Nonlinear optics: taking full dispersion and ionization into account

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Workshop "Modified dispersion for dispersive equations and systems"

Joint work with D. Lannes and J. Szeftel

Goal

Analyze laser-matter interaction,

and understand how singularities can appear, or can be avoided:

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and understand how singularities can appear, or can be avoided:

- self-focusing (associated focusing NLS),
- optical shocks (self-steepening of the pulse),
- ...

Modelization of ionization effects inspired by [BERGÉ-SKUPIN].

Maxwell's equations

- Maxwell's equations: magnetic field B, electric induction D,

$$\begin{cases} \partial_t B + \text{curl } E = 0, \\ \partial_t D - \dfrac{1}{\mu_0}\text{curl } B = 0 \end{cases}$$

- Coupling to matter *via* the polarization P,

$$D = \epsilon_0 E + P.$$

- Together with the constitutive laws

$$\nabla \cdot D = 0, \qquad \nabla \cdot B = 0.$$

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Maxwell's equations in (B, E):

$$\begin{cases} \partial_t B + \text{curl } E = 0, \\ \partial_t E - c^2\text{curl } B = \boxed{-\dfrac{1}{\epsilon_0}\partial_t P}. \end{cases}$$

The polarization

- The "anharmonic oscillator model" model [LORENTZ]

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- The "nonlinear anharmonic oscillator model" [OWYOUNG,BLOEMBERGEN,DONNAT-JOLY-METIVIER-RAUCH,...]

$$\partial_t^2 P + \omega_1 \partial_t P + \omega_0^2 P + \boxed{\nabla V_{NL}(P)} = \epsilon_0 bE.$$

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Example

(i) Cubic nonlinearity:

$$\nabla V_{NL}(P) = a_3 |P|^2 P.$$

(ii) Cubic-quintic nonlinearity:

$$\nabla V_{NL}(P) = a_3 |P|^2 P - a_5 |P|^4 P.$$

(iii) Saturated nonlinearity: there exists a function $v_{sat} : \mathbb{R}^+ \to \mathbb{R}$, with

$$v_{sat}(x) \sim_0 a_3 x \quad \text{and} \quad \nabla V_{NL}(P) = v_{sat}(|P|^2) P.$$

Remark

Global well-posedness for some perturbed (focusing) cubic NLS,

$$i\partial_t v + \Delta v + \big(1 + f(|v|^2)\big)|v|^2 v = 0, \quad t > 0, \, x \in \mathbb{R}^d.$$

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- *Saturated nonlinearity:* $(1 + f(s))s$ *bounded on* $\mathbb{R}^+$.

$$\|(1 + f(|v|^2))|v|^2 v\|_{L^2} \lesssim \|v\|_{L^2}$$

ensures global L^2*-wellposedness.*

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- *Focusing cubic-defocusing quintic NLS:* $f(s) = -s$*.*

$$\begin{aligned} E(v(t)) &= \frac{1}{2}\int |\nabla v(t,x)|^2 \mathrm{d}x - \frac{1}{4}\int |v(t,x)|^4 \mathrm{d}x + \frac{1}{6}\int |v(t,x)|^6 \mathrm{d}x \\ &\geq \frac{1}{2}\|v(t)\|_{H^1}^2 - \left(\frac{1}{2} + \frac{3}{32}\right)\|v(t)\|_{L^2}^2. \end{aligned}$$

  Globally well-posed in $H^1(\mathbb{R}^d)$ *when* $d = 2$*.*

The Maxwell-Lorentz equations, nondimensionalized

With $Q = \partial_t P$,

$$\begin{cases} \partial_t B \boxed{+\mathrm{curl}\ E} = 0, \\ \partial_t E \boxed{-\mathrm{curl}\ B} \boxed{+\frac{1}{\varepsilon}\sqrt{\gamma} Q} = 0, \\ \partial_t Q + \varepsilon^{1+p}\omega_1 Q \boxed{-\frac{1}{\varepsilon}\sqrt{\gamma} E + \frac{1}{\varepsilon}\omega_0 P} = \boxed{\varepsilon}\frac{\gamma}{\omega_0^3}\left(1 + f(\varepsilon^r |P|^2)\right)|P|^2 P, \\ \partial_t P \boxed{-\frac{1}{\varepsilon}\omega_0 Q} = 0, \end{cases}$$

where

$$\varepsilon = \frac{\text{period of the laser}}{\text{duration of the pulse}}.$$

