K}\big(\{s_n\}_{n=1}^N\big) \subset \mathbb{R}^3$, the complete graph whose vertices are the set $\{s_n\}_{n=1}^N$. Furthermore, let $E_{n,m} \subset \mathcal{K}_N$ denote the edge of $\mathcal{K}_N$ with endpoints s_n and s_m. Then $E_{n,m} \subset \Omega_{\text{crit}}$ if and only if $|u_n - u_m| = |s_n - s_m|$, and in that case we have $u(ts_n + (1-t)s_m) = tu_n + (1-t)u_m$ for $t \in [0, 1]$.*

3.2. Proof that $\Omega_{\text{crit}} = \{s_n\}_{n=1}^N$. We recall that any distributional solution $\in W^{1,2}$ of (1) satisfies the weak maximum principle, Theorem 8.1 in [GiTr1983]. Therefore $v_\infty(s) \equiv u(s)$ has a local maximum at s_n whenever $a_n > 0$ and a local minimum when $a_n < 0$, and no extremum in $\mathbb{R}^3 \backslash \{s_n\}_{n=1}^N$. This together with Proposition 3.1 right away gives us:

Corollary 3.2. *Let $a_n a_m > 0$. Then $E_{nm} \backslash \{s_n, s_m\} \not\subset \Omega_{\text{crit}}$.*

Thus, the only potentially critical edges $E_{n,m}$ are those whose end points s_n and s_m sport amplitudes a_n and a_m of different sign. To show that also those edges, save their endpoints, are not critical requires a different argument. We shall invoke a convex duality argument which rules out all the edges, save their endpoints, from the critical set.

Proposition 3.3. *For all* $1 \leq n < m \leq N$ *we have that* $E_{nm} \backslash \{s_n, s_m\} \not\subset \Omega_{\text{crit}}$.

Proof. Since $C_0^\infty(\mathbb{R}^3)$ is dense in $C_0^{0,1}(\mathbb{R}^3)$, we can substitute $v_\infty = u$ for ψ in (7) and, for the solution u of (7), obtain the identity

$$0 = \int_{\mathbb{R}^3} \Big(\frac{|\nabla u(s)|^2}{\sqrt{1 - |\nabla u(s)|^2}} - 4\pi u(s) \sum_{1 \leq n \leq N} a_n \delta_{s_n}(s) \Big) \mathrm{d}^3 s. \tag{23}$$

A simple rewriting of (23) yields

$$\mathcal{F}(u) = \int_{\mathbb{R}^3} \Big(1 - \frac{1}{\sqrt{1 - |\nabla u(s)|^2}} \Big) \mathrm{d}^3 s. \tag{24}$$

Defining

$$U(s) = \frac{-\nabla u(s)}{\sqrt{1 - |\nabla u(s)|^2}}, \tag{25}$$

where u is still the solution of (7), another elementary rewriting of (24) yields that $\mathcal{F}(u) = -\mathcal{G}(U)$, where

$$\mathcal{G}(V) = \int_{\mathbb{R}^3} \Big[\sqrt{1 + |V|^2} - 1 \Big] \mathrm{d}^3 s \tag{26}$$

is well-defined for any vector field V for which $|V| \in L^1_{\text{loc}}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3 \backslash B_R)$, where B_R is some ball of large radius R. Note that $\mathcal{G}(V)$ is related to $\mathcal{F}(v)$ by a Legendre–Fenchel transform, viz.

$$\mathcal{G}(V) = \max_{v \in C_0^{0,1}} \int_{\mathbb{R}^3} \Big(\Big[\sqrt{1 - |\nabla v|^2} - 1 \Big] - V \cdot \nabla v \Big) \mathrm{d}^3 s; \tag{27}$$

the dual variables of the transformation are $\nabla v \leftrightarrow V$. Thus, $\mathcal{G}(V)$ is strictly convex in V. But we have seen that also $\mathcal{F}(v)$ is strictly convex for $v \in \mathscr{A}$, so $\mathcal{F}(v)$ — or rather the source-free part of $\mathcal{F}(v)$ — is given as a Legendre–Fenchel transform of $\mathcal{G}(V)$. As a result, we can also obtain the minimum of $\mathcal{F}(v)$ and its minimizer $v_\infty = u$ in terms of a constrained minimum principle for $\mathcal{G}(U)$. Explicitly,

$$\mathcal{G}(U) = \min \Big\{ \mathcal{G}(V) \Big| \nabla \cdot V = 4\pi \sum_{n=1}^{N} a_n \delta_{s_n};\ |V| \in L^1_{\text{loc}}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3 \backslash B_R) \Big\}; \tag{28}$$

in (28) it is understood that $\nabla \cdot V$ is well-defined in the sense of distributions and that R is big enough so that $\{s_n\}_{n=1}^N \subset B_R$.

