A new definition of a coordinate- and frame-free observer in relativistic electrodynamics

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Situating our Work

1. Formulate a model in intuitive terms
2. Recast the model in mathematical terms
3. Analyze the mathematical structures
4. Apply the framework to relativistic electrodynamics
5. Generalize the framework beyond the initial scope

Here are our contributions.

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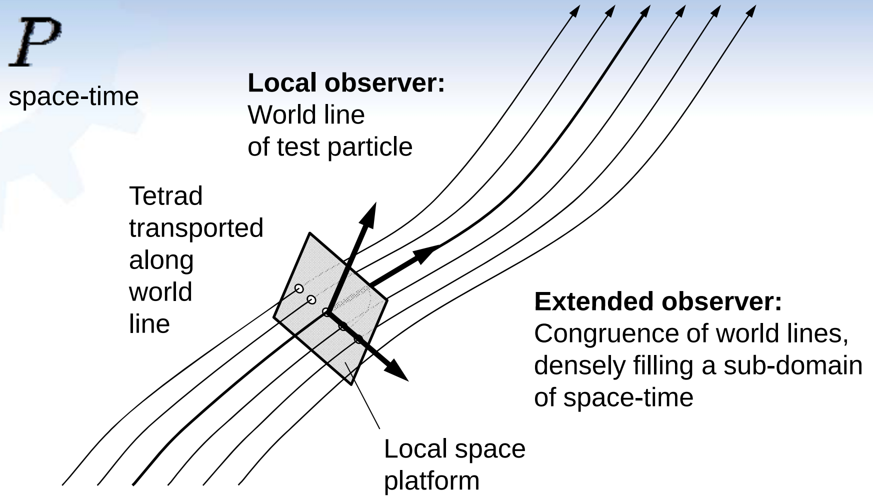
The Model in Intuitive Terms (i)
Local Observer versus Extended Observer



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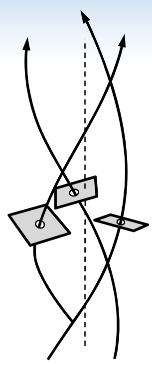
The Model in Mathematical Terms Proposal

- A space-time densely filled with world lines is a fiber bundle
- Time translation implies: the fibers are diffeomorphic to the Lie group of one-dimensional translations
- An observer's space is modeled by the bundle's base manifold
- A time synchronization is modeled by a section of the bundle
- The splitting into spatial and temporal pieces is modeled by an Ehresmann connection



Situating our Work

1. Formulate a model in intuitive terms
2. Recast the model in mathematical terms
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   1. Space and time – setting
   2. Space and time – splitting
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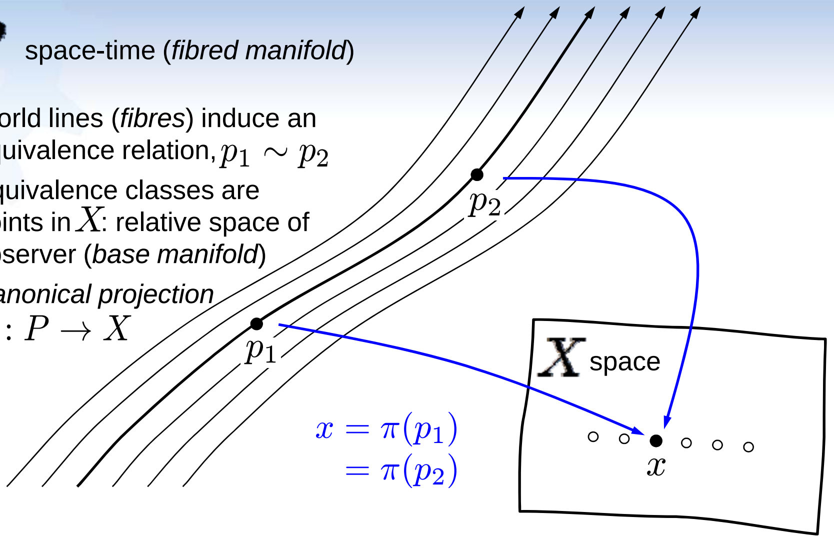
Space and Time – Setting (i)
Space as Base Manifold

P space-time (*fibred manifold*)

- World lines (*fibres*) induce an equivalence relation, $p_1 \sim p_2$
- Equivalence classes are points in X: relative space of observer (*base manifold*)
- *Canonical projection* $\pi : P \to X$