Abstract formulation

Under the form of a first order semilinear hyperbolic system,

$$\partial_t \mathbf{U} + A(\partial)\mathbf{U} + \frac{1}{\varepsilon} E\mathbf{U} + \varepsilon B\mathbf{U} = \varepsilon F(\varepsilon, \mathbf{U}),$$

where

$$\mathbf{U} : \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^n, \qquad A(\partial) = \sum_{j=1}^{d} A_j \partial_j .$$

Assumption

(i) The matrices A_j are real valued, symmetric matrices.
(ii) The matrix E is a real valued, skew symmetric matrix.
(iii) There exists a quadratic form Q and trilinear mapping T such that

$$\forall U \in \mathbb{C}^n, \qquad F(\varepsilon, U) = \big(1 + f(\varepsilon^r Q(U))\big)\, T(U, \overline{U}, U).$$

Three scale WKB analysis

$$Res(\mathbf{U}) := \partial_t \mathbf{U} + A(\partial)\mathbf{U} + \frac{1}{\varepsilon} E\mathbf{U} - \varepsilon |\mathbf{U}|^2 \mathbf{U}.$$

At diffractive scale:

$$\mathbf{U}(t,x) \sim U(\varepsilon t, t, x) e^{i \frac{\mathbf{k} \cdot x - \omega t}{\varepsilon}} \qquad \text{(slowly modulated wave packets).}$$

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$$\begin{aligned} Res(\mathbf{U}) & \quad e^{-i\frac{\mathbf{k}\cdot x - \omega t}{\varepsilon}} \sim \\ \frac{1}{\varepsilon} & \quad (-i\omega \mathrm{Id} + iA(\mathbf{k}) + E)U \end{aligned}$$

(dispersion relation & polarization condition)

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$$+ \quad (\partial_t + \nabla\omega(\mathbf{k}) \cdot \nabla)U$$ (transport at the group velocity)

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\frac{1}{\varepsilon} \quad & (-i\omega \mathrm{Id} + iA(\mathbf{k}) + E)U \\
& \qquad \text{(dispersion relation \& polarization condition)} \\
+ \quad & (\partial_t + \nabla\omega(\mathbf{k})\cdot\nabla)U \quad \text{(transport at the group velocity)} \\
+ \quad & \varepsilon NLS(U) + O(\varepsilon^2).
\end{aligned}
$$

Notations: $\omega = \omega(\mathbf{k})$, $c_g = \nabla\omega(\mathbf{k})$.

With $U(\varepsilon t, t, x) = V(\varepsilon t, x - c_g t) \in Ker(-\omega \mathrm{Id} + A(\mathbf{k}) + E)$,

residual of order $O(\varepsilon^2)$ if V solves the NLS equation

$$i\partial_\tau V + \varepsilon \frac{1}{2}\omega''(\mathbf{k})(\partial, \partial) V = 3i|V|^2 V.$$

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Theorem (Donnat-Joly-Métivier-Rauch, Kalyakin, Schneider, Lannes. . .)

The NLS approximation can be justified “far from singularities”.

The Cauchy problem

Theorem

Let $s > d/2$, $\mathbf{U}^0 \in H^s(\mathbb{R}^d)^n$. *Then there exists* $T = T(|\mathbf{U}^0|_{H^s})$ *and a unique solution* $\mathbf{U} \in C([0, T/\varepsilon]; H^s)$ *of*

$$\partial_t \mathbf{U} + A(\partial)\mathbf{U} + \frac{1}{\varepsilon} E\mathbf{U} + \varepsilon B\mathbf{U} = \varepsilon F(\varepsilon, \mathbf{U}),$$

with initial condition $\mathbf{U}^0$.

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Initial conditions

Initial conditions corresponding to laser pulses:

$$\mathbf{U}_{|t=0} = u^0(x) e^{i\frac{\mathbf{k}\cdot x}{\varepsilon}} + \text{c.c.},$$

whith $\mathbf{k} \in \mathbb{R}^d$ the (spatial) wave-number of the oscillations.