Next, we define the almost everywhere harmonic field

$$V_h(s) = -\sum_{n=1}^{N} a_n \nabla \frac{1}{|s - s_n|}. \tag{29}$$

Note that $V_h(s) \in L^1_{\rm loc}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3 \backslash B_R)$ whenever $\{s_n\}_{n=1}^N \subset B_R$. Furthermore,

$$\nabla \cdot V_h = 4\pi \sum_{n=1}^{N} a_n \delta_{s_n}. \tag{30}$$

So V_h is in the set of admissible vector fields for our variational principle (28).

We are now ready for our main argument. Namely, suppose that for some n, m the edge $E_{n,m} \subset \Omega_{\rm crit}$. Without loss of generality we may assume that no other s_k lies on $E_{n,m}$. Then $\lim_{s \to E_{n,m}} |\nabla u(s)| = 1$, and so $\lim_{s \to E_{n,m}} |U(s)| = \infty$. But since u is analytic away from $\Omega_{\rm crit}$, so is U, hence we conclude that there is some tubular neighborhood of $E_{n,m}$ in which $|U(s)| > |V_h(s)|$. Since $s_n \neq s_m$ we can intersect our tubular neighborhood of $E_{n,m}$ with two small balls centered at s_n and s_m, respectively, and delete the intersection domain from it. Denote the resulting truncated tubular neighborhood by $E^\circ_{n,m}$; it is a bounded open set. Mollify its boundary $\partial E^\circ_{n,m}$ a little bit to obtain an open corridor $C^\circ_{n,m}$; it needs to have a finite distance from any s_k. Now let $V_*(s)$ be given by $U(s)$ for $s \notin E^\circ_{n,m} \cup C^\circ_{n,m}$, and by $V_*(s) = V_h(s)$ for $s \in E^\circ_{n,m} \backslash C^\circ_{n,m}$. We need to connect these fields smoothly across $C^\circ_{n,m}$, but this is easy. Since away from $\{s_n\}_{n=1}^N$ the fields U and V_h are divergence-free, we can represent each field as the curl of some vector field. We can choose a C^∞ deformation of one such vector field into the other across the transition region $C^\circ_{n,m}$, and in $C^\circ_{n,m}$ we define V_* to be the curl of this deformed field. Thus V_* is in the set of admissible vector fields for our variational principle. But then we have $\mathcal{G}(V_*) < \mathcal{G}(U)$, which is a contradition to our variational principle (28).

Thus $|U(s)| < \infty$ for $s \in \mathbb{R} \backslash \{s_n\}_{n=1}^N$, and therefore $\Omega_{\rm crit} = \{s_n\}_{n=1}^N$. □

Remark 3.4. We close this section with the remark that our convex duality argument can also be adapted to show that $|\nabla v_\infty(s)| < 1$ away from $\{s_n\}_{n=1}^N$ without invoking Proposition 3.1. But then a Nash-Moser argument has to be supplied to bootstrap the regularity of v_∞ from Lipschitz continuity to real analyticity in $\mathbb{R}^3 \backslash \{s_n\}_{n=1}^N$.