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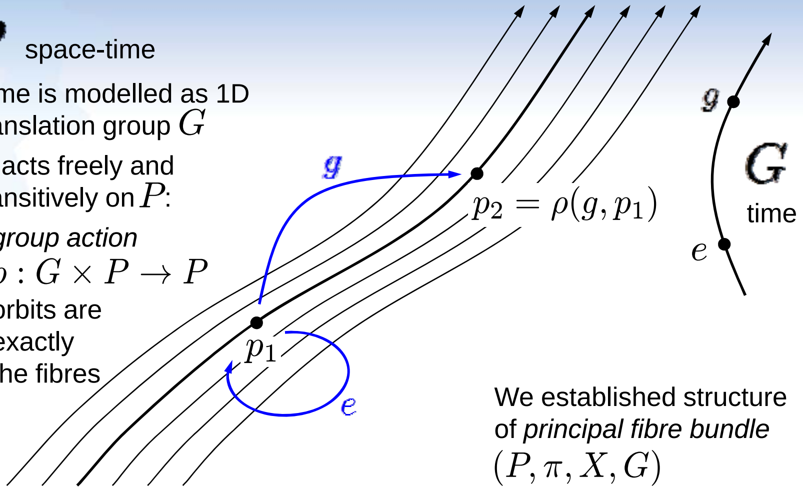
Space and Time – Setting (ii)
Time as Translation Group

P space-time

- Time is modelled as 1D translation group G
- G acts freely and transitively on P:
  - *group action* $\rho : G \times P \to P$
  - orbits are exactly the fibres



We established structure of *principal fibre bundle* (P, π, X, G)

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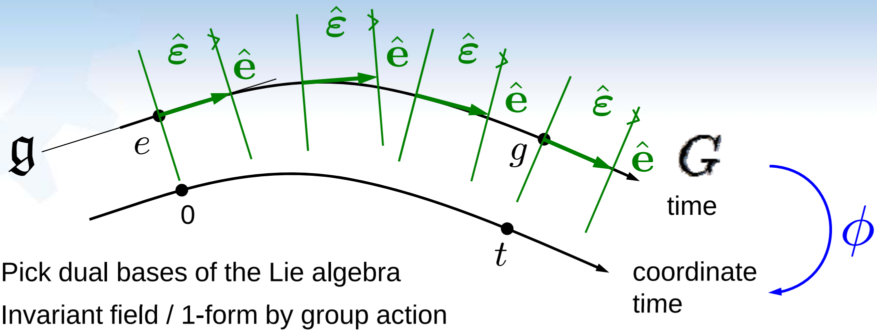
Space and Time – Setting (iii)
Lie Algebra, Time Scale, Coordinate Time



- Pick dual bases of the Lie algebra
- Invariant field / 1-form by group action
- $\hat{\varepsilon}$ fixes a time scale
- $\hat{\varepsilon}$ fixes a chart (coordinate time): $g \mapsto \phi(g) = \int_e^g \hat{\varepsilon}$
- In the chart, $(\hat{\mathbf{e}}, \hat{\varepsilon})$ is represented by $(\partial_t, \mathrm{d}t)$

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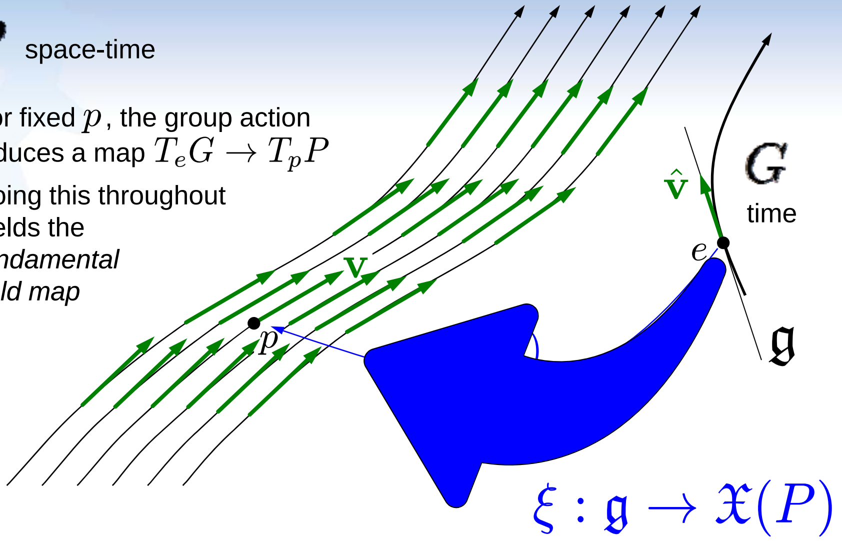
Space and Time – Setting (iv)
Fundamental Field Map

P space-time

- For fixed p, the group action induces a map $T_eG \to T_pP$
- Doing this throughout yields the *fundamental field map*



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Space and Time – Setting (v)
Fundamental Field

Fundamental field	$\mathbf{w} = \xi(\hat{\mathbf{e}}) \otimes \hat{\varepsilon} \in \mathfrak{X}(P; \mathfrak{g}^*)$
Independent of dual bases of Lie algebra ("time scale")	✓