These initial conditions are $O(\varepsilon^{-s})$ in H^s!

The profile equation

Original equation: $\partial_t \mathbf{U} + A(\partial)\mathbf{U} + \frac{1}{\varepsilon} E\mathbf{U} = \varepsilon F(\varepsilon, \mathbf{U}).$

- Add an extra variable θ

$$\mathbf{U}(t,x) = U\left(t, x, \frac{\mathbf{k}\cdot x - \omega t}{\varepsilon}\right),$$

with the profile $U(t,x,\theta)$ periodic with respect to θ (and any ω).

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  with the profile $U(t,x,\theta)$ periodic with respect to θ (and any ω).
- Now, solve the profile equation (with nonsingular initial data),

$$\begin{cases} \partial_t U + A(\partial) U + \frac{i}{\varepsilon}\mathcal{L}(\omega D_\theta, \mathbf{k} D_\theta) U = \varepsilon F(\varepsilon, U), \\ U_{|t=0}(x,\theta) = u^0(x) e^{i\theta} + \text{c.c.} \end{cases}$$

Notation:

$$\mathcal{L}(\omega D_\theta, \mathbf{k} D_\theta) = -\omega D_\theta + A(\mathbf{k}) D_\theta + \frac{E}{i}, \qquad D_\theta = -i\partial_\theta.$$

$$\partial_t \mathbf{U} + A(\partial)\mathbf{U} + \frac{1}{\varepsilon} E\mathbf{U} = \varepsilon F(\varepsilon, \mathbf{U}), \qquad \mathbf{U}_{|_{t=0}} = u^0(x) e^{i\frac{\mathbf{k}\cdot x}{\varepsilon}} + \text{c.c.}$$

Theorem

Let $s > d/2$. There exists $T = T(|u^0|_{H^s})$ and a unique solution $\mathbf{U} \in C([0, T/\varepsilon]; H^s)$. Moreover,

$$\mathbf{U}(t,x) = U\left(t, x, \frac{\mathbf{k}\cdot x - \omega t}{\varepsilon}\right),$$

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Functional setting: $U \in C([0, T/\varepsilon]; H^k(\mathbb{T}; H^s))$ $(k \geq 1)$,

$$H^k(\mathbb{T}; H^s) = \left\{ f = \sum_{n\in\mathbb{Z}} f_n e^{in\theta}, |f|_{H^k(\mathbb{T},H^s)} < \infty \right\},$$

$$|f|^2_{H^k(\mathbb{T},H^s)} = \sum_{n\in\mathbb{Z}} (1+n^2)^k |f_n|^2_{H^s}.$$

The slowly varying envelope approximation

Profile equation: $\partial_t U + A(\partial)U + \frac{i}{\varepsilon}\mathcal{L}(\omega D_\theta, \mathbf{k} D_\theta)U = \varepsilon F(\varepsilon, U).$

$$U(t, x, \theta) \sim u_{env}(t, x)e^{i\theta} + \text{c.c.}$$

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The envelope $u = u_{env}$ formally satisfies the envelope equation

$$\partial_t u + A(\partial)u + \frac{i}{\varepsilon}(-\omega \mathrm{Id} + A(\mathbf{k}) + \frac{E}{i})u = \varepsilon F^{env}(\varepsilon, u), \quad u_{|_{t=0}} = u^0,$$

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where

$$F^{env}(\varepsilon, u) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta} F(\varepsilon, ue^{i\theta} + \text{c.c.})\mathrm{d}\theta.$$

Example

With $F(u) = |u|^2 u$, one gets $F^{env}(u) = (u \cdot u)\overline{u} + 2|u|^2 u$.

Fast oscillations in the envelope must be avoided. But:

- The singular linear term creates fast oscillations with frequencies $\omega - \omega_j(\mathbf{k})$, where the $\omega_j(\mathbf{k})$ are the eigenvalues of

$$\mathcal{L}(0, \mathbf{k}) = A(\mathbf{k}) + \frac{1}{i}E = \sum_{j=1}^{m} \omega_j(\mathbf{k})\pi_j(\mathbf{k}).$$

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Solution:

- A good choice of ω:
  $\mathcal{L}(\omega, \mathbf{k}) = 0 \quad (\omega = \omega_1(\mathbf{k})), \quad |\omega - \omega_j(\mathbf{k}')|$ bounded from below.
- Polarization condition on u^0: $\quad \pi_1(\mathbf{k}) u^0 = u^0$.
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Approximation theorem:

[(T.)Colin-Gallice-Laurioux, (M.)Colin-Lannes].