4. Applications to Geometry and Physics

4.1. Spacetime interpretation of Theorem 1.1. A smooth space-like hypersurface Σ in Minkowski spacetime $\mathbb{M}^4 \cong \mathbb{R} \times \mathbb{R}^3$ is a three-dimensional simply connected subset of $\mathbb{M}^4$ with a time-like normal vector at every point in Σ. Thus $\Sigma = \{\varpi \in \mathbb{M}^4 : \mathcal{T}(\varpi) = 0\}$ is the boundary of the zero level set of a differentiable function $\mathcal{T} : \mathbb{M}^4 \to \mathbb{R}$ with $\mathrm{ran}(\mathcal{T}) = \mathbb{R}$ and $\mathbf{d}\mathcal{T}(\varpi)$ time-like, i.e. $\mathbf{g}^{-1}(\mathbf{d}\mathcal{T}(\varpi), \mathbf{d}\mathcal{T}(\varpi)) < 0$ for all $\varpi \in \mathbb{M}^4$; here $\mathbf{d}$ is E. Cartan's exterior derivative on $\mathbb{M}^4$ and $\mathbf{g}$ the Minkowski metric with signature +2, a 2-covariant tensor acting on $T(\mathbb{M}^4) \times T(\mathbb{M}^4)$, where $T(\mathbb{M}^4)$ is the tangent bundle of $\mathbb{M}^4$. Topologically, $\Sigma \sim \mathbb{R}^3$.

Since any such hypersurface is a graph over $\mathbb{R}^3$, without loss of generality we may assume that the generating function T is of the form $T(\varpi) = t - c^{-1}S(s)$, where $\varpi \cong (ct, s)$ defines a Lorentz frame, where t is time and s is a vector in Euclidean space $\mathbb{R}^3$. Then $\Sigma = \{(ct, s) : t = c^{-1}S(s)\}$. Since $\mathbf{g}^{-1}(\mathbf{d}T(\varpi), \mathbf{d}T(\varpi)) = -1 + |\nabla S|^2$, and since Σ is space-like, we need to have $1 - |\nabla S|^2 > 0$ everywhere.

For those Σ which are asymptotically flat, more precisely if $\Sigma \asymp \Sigma_0$ with $\Sigma_0 \cong \mathbb{R}^3$, the volume difference $\triangle\mathrm{vol}(\Sigma|\Sigma_0)$ of Σ versus Σ_0 is well-defined. After at most a Lorentz transformation we can choose $\Sigma_0 = \{(ct, s) : t = 0\} \cong \mathbb{R}^3$, in which case

$$\triangle\mathrm{vol}(\Sigma|\Sigma_0) = \int_{\mathbb{R}^3} \left(\sqrt{1 - |\nabla S|^2} - 1\right) \mathrm{d}^3 s. \tag{31}$$

Note that $\triangle\text{vol}(\Sigma|\Sigma_0) \leq 0$. A hypersurface Σ is called *maximal* if any compact variation leads to a decrease of volume. In particular, $\Sigma_0 \cong \mathbb{R}^3$ is a maximal entire spacelike hypersurface in $\mathbb{M}^4$. By a Bernstein theorem of Cheng and Yau [ChYa1976], *any* entire space-like maximal hypersurface in $\mathbb{M}^4$ is flat; see also [Yan2000]. Thus, to be interesting a maximal hypersurface cannot be entirely space-like but at best only space-like almost everywhere. If in particular Σ has isolated defects then by Ecker's theorem these are lightcone singularities, i.e. isolated points in Σ where the normal vector touches the lightcone. Any such almost-everywhere space-like maximal hypersurface with point defects in $\mathbb{M}^4$ is the graph $\Sigma = \{(ct, s) : t = c^{-1}S(s)\}$ of a function $S(s)$ satisfying $1 - |\nabla S|^2 > 0$ away from the defects, such that $1 - |\nabla S|^2$ extends continuously into the defects, where it vanishes.

Prescribing the locations $s_k \in \mathbb{R}^3$ of the lightcone singularities does not uniquely determine an asymptotically flat maximal hypersurface with defects. In addition, the particular asymptotically linear behavior of $S(s)$, and also the integral mean curvatures $\mu_k \in \mathbb{R}\backslash\{0\}$ which are associated with each lightcone singularity of the hypersurface have to be prescribed. Maximizing $\triangle\text{vol}(\Sigma|\Sigma_0)$ for such a hypersurface with lightcone singularities of prescribed integral mean curvatures is a variational problem with constraints. The Euler-Lagrange equation for this problem reads[5]

$$-\nabla \cdot \frac{\nabla S}{\sqrt{1 - |\nabla S|^2}} = 3 \sum_{k=1}^{N} \mu_k \delta_{s_k}. \tag{32}$$

Identifying $S = u$ and $\mu_k = (4\pi/3)a_k$ yields (1).