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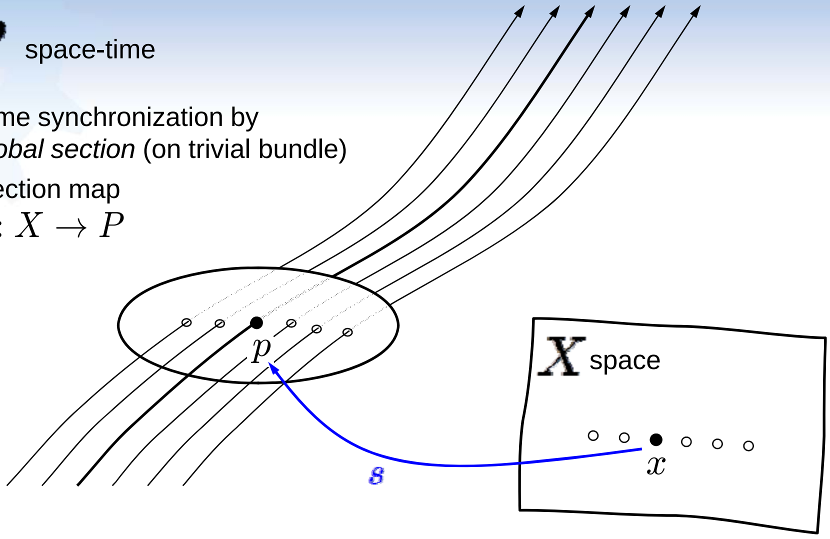
Space and Time – Setting (vi)
Time Synchronization as Section

P space-time

- Time synchronization by *global section* (on trivial bundle)
- Section map
  $s : X \to P$



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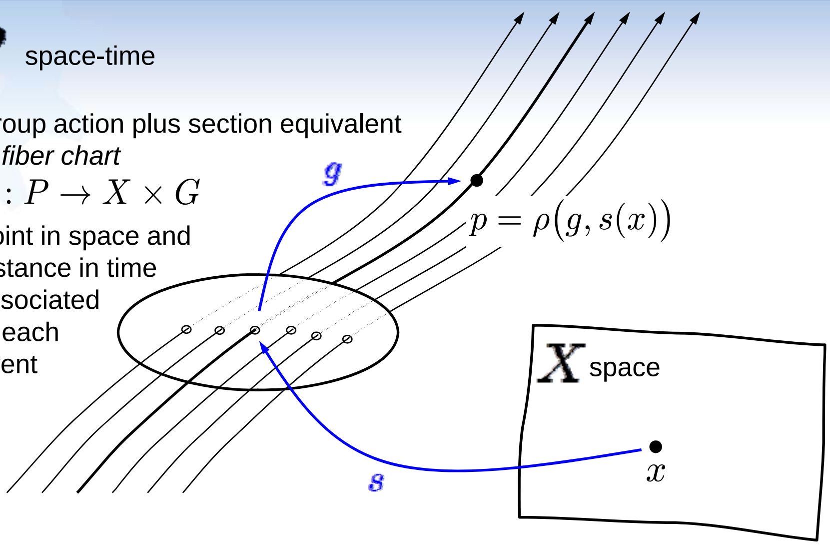
Space and Time – Setting (vii)
Fiber Chart and Simultaneity Structure

P space-time

- Group action plus section equivalent to *fiber chart*
  $\varphi : P \to X \times G$
- Point in space and instance in time associated to each event



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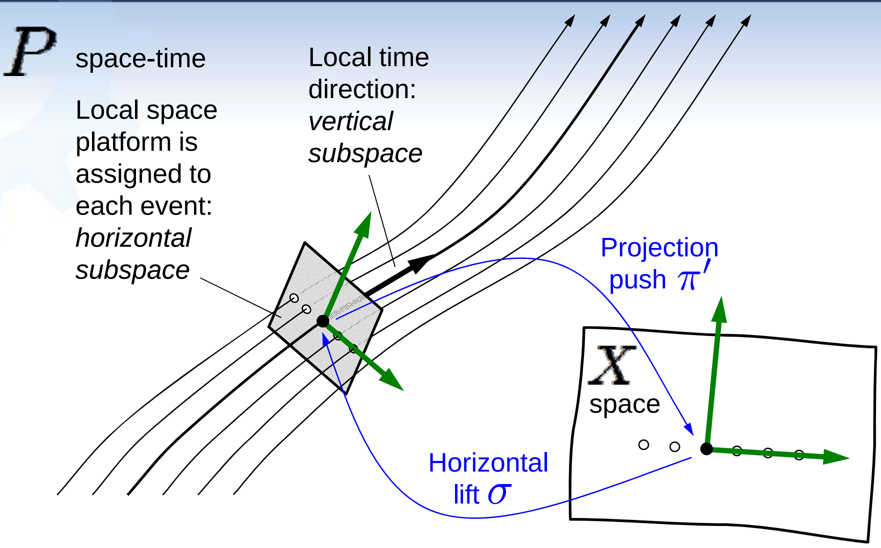
Space and Time – Splitting (i)
Local Space Platforms as Horizontal Lift



