Characterisation of weak solutions to gradient flows of general linear growth functionals

Wojciech Górny

University of Vienna, University of Warsaw

(joint work with J.M. Mazón)

Degenerate and Singular PDEs
ESI (Vienna), 24-28 February 2025

Problem 1: time-dependent minimal surface equation

Consider the model problem

$$\begin{cases} u_t(t,x) = \mathrm{div}\left(\frac{Du(t,x)}{\sqrt{1+|Du(t,x)|^2}}\right) & \text{in } (0,T)\times\Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

This corresponds to the gradient flow of the functional

$$F(u) = \int_\Omega \sqrt{1+|Du|^2},$$

which has linear growth; how to define a notion of solutions?

Problem 1: time-dependent minimal surface equation

Due to the linear growth of the Lagrangian, the natural energy space for the right-hand side (for fixed time) is $BV(\Omega)$, i.e.

$$BV(\Omega) = \Big\{ u \in L^1(\Omega) : \ Du \text{ is a Radon measure} \Big\},$$

where Du denotes the distributional gradient of u. It is only a measure, so we need to give meaning to the expression on the right-hand side of

$$u_t(t,x) = \mathrm{div} \left(\frac{Du(t,x)}{\sqrt{1 + |Du(t,x)|^2}} \right)$$

and the expression $\int_\Omega \sqrt{1 + |Du|^2}$ in the functional F.

Problem 1: time-dependent minimal surface equation

For $u \in BV(\Omega)$, write

$$Du = \nabla u \, \mathcal{L}^N + D^s u,$$

i.e. ∇u is the absolutely continuous part of Du and $D^s u$ is its singular part. In place of the functional F, we consider its relaxation $\mathcal{F}$, i.e.

$$\mathcal{F}(u) = \inf \left\{ \liminf_{n \to \infty} F(u_n) : \quad u_n \in W^{1,1}(\Omega), \quad u_n \to u \text{ in } L^1(\Omega) \right\}.$$

A direct computation shows that

$$\mathcal{F}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} \, dx + \int_\Omega |D^s u|.$$

Problem 1: time-dependent minimal surface equation

We now denote

$$a(t,x) = \frac{\nabla u(t,x)}{\sqrt{1+|\nabla u(t,x)|^2}},$$

which defines $\mathcal{L}^N$-a.e. in Ω the vector field a. Therefore, $\|a(\cdot,t)\|_\infty \leq 1$ and $\mathrm{div}(a(\cdot,t)) \in L^2(\Omega)$. In other words, if we denote

$$X_2(\Omega) = \left\{ \mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N) : \mathrm{div}(\mathbf{z}) \in L^2(\Omega) \right\},$$

we necessarily have $a(\cdot,t) \in X_2(\Omega)$.

Problem 1: time-dependent minimal surface equation

Therefore, we understand the equation

$$u_t(t,x) = \mathrm{div}\left(\frac{Du(t,x)}{\sqrt{1+|Du(t,x)|^2}}\right)$$

in the following sense: for $a(t,x) = \frac{\nabla u(t,x)}{\sqrt{1+|\nabla u(t,x)|^2}}$, we have

$$u_t(t) = \mathrm{div}(a(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$a(t) \cdot D^s u(t) = |D^s u(t)| \quad \text{as measures.}$$

The last equation is understood in a suitable weak sense due to Anzellotti. We also need to add the boundary and initial conditions and apply the classical theory of semigroup solutions to get existence of solutions.