In this spacetime interpretation our Theorem 1.1 becomes:

Corollary 4.1. *For any set $\{s_k\}_{k=1}^N \subset \mathbb{R}^3$ of distinct points and any set of integral mean curvatures $\{\mu_k\}_{k=1}^N \subset \mathbb{R}\backslash\{0\}$ assigned to these points, there exists a unique asymptotically flat hypersurface $\Sigma = \{(ct, s) : t = c^{-1}S(s)\}$ with $S \in C_0^{0,1}(\mathbb{R}^3, \mathbb{R}) \cap C^\omega(\mathbb{R}^3\backslash\{s_k\}_{k=1}^N, \mathbb{R})$ solving (32); moreover, $|\nabla S(s)| \to 1$ as $s \to s_k$. Thus Σ is an almost everywhere space-like maximal hypersurface, having N lightcone singularities with prescribed integral mean curvatures μ_k located at the s_k.*

4.2. Electrostatic interpretation of Theorem 1.1. In classical electromagnetic field theory, the Coulomb law states that an electric (point-)charge "density" in $\mathbb{R}^3$ is the source of the electric displacement field $\boldsymbol{D}$,

$$\nabla \cdot \boldsymbol{D} = 4\pi \sum_{k=1}^{N} z_k \delta_{s_k} \qquad \text{Coulomb's law}, \tag{33}$$

with[6] $z_k \in \mathbb{Z}\backslash\{0\}$, while Faraday's law says that an electrostatic field $\boldsymbol{E}$ is curl-free,

$$\nabla \times \boldsymbol{E} = \boldsymbol{0} \qquad \text{Faraday's law (stationary)}. \tag{34}$$

[5] We follow the convention of [GiTr1983] which differs from that in [BaSi1982] by the factor 3.

[6] Empirically, all nuclear and electron charges are integer multiples of the elementary charge, which is unity in our units.

These two laws need to be complemented by a law to relate $\boldsymbol{E}$ and $\boldsymbol{D}$, for which Max Born [Bor1933] proposed

$$\boldsymbol{D} = \frac{\boldsymbol{E}}{\sqrt{1-\beta^4|\boldsymbol{E}|^2}} \qquad \text{Born's law} \tag{35}$$

with $\beta \in (0, \infty)$ (we use the dimensionless notation of [Kie2004a]). In the limit $\beta \to 0$ Born's law (35) goes over into Maxwell's law of the "pure aether", $\boldsymbol{D} = \boldsymbol{E}$. We remark that (35) is the electrostatic limit of both, the electromagnetic law proposed by Born [Bor1933, Bor1969] and the more elaborate law proposed by Born and Infeld [BoIn1933, BoIn1934]. The latter has received much attention in recent years, see the references in [Gib1998, Kie2004a, Kie2004b, Kie2012].

Clearly, (34) implies that $\boldsymbol{E} = -\nabla A$ for some scalar potential field A. Inserting this representation for $\boldsymbol{E}$ into (35), which in turn is inserted in (33), we obtain

$$-\nabla \cdot \frac{\nabla A}{\sqrt{1-\beta^4|\nabla A|^2}} = 4\pi \sum_{k=1}^{N} z_k \delta_{s_k}\,. \tag{36}$$

Multiplying (36) by β^2 and identifying $\beta^2 A = u$ and $\beta^2 z_k = a_k$ we retrieve (1).

In this electrostatic interpretation our Theorem 1.1 yields:

Corollary 4.2. *For any finite number N of point charges $\{z_k\}_{k=1}^N \subset \mathbb{Z}\backslash\{0\}$ located at distinct points $\{s_k\}_{k=1}^N \subset \mathbb{R}^3$, there exists a unique electrostatic field $\mathbf{E}$ in $\mathbb{R}^3\backslash\{s_k\}_{k=1}^N$ which solves (33), (34), (35) and has finite field energy*[7]

$$\mathcal{E}_{\text{field}}(\boldsymbol{D}) = \frac{1}{4\pi}\frac{\alpha}{\beta^4}\int_{\mathbb{R}^3}\left(\sqrt{1+\beta^4|\boldsymbol{D}|^2}-1\right)\mathrm{d}^3s. \tag{37}$$

The solution $\mathbf{E} \in C^\omega(\mathbb{R}^3\backslash\{s_k\}_{k=1}^N, \mathbb{R}^3)$, but it cannot be continuously extended into the s_k. It is bounded, with $\beta^2|\mathbf{E}(s)| \to 1$ for $s \to s_n$, and it vanishes for $|s| \to \infty$.