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Space and Time – Splitting (ii)
Vector Fields and Horizontal Lift

$\mathfrak{X}(P)$ Vector fields in space-time

$\mathfrak{X}(X,G)$ Time-dependent vector fields in space

$$\left.\begin{matrix}\pi' \\ \sigma\end{matrix}\right\} \text{induce with fiber chart} \left\{\begin{aligned}\Pi &: \mathfrak{X}(P) \to \mathfrak{X}(X,G) \\ \Sigma &: \mathfrak{X}(X,G) \to \mathfrak{X}(P)\end{aligned}\right.$$

$$\Pi \circ \Sigma = \mathrm{Id}$$

$$\Sigma \circ \Pi \neq \mathrm{Id}$$ in general

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Space and Time – Splitting (iii)
Horizontal Lift Fixes Ehresmann Connection

$$\Sigma \circ \Pi = \mathrm{Id} - \mathbf{w} \otimes \boldsymbol{\omega}$$

uniquely defines the *connection form* $\boldsymbol{\omega} \in \mathcal{F}^1(P;\mathfrak{g})$ of an Ehresmann connection

$\boldsymbol{\omega} \circ \Sigma = 0$ kills horizontal vectors

$\boldsymbol{\omega}(\mathbf{w}) = 1$ normalized

Space and Time – Splitting (iv)
Differential Forms and Horizontal Lift

$\mathcal{F}(P)$ Differential forms in space-time

$\mathcal{F}(X,G)$ Time-dependent differential forms in space

$\mathcal{F}, \mathcal{F}_\times$ ordinary, twisted

	Vector fields	Differential forms
Space-time	$\mathfrak{X}(P)$	$\mathcal{F}(P)$
	$\Pi \downarrow \quad \uparrow \Sigma$	$\Sigma^* \downarrow \quad \uparrow \Pi^*$
Space, Time	$\mathfrak{X}(X,G)$	$\mathcal{F}(X,G)$

Space and Time – Splitting (vi)
Description of Connection in Base Manifold

- Covariant exterior derivative $D = \Sigma^* \circ \mathrm{d} \circ \Pi^*$
- Group derivative $\boldsymbol{\partial}_t = \Sigma^* \circ \mathcal{L}_{\mathbf{w}} \circ \Pi^*$

$$\boldsymbol{\partial}_t : \mathcal{F}^p(X,G) \to \mathcal{F}^p(X,G;\mathfrak{g}^*)$$
$$\gamma(x,t) \mapsto \partial_t \gamma(x,t) \otimes \mathrm{dt}$$

$\mathbf{d}$ exterior derivative, $\mathcal{L}$ Lie derivative

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Space and Time – Splitting (vi)
Description of Connection in Base Manifold

- Covariant exterior derivative $D = \Sigma^* \circ \mathrm{d} \circ \Pi^*$
- Group derivative $\boldsymbol{\partial}_t = \Sigma^* \circ \mathcal{L}_{\mathbf{w}} \circ \Pi^*$
- Curvature 2-form $\boldsymbol{\Omega} = \Sigma^* \mathrm{d}\boldsymbol{\omega}$

  $\boldsymbol{\Omega} = 0$: connection flat / integrable
- Non-principality 1-form $\boldsymbol{\chi} = \Sigma^* \mathcal{L}_{\mathbf{w}} \boldsymbol{\omega}$

  $\boldsymbol{\chi} = \mathbf{0}$: connection principal

$\mathbf{d}$ exterior derivative, $\mathcal{L}$ Lie derivative

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   2. Metric and constitutive relations
   3. Normalization
   4. Rotating platform
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Pre-metric Electrodynamics (i)
Maxwell's Equations in 4D Space-Time

$\mathbf{F} \in \mathcal{F}^2(P)$	Electromagnetic field
$\mathbf{G} \in \mathcal{F}^2_\times(P)$	Electromagnetic excitation
$\mathcal{J} \in \mathcal{F}^3_\times(P)$	Electric charge-current
$\mathrm{d}\mathbf{F} = 0$	Maxwell-Faraday's law: Flux conservation
$\mathrm{d}\mathbf{G} = \mathcal{J}$	Maxwell-Ampère's law: Charge conservation

Pre-metric Electrodynamics (ii)
Connection Fixes Splitting Map

$$S : \mathcal{F}^p(P) \xrightarrow{\sim} \mathcal{F}^p(X, G) \times \mathcal{F}^{p-1}(X, G; \mathfrak{g}^*) :$$

$$S = \Sigma^* \circ \begin{pmatrix} \mathrm{Id} \\ \mathrm{i}(\mathbf{w}) \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} \mathrm{Id} & \mathrm{e}(\boldsymbol{\omega}) \end{pmatrix} \circ \Pi^*$$