Other approximations

- Full dispersion: $u_{FD} = \pi_1(\mathbf{k} + \varepsilon D) u_{env}$,

$$\partial_t u + \frac{i}{\varepsilon}(\omega_1(\mathbf{k} + \varepsilon D) - \omega) u = \varepsilon \pi_1(\mathbf{k} + \varepsilon D) F^{env}(\varepsilon, u).$$

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- NLS: Taylor expansions ($H_{\mathbf{k}}$ Hessian of ω_1 at $\mathbf{k}$)

$$\begin{aligned} \frac{i}{\varepsilon}(\omega_1(\mathbf{k} + \varepsilon D) - \omega) &= \nabla\omega_1(\mathbf{k}) \cdot \nabla - \varepsilon \frac{i}{2} \nabla \cdot H_{\mathbf{k}}(\omega_1) \nabla + O(\varepsilon^2), \\ \pi_1(\mathbf{k} + \varepsilon D) &= \pi_1(\mathbf{k}) + O(\varepsilon), \end{aligned}$$

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- Frequency improved NLS: better approximates the dispersion relation $\omega_{NLS}(\mathbf{k}')$ than $\omega_1(\mathbf{k}) + c_g \cdot (\mathbf{k}' - \mathbf{k}) + \frac{1}{2}(\mathbf{k}' - \mathbf{k}) \cdot H_{\mathbf{k}}(\mathbf{k}' - \mathbf{k})$.

- Frequency improved NLS:

$$\left(1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla\right) \partial_t u_{imp} + \left(c_g(\mathbf{k}) \cdot \nabla - i\varepsilon \nabla \cdot \left(\frac{1}{2} H_{\mathbf{k}} + \nabla_{\mathbf{k}} \omega \mathbf{b}^T \right) \nabla + \varepsilon^2 \mathbf{C}(\nabla) \right) u_{imp} = \varepsilon \pi_1(\mathbf{k}) F(\varepsilon, u_{imp}).$$

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- NLS with frequency dependent polarization: to capture slight changes of polarization,

$$\begin{aligned} \pi_1(\mathbf{k} + \varepsilon D) &= \sout{\pi_1(\mathbf{k}) + O(\varepsilon)} \\ &\simeq (1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla)^{-1} (\pi_1(\mathbf{k}) + \varepsilon Q(D)), \end{aligned}$$

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$$Q(D) = \pi_1'(\mathbf{k})\cdot D - i(\mathbf{b}\cdot\nabla)\pi_1(\mathbf{k}) :$$

approximation of $\pi_1(\mathbf{k}+\varepsilon D)$ up to order ε.

Then,

$$(1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla)\partial_t u - \frac{i}{2}\left(\frac{\omega_1'(k)}{k}\Delta_\perp + \omega_1''(k)\partial_z^2\right) u$$
$$= \varepsilon((1 - i\varepsilon \mathbf{b} \cdot \nabla)\pi_1(\mathbf{k}) + \varepsilon \pi_1'(\mathbf{k}) \cdot D)F^{env}(\varepsilon, u).$$

Then,

$$(1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla)\partial_t u - \frac{i}{2}\left(\frac{\omega_1'(k)}{k}\Delta_\perp + \omega_1''(k)\partial_z^2\right) u$$
$$= \varepsilon((1 - i\varepsilon \mathbf{b} \cdot \nabla)\pi_1(\mathbf{k}) + \varepsilon \pi_1'(\mathbf{k}) \cdot D)F^{env}(\varepsilon, u).$$

This equation looks quasilinear!

Possibility of optical shocks, induced by the "steepening operator" in front of the nonlinearity?