Remark 4.3. Presumably inspired by Theorem 4.1 and Corollary 4.2 in [BaSi1982], at the beginning of Sect. 4 in [Gib1998][8] Gibbons contemplates the following: "It is well known that one can construct explicit multi-black hole solutions held apart by struts, the struts being the sites of conical singularities representing distributional stresses. One should be able to construct analogous multi-BIon solutions." (What Gibbons calls "multi-BIon" solutions are but electrostatic solutions to the Maxwell–Born–Infeld equations with many point charge sources. In particular, Born's solution, for Gibbons, is "the BIon.") Our Theorem 1.1 and its Corollary 4.2 show that struts between the point charges do not occur in the electrostatic solutions to the Maxwell–Born–Infeld field equations with point charge sources.

[7] Here, α is Sommerfeld's fine structure constant, inherited from the units in [Kie2004a].

[8] In the same paragraph in [Gib1998] the results of Bartnik and Simon [BaSi1982] (Gibbons' reference [Bar1987]) for the weak solvability of the Dirichlet problem are somewhat misquoted: the necessary condition (in our notation) $|u_n - u_m| \le |s_n - s_m|$ on the Dirichlet data at the distinct points s_n and s_m, which in Gibbons' notation would have to read $|\Phi^a - \Phi^b| \le |\mathbf{x}_a - \mathbf{x}_b|$, is missing. Whenever $|\Phi^a - \Phi^b| = |\mathbf{x}_a - \mathbf{x}_b|$, then a strut between $\mathbf{x}_a$ and $\mathbf{x}_b$ does occur.

5. Extensions of our Results

The geometric interpretation of u as a time function of a maximal almost-everywhere space-like hypersurface with lightcone singularities in Minkowski spacetime allows one to exploit the Poincaré group of $\mathbb{M}^4$ to generate solutions u to (1) with different linear asymptotics at space-like infinity than (2). Since there is a unique Poincaré transformation for the transition from asymptotically vanishing to non-zero asymptotically linear conditions, we can therefore conclude:

Corollary 5.1. *For any finite sets $\{s_n\}_{n=1}^N \subset \mathbb{R}^3$ and $\{a_n\}_{n=1}^N \subset \mathbb{R}\backslash\{0\}$ and a vector $e \in \mathbb{R}^3$ of magnitude $|e| < 1$ there exists a unique real function $u \in C^{0,1}(\mathbb{R}^3)$ satisfying*

$$u(s) - e \cdot s \to 0 \quad \text{as} \quad |s| \to \infty, \tag{38}$$

and solving (1) in the sense of distributions. Furthermore, $|\nabla u(s)| < 1$ for $s \in \mathbb{R}^3\backslash\{s_n\}_{n=1}^N$, and $\lim_{s\to s_n} |\nabla u(s)| = 1$ for each s_n. Thus, $u \in C^\omega(\mathbb{R}^3\backslash\{s_n\}_{n=1}^N)$.

The equivalence between the mathematical theories of maximal space-like hypersurfaces with point defects in Minkowski spacetime on the one hand and the electrostatic Maxwell–Born-Infeld potentials generated by point charge sources on the other allows us furthermore to re-interpret these asymptotically nontrivially linear hypersurfaces as electrostatic solutions with asymptotically (at spacelike infinity) constant electric fields. It is worth stressing that, in the notation of our previous subsection, one thus interprets $(\beta^2 A, s)$, rather than the spacetime point (ct, s), as the Minkowski four-vector to generate new solutions by Poincaré transformations. This "hidden Poincaré symmetry" seems to have been exploited first by Gibbons, see Sects. 3.3 and 3.7 of [Gib1998]; in particular, in Sect. 3.7 Gibbons transforms Born's solution into an electrostatic solution with a single point charge and an asymptotically constant electric field whose magnitude is below Born's critical field strength.