$\mathrm{i}(\mathbf{w})$ contraction, $\mathrm{e}(\boldsymbol{\omega})\boldsymbol{\gamma} = \boldsymbol{\omega} \wedge \boldsymbol{\gamma}$ multiplication

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Pre-metric Electrodynamics (iii)
Formally Define Field Entities in (1+3)D

$$S\mathbf{F} = \begin{pmatrix} \mathbf{B} \\ -\tilde{\mathbf{E}} \end{pmatrix} \in \mathcal{F}^2(X,G) \times \mathcal{F}^1(X,G;\mathfrak{g}^*)$$

$$S\mathbf{G} = \begin{pmatrix} \mathbf{D} \\ \tilde{\mathbf{H}} \end{pmatrix} \in \mathcal{F}^2_\times(X,G) \times \mathcal{F}^1_\times(X,G;\mathfrak{g}^*)$$

$$S\boldsymbol{\mathcal{J}} = \begin{pmatrix} \boldsymbol{\rho} \\ -\tilde{\mathbf{J}} \end{pmatrix} \in \mathcal{F}^3_\times(X,G) \times \mathcal{F}^2_\times(X,G;\mathfrak{g}^*)$$

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Pre-metric Electrodynamics (iv)
Decomposition of Exterior Derivative

$$S \circ \mathrm{d} \circ S^{-1} = \begin{pmatrix} D & \mathrm{e}(\boldsymbol{\Omega}) \\ \boldsymbol{\partial}_t & -D + \mathrm{e}(\boldsymbol{\chi}) \end{pmatrix}$$

$$\begin{aligned} D \circ D &= -\mathrm{e}(\boldsymbol{\Omega}) \circ \boldsymbol{\partial}_t \neq 0 \\ D \circ \boldsymbol{\partial}_t - \boldsymbol{\partial}_t \circ D &= \quad \mathrm{e}(\boldsymbol{\chi}) \circ \boldsymbol{\partial}_t \neq 0 \end{aligned}$$

Pre-metric Electrodynamics (v)
Structure of Maxwell's Equations Emerges

$$\left.\begin{aligned} D\tilde{\mathbf{E}} &= -\boldsymbol{\partial}_t\mathbf{B} + \mathrm{e}(\boldsymbol{\chi})\,\tilde{\mathbf{E}} \\ D\mathbf{B} &= \mathrm{e}(\boldsymbol{\Omega})\tilde{\mathbf{E}} \end{aligned}\right\}\ \text{Maxwell-Faraday}$$

$$\left.\begin{aligned} D\tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + \boldsymbol{\partial}_t\mathbf{D} + \mathrm{e}(\boldsymbol{\chi})\,\tilde{\mathbf{H}} \\ D\mathbf{D} &= \boldsymbol{\rho} - \mathrm{e}(\boldsymbol{\Omega})\tilde{\mathbf{H}} \end{aligned}\right\}\ \text{Maxwell-Ampère}$$

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Metric and Constitutive Relations (i)
Definition of Observer

- Consider space-time as Lorentzian manifold[1)] (P, h)
- An extended observer is modeled by a principal bundle with Ehresmann connection, such that:
  1. The fibers of the bundle coincide with the observer's world lines.
  2. *Hypothesis of locality*. The horizontal subspaces are orthogonal to the fibers.

[1)] orientable, time-orientable, signature $(+,-,-,-)$ globally hyperbolic

Metric and Constitutive Relations (ii)
Definition of Observer (cont'd)

- Observer models are equivalent if they define the same fibration of P
  - Equivalent observer models may differ by the choice of the group action ρ along the fibers
- We use the term *gauge* for the selection of a specific fiber chart
  - Equivalent to selection of section and group action
  - Determines simultaneity structure
  - Selects a specific representative of the class of equivalent observer models

Metric and Constitutive Relations (iii)
Observer Metric as Horizontal Metric

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- The observer metric is the *horizontal metric*

$$\bar{h} = -\Sigma^* h \quad \text{positive definite}$$

- Relative space of the observer is turned into a family of Riemannian manifolds

$$\{(X, \bar{h})\}_{g \in G}$$

- If $\bar{h}$ is independent of $g \in G$, the observer is called Born rigid.