Ionization

Taking current density into account:

$$\begin{cases} \partial_t B + \text{curl } E = 0, \\ \partial_t E - c^2 \text{curl } B + \dfrac{1}{\epsilon_0} Q = -\dfrac{1}{\epsilon_0} J, \\ \partial_t Q - \epsilon_0 b E + \omega_0^2 P + \omega_1 Q = -\nabla V_{NL}(P), \\ \partial_t P - Q = 0. \end{cases}$$

Ionization

Taking current density into account:

$$\begin{cases} \partial_t B + \text{curl } E = 0, \\ \partial_t E - c^2 \text{curl } B + \frac{1}{\epsilon_0} Q = -\frac{1}{\epsilon_0} J, \\ \partial_t Q - \epsilon_0 b E + \omega_0^2 P + \omega_1 Q = -\nabla V_{NL}(P), \\ \partial_t P - Q = 0. \end{cases}$$

$$\boxed{J = J_{\mathrm{e}} + J_{\mathrm{i}},}$$

$$\begin{cases} \partial_t J_{\mathrm{e}} + \nu_{\mathrm{e}} J_{\mathrm{e}} = \frac{q_{\mathrm{e}}^2}{m_{\mathrm{e}}} \rho_{\mathrm{e}} E, \\ \partial_t \rho_{\mathrm{e}} = \sigma_K \rho_{\mathrm{nt}} |E|^{2K} + \frac{\sigma}{U_{\mathrm{i}}} \rho_{\mathrm{e}} |E|^2, \end{cases}$$

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Usual approximation, when $E \sim E_{01} e^{i(\mathbf{k}\cdot x - \omega t)} + \text{c.c.}$:

$$J_{\mathrm{e}} \sim J_{01} e^{i(\mathbf{k}\cdot x - \omega t)} + \text{c.c.} \quad \text{with} \quad J_{01} = i \frac{q_{\mathrm{e}}^2}{\omega m_{\mathrm{e}}} \rho_{\mathrm{e}} E_{01}.$$

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Approximated relation ($k = |\mathbf{k}|$):

$$J_{\mathrm{e}} = \frac{q_{\mathrm{e}}^2}{\omega m_{\mathrm{e}}} \rho_{\mathrm{e}} \mathcal{H}\left(\frac{1}{k} D_z\right) E, \quad \mathcal{H}(D_z) = \frac{\sqrt{2}\; i D_z}{(1 + D_z^2)^{1/2}}.$$

- After nondimensionalization,

$$
\begin{cases}
\partial_t B + \text{curl } E = 0, \\
\partial_t E - \text{curl } B + \dfrac{1}{\varepsilon}\sqrt{\gamma} Q = -\varepsilon \rho \mathcal{H}\left(\varepsilon \mathbf{k} \cdot D_x\right) E \\
\qquad\qquad\qquad\qquad\qquad -\varepsilon c_0 \left(c_1 |E|^{2K-2} + c_2 \rho\right) E, \\
\partial_t Q + \varepsilon^{1+p} \omega_1 Q - \dfrac{1}{\varepsilon}(\sqrt{\gamma} E - \omega_0 P) = \varepsilon \dfrac{\gamma}{\omega_0^3}\left(1 + f(\varepsilon^r |P|^2)\right)|P|^2 P, \\
\partial_t P - \dfrac{1}{\varepsilon}\omega_0 Q = 0, \\
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- NLS:

$$\begin{cases} i(\partial_t + c_g \cdot \nabla)E + \varepsilon(\Delta_\perp + a_1 \partial_z^2)E + \varepsilon a_2 |E|^2 E = \\ \qquad\qquad a_3 \rho E - ic(a_4 |E|^{2K-2} E + a_5 \rho E), \\ \partial_t \rho = \varepsilon a_4 |E|^{2K} + \varepsilon a_5 \rho |E|^2. \end{cases}$$

Does blow-up occur?

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Energy estimates, $N > 3/2$:

$$\begin{aligned} \frac{d}{dt}[\|E\|_{H^N}^2 + \|\rho\|_{H^N}^2] &\lesssim \varepsilon\,(\|E\|_{H^N}^2 + \|\rho\|_{H^N}^2)(\|E\|_{L^\infty} + \|E\|_{L^\infty}^{2K-2} + \|\rho\|_{L^\infty}) \\ &\lesssim \varepsilon\,(\|E\|_{H^N}^2 + \|\rho\|_{H^N}^2)(1 + \|E\|_{H^N}^2 + \|\rho\|_{H^N}^2)^{K-1} \end{aligned}$$

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Global existence?