Lastly, as already announced in the Introduction, there is an analogue of our Theorem 1.1 for the Dirichlet problem in bounded domains with nice boundary. This is not directly a corollary of our proof, yet its proof follows by a straightforward adaptation of our proof to the Dirichlet problem. Thus we claim:

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain containing the finite point set $\{s_n\}_{n=1}^N \subset \mathbb{R}^3$. Then for any $\phi : \partial\Omega \to \mathbb{R}$ satisfying $|\phi(s) - \phi(s')| < |s - s'|$ for $s \neq s'$, and any set $\{a_n\}_{n=1}^N \subset \mathbb{R}\backslash\{0\}$, there exists a unique real $u \in C^{0,1}(\overline{\Omega})$ solving*

$$\nabla \cdot \frac{\nabla u(s)}{\sqrt{1 - |\nabla u(s)|^2}} + 4\pi \sum_{n=1}^{N} a_n \delta_{s_n}(s) = 0 \qquad \text{for } s \in \Omega, \tag{39}$$

$$u(s) = \phi(s) \quad \text{for } s \in \partial\Omega \tag{40}$$

in the sense of distributions. Furthermore, $|\nabla u(s)| < 1$ for $s \in \Omega\backslash\{s_n\}_{n=1}^N$, and $\lim_{s\to s_n} |\nabla u(s)| = 1$ for each s_n. Thus, $u \in C^\omega(\Omega\backslash\{s_n\}_{n=1}^N)$.

Remark 5.3. Bernd Kawohl kindly explained to me that for such a bounded domain the detour via the $\mathcal{F}_K(v)$ should be unnecessary to minimize the convex functional $\mathcal{F}(v)$ over the convex set $\{v \in W_0^{1,\infty}(\Omega) : |\nabla v| \leq 1\, a.e.\ in\ \Omega\}$.

6. Desiderata

For matters of a quantitative nature it is important to have efficient algorithms to actually compute the solutions which in this paper we have proved do exist. For Hölder-continuous regularizations of the point charge sources such an algorithm has been developed in [CaKi2010, Kie2011], but so far none seems available which would cover the point charge and other singular sources in $\mathbb{R}^3$. The situation is better for the lower-dimensional problem in $\mathbb{R}^2$, see [Pry1935a, Kob1988, Fer2010], and it is desirable also for the solutions in $\mathbb{R}^3$ to have explicit formulas in terms of, say, quadratures and such. For the time being, the variational arguments allow one to work with numerical discretizations and to run minimization routines.

Another question is whether our Minkowski space results extend to certain curved Lorentz manifolds, in particular to asymptotically flat Lorentz manifolds [Bar1984]. If the Lorentz manifold is given (a so-called background spacetime), then the essence of the results of [BaSi1982] remains true under appropriate conditions, as shown by Bartnik in [Bar1988] with quite different arguments than those in [BaSi1982]. Moreover, in [Bar1989] Bartnik also extended Ecker's singularity theorem to certain Lorentz manifolds. For those Lorentz manifolds for which the analogue of the flat spacetime theorems of Bartnik–Simon hold we expect that analogues of our theorems will hold as well. We remark that Bartnik's theorems in [Bar1988, Bar1989] do not require the manifold to be asymptotically flat.

Another question, of prime importance as explained in [Kie2012], is whether the extension of our electrostatic results to a general-relativistic setting is possible in which an asymptotically flat Lorentz manifold is to be found by solving Einstein's field equations, with an electrostatic energy(-density)-momentum(-density)-stress tensor as curvature source for the metric, along with solving the Maxwell–Born–Infeld equations for the electrostatic field in the curved spacetime. The problem with a single point charge source was treated already by Hoffmann [Hof1933ff] but only recently with complete rigor, by Tahvildar-Zadeh [TaZa2011]; there the reader is also directed to the large amount of literature on the subject. The general sentiment, as expressed in the quote from Gibbons at the end of Sect. 4, seems to be that in the multi-point-charge problem struts will occur between the point charges; see also [Wei1996]. We hope to offer a definitive answer in the foreseeable future.

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