Metric and Constitutive Relations (iv)
Four Velocity and Lapse Field

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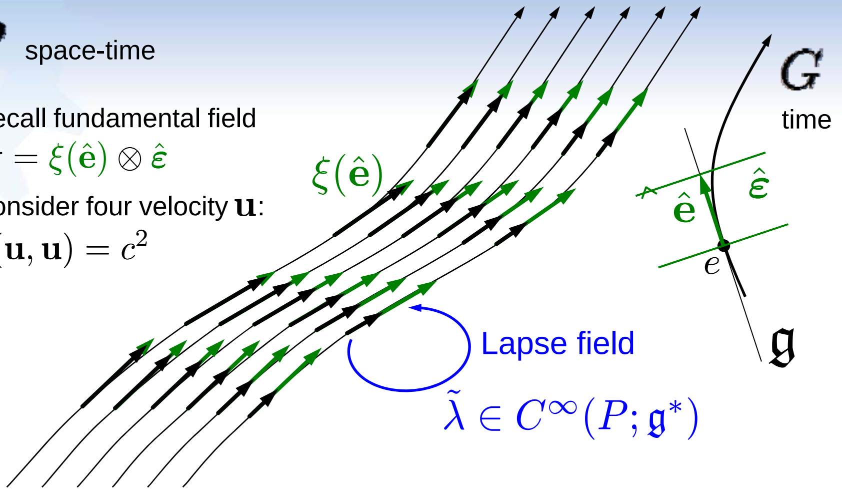
P space-time

- Recall fundamental field
  $\mathbf{w} = \xi(\hat{\mathbf{e}}) \otimes \hat{\varepsilon}$
- Consider four velocity $\mathbf{u}$:
  $g(\mathbf{u}, \mathbf{u}) = c^2$



Metric and Constitutive Relations (v)
Decomposition of Hodge

- Lorentzian metric h induces Hodge

$$* : \mathcal{F}(P) \to \mathcal{F}_\times(P)$$

- Riemannian metric $\bar{h}$ induces Hodge

$$\bar{*} : \begin{cases} \mathcal{F}(X,G) \to \mathcal{F}_\times(X,G) \\ \mathcal{F}(X,G,\mathfrak{g}^*) \to \mathcal{F}_\times(X,G,\mathfrak{g}^*) \end{cases}$$

Metric and Constitutive Relations (vi)
Constitutive Relations for Empty Space

4D

$$\mathbf{G} = \frac{1}{Z} * \mathbf{F} \qquad Z = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

(1+3)D

$$\left.\begin{aligned} \mathbf{D} &= \varepsilon_0 \,\bar{*}\tilde{\lambda}^{-1}\tilde{\mathbf{E}} \\ \mathbf{B} &= \mu_0 \bar{*}\tilde{\lambda}^{-1}\tilde{\mathbf{H}} \end{aligned}\right\}$$

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Normalization (i)
Retrieve Ordinary Differential Forms

- Convert all Lie (co-)algebra valued differential forms into ordinary differential forms by $\tilde{\lambda}$
- Example:

$$\tilde{\mathbf{E}} = \tilde{\lambda}\,\mathbf{E}$$

no “tail”: ordinary 1-form

“tail” $\hat{\boldsymbol{\varepsilon}}$: Lie co-algebra valued 1-form

“tail” $\hat{\boldsymbol{\varepsilon}}$: Lie co-algebra valued scalar

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Normalization (ii)
Maxwell's Equations Revisited

$$
\begin{aligned}
D\mathbf{E} &= -\partial_\tau \mathbf{B} + \mathrm{e}(\boldsymbol{\delta})\mathbf{E} \\
D\mathbf{B} &= \mathrm{e}(\boldsymbol{\eta})\mathbf{E} \\
D\mathbf{H} &= \mathbf{J} + \partial_\tau \mathbf{D} + \mathrm{e}(\boldsymbol{\delta})\mathbf{H} \\
D\mathbf{D} &= \boldsymbol{\rho} - \mathrm{e}(\boldsymbol{\eta})\mathbf{H}
\end{aligned}
$$

- $\partial_\tau = \tilde{\lambda}^{-1}\boldsymbol{\partial}_t$ proper time derivative
- $\boldsymbol{\delta} = \boldsymbol{\chi} - \tilde{\lambda}^{-1} D\tilde{\lambda}$ acceleration 1-form
- $\boldsymbol{\eta} = \tilde{\lambda}\boldsymbol{\Omega}$ vorticity 2-form

kinematic quantities, up to factor c^2

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Rotating Platform (i)
What is a Stationary Problem?

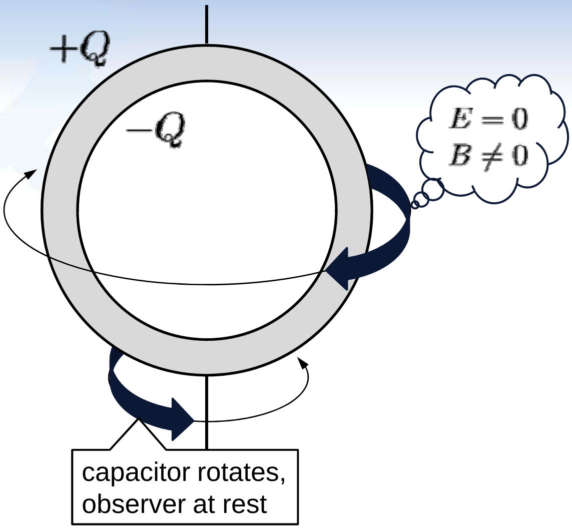
- Stationary problem
  - Fibers admit Killing field $\mathbf{k}$
  - Charge current symmetric, $\mathcal{L}_{\mathbf{k}}\mathcal{J} = 0$
- Then $\partial_\tau = 0$ ("time independent"), $D = \mathrm{d}$:

$$\begin{aligned} \mathrm{d}\mathbf{E} &= \mathrm{e}(\boldsymbol{\delta})\mathbf{E} \\ \mathrm{d}\mathbf{D} &= \boldsymbol{\rho} - \mathrm{e}(\boldsymbol{\eta})\mathbf{H} \end{aligned} \quad \text{electro-static}$$

coupled via vorticity

$$\begin{aligned} \mathrm{d}\mathbf{B} &= \mathrm{e}(\boldsymbol{\eta})\mathbf{E} \\ \mathrm{d}\mathbf{H} &= \mathbf{J} + \mathrm{e}(\boldsymbol{\delta})\mathbf{H} \end{aligned} \quad \text{magneto-static}$$

Rotating Platform (iii)
Definition of Schiff's Paradox



Schiff 1939: A question in general relativity. *Proc. Nat. Acad. Sci. USA* 25, 391-395

Rotating Platform (iii)
Definition of Schiff's Paradox

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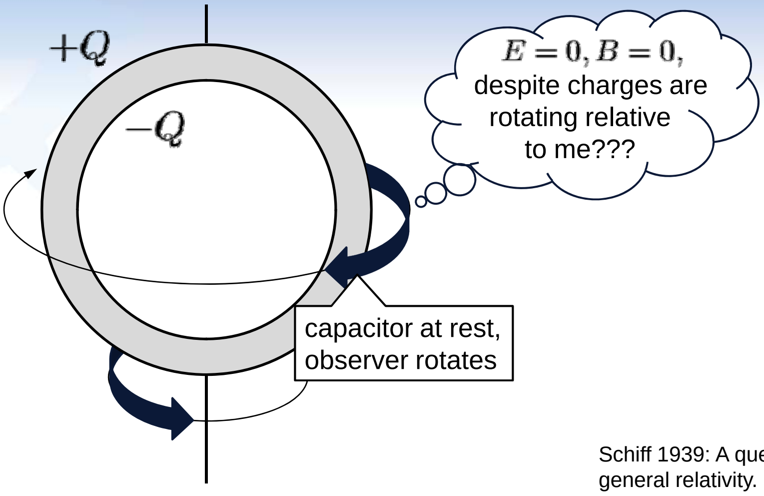

Schiff 1939: A question in general relativity. *Proc. Nat. Acad. Sci. USA* 25, 391-395

Rotating Platform (iv)
Resolution of Schiff's Paradox

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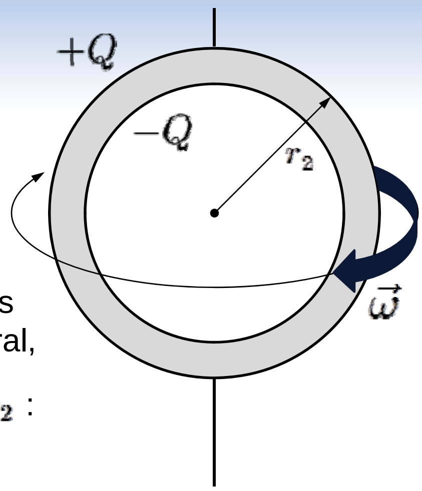
- Stationary problem
- Consider first order in v/c
- Maxwell's equations have standard form, except

  $\operatorname{div} \vec{B} = \frac{2}{c^2} \vec{\omega} \cdot \vec{E}$

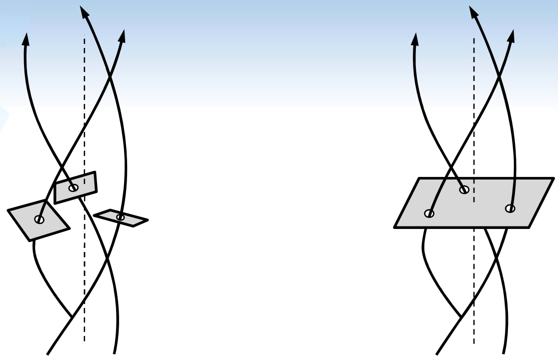
- Solution in spherical coordinates (r, θ, φ): E_r by Coulomb integral,

  $B_\theta = \dfrac{\omega r \cos\theta}{c^2} E_r = 0$ for $r > r_2$:

  paradox resolved



Rotating Platform (v)
Schiff's Treatment: Fields not Observable



Our treatment:
Orthogonal
curved connection

Schiff's treatment:
Non-orthogonal
flat connection

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    3. normalization
    4. Schiff paradox
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Beyond the Initial Scope (i)
Extension of the Pre-Metric Framework

- The pre-metric framework can be extended with minimal adjustments to allow for:
  - Bundle- and base manifolds of arbitrary dimensions.
  - Arbitrary Lie groups (including non-Abelian Lie groups).
- Splitting maps, decomposition of operators, and gauge transformations all remain operational.
- Any extension to a metric setting appears to be specific to an application.

Beyond the Initial Scope (ii)
What about a Frame-Field Approach?

1. Formulate a model in intuitive terms
   - Idem.
2. Recast the model in mathematical terms
   - Introduce observers as sections of the orthonormal frame bundle with an adapted coordinate system for the fibration.
3. Analyze the mathematical structures
   - Tensor calculus, anholonomic frames, Levi-Civita connections.
4. Apply the emerging framework
   - Frame-based splitting and decomposition of operators.
5. Generalize the framework beyond the initial scope
   - General frame-bundles.

Summary
What are the Takeaways?

- We model relativistic observers by principal bundles and Ehresmann connections.
- We define a concise set of mathematical structures to discuss observers in an entirely frame- and coordinate-free way.
  - The presented framework subsumes, characterizes, and extends various observer models.
  - The framework enables discussing anholonomic observers without conceptual or technical difficulty.
- The generalization to arbitrary principal bundles contributes to mathematical physics.

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Addendum (i)
Remarks about Poynting Form

- Note that $\tilde{\mathbf{E}} \wedge \tilde{\mathbf{H}}$ is not defined
- Energy-momentum tensor
$$\mathbf{T} : \mathfrak{X}(P) \to \mathcal{F}^3_\times(P) :$$
$$\mathbf{n} \mapsto \tfrac{1}{2}\big(\mathrm{i}(\mathbf{n})\mathbf{F} \wedge \mathbf{G} - \mathrm{i}(\mathbf{n})\mathbf{G} \wedge \mathbf{F}\big)$$
- Splitting yields
$$\tilde{\mathbf{S}} : C^\infty(P, \mathfrak{g}) \to \mathcal{F}^2_\times(P; \mathfrak{g}^*) :$$
$$\upsilon \mapsto \tfrac{1}{2}\big((\upsilon\tilde{\mathbf{E}}) \wedge \tilde{\mathbf{H}} - (\upsilon\tilde{\mathbf{H}}) \wedge \tilde{\mathbf{E}}\big)$$
- Poynting form $\mathbf{S} = \lambda^{-1}\tilde{\mathbf{S}}(\lambda^{-1})$

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Addendum (ii)
Relation to Object of Anholonomity

- Exterior derivative in terms of (co-)frame $(\boldsymbol{\varepsilon}^\alpha, \mathbf{e}_\alpha)$
$$\mathrm{d}\boldsymbol{\gamma} = \boldsymbol{\varepsilon}^\alpha \wedge \mathcal{L}_{\mathbf{e}_\alpha}\boldsymbol{\gamma} - \mathrm{d}\boldsymbol{\varepsilon}^\alpha \wedge \mathrm{i}(\mathbf{e}_\alpha)\boldsymbol{\gamma}$$
$$\mathrm{d}\boldsymbol{\varepsilon}^\alpha = \tfrac{1}{2} C_{\beta\gamma}{}^\alpha \boldsymbol{\varepsilon}^\beta \wedge \boldsymbol{\varepsilon}^\gamma$$
object of anholonomity
- Pick adapted coordinates
  x^0 proper time along world line from sync event
  x^a labels world lines
$$\Gamma_a = h(\partial x_a, \partial x_0)$$

preliminary

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Addendum (iii)
Relation to Object of Anholonomity (cont'd)

- Establish time-orthogonal (co-)frame

$$\mathbf{e}_0 = \partial x_0 \qquad \boldsymbol{\varepsilon}^0 = \mathrm{d}x^0 + \Gamma_a \mathrm{d}x^a$$

$$\mathbf{e}_a = \partial x_a - \Gamma_a \partial x_0 \qquad \boldsymbol{\varepsilon}^a = \mathrm{d}x^a$$

$$h(\mathbf{e}_0, \mathbf{e}_0) = 1, \quad h(\mathbf{e}_a, \mathbf{e}_0) = 0$$

$$\left.\begin{aligned} C_{\beta\gamma}{}^a &= 0 \\ cC_{0c}{}^0 &= \delta_c \\ C_{bc}{}^0 &= \eta_{bc} \end{aligned}\right\}$$

- Kinematic quantities by splitting object of anholonomity
- Anholonomic frame formalism yields identical form of Maxwell's equations, cf. H&O eq. (B.4.31)