Problem 1: time-dependent minimal surface equation

Then, given any $u_0 \in L^2(\Omega)$, there exists a unique weak solution u to the time-dependent MSE in the following sense:

$$u \in C([0,T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega));$$

$$u(0,\cdot) = u_0;$$

and for almost all $t \in (0,T)$ we have $u(t) \in BV(\Omega)$ and there exist vector fields $a(t) \in X_2(\Omega)$ such that the following conditions hold:

$$a(t) = \frac{\nabla u(t,x)}{\sqrt{1 + |\nabla u(t,x)|^2}} \quad \mathcal{L}^N - \text{a.e. in } \Omega;$$

$$u_t(t) = \mathrm{div}(a(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$a(t) \cdot D^s u(t) = |D^s u(t)| \quad \text{as measures};$$

$$[a(t), \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Problem 1: time-dependent minimal surface equation

This definition is based on the ideas of Demengel and Temam (1984), and it was proved by Andreu, Ballester, Caselles and Mazón (1999-2004) that it can be extended to other functionals of linear growth in the following way. For $u_0 \in L^2(\Omega)$, we consider the equation

$$\begin{cases} u_t(t,x) = \mathrm{div}\, a(x, Du(t,x)) & \text{in } \ (0,T) \times \Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \ (0,T) \times \partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \ \Omega, \end{cases}$$

where a is the gradient of a differentiable function with linear growth, i.e. $f \in C^1(\overline{\Omega} \times \mathbb{R}^N)$ and $a(x,\xi) = \partial_\xi f(x,\xi)$. Then, one can give a similar definition of solutions and prove their existence and uniqueness.

Problem 2: total variation flow

Now, consider a different problem

$$\begin{cases} u_t(t,x) = \mathrm{div}\left(\frac{Du(t,x)}{|Du(t,x)|}\right) & \text{in } (0,T)\times\Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

appearing in relation to image processing. It corresponds to the gradient flow of the functional

$$\mathcal{F}(u) = \int_\Omega |Du|.$$

Observe that the corresponding integrand $f(x,\xi) = |\xi|$ is not differentiable, so we cannot apply the previous definition. How to define solutions?

Problem 2: total variation flow

Now, consider a different problem

$$\begin{cases} u_t(t,x) = \mathrm{div}\left(\frac{Du(t,x)}{|Du(t,x)|}\right) & \text{in } (0,T)\times\Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

appearing in relation to image processing. It corresponds to the gradient flow of the functional

$$\mathcal{F}(u) = \int_\Omega |Du|.$$

Observe that the corresponding integrand $f(x,\xi) = |\xi|$ is not differentiable, so we cannot apply the previous definition. How to define solutions?

Also: the corresponding 1-Laplacian operator is not μ-elliptic for any μ!

Problem 2: total variation flow

Again, the natural energy space for the right-hand side is $BV(\Omega)$, and we need to give meaning to the expression on the right hand side of

$$u_t(t,x) = \operatorname{div}\left(\frac{Du(t,x)}{|Du(t,x)|}\right).$$

We will replace the expression $\frac{Du}{|Du|}$ by a non-uniquely defined vector field. To be exact, we require that there exists $\mathbf{z} \in X_2(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$ such that

$$u_t(t) = \operatorname{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$(\mathbf{z}(t), Du(t)) = |Du(t)| \quad \text{as measures.}$$

The last equation is again understood in a suitable weak sense due to Anzellotti. We also add the initial and boundary condition.

Problem 2: total variation flow

Then, given any $u_0 \in L^2(\Omega)$, there exists a unique weak solution u to the Neumann problem for the total variation flow in $[0, T]$, i.e.

$$u \in C([0, T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0, T; L^2(\Omega));$$

$$u(0, \cdot) = u_0;$$

and for almost all $t \in (0, T)$ we have $u(t) \in BV(\Omega)$ and there exist vector fields $\mathbf{z}(t) \in X_2(\Omega)$ with $\|\mathbf{z}(t)\|_\infty \leq 1$ such that

$$u_t(t) = \mathrm{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$(\mathbf{z}(t), Du(t)) = |Du(t)| \quad \text{as measures;}$$

$$[\mathbf{z}(t), \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Comparison between the two approaches

We can write both problems as gradient flows of

$$\int_\Omega f(x, Du)$$

with f of linear growth (with given initial and boundary conditions).

Time-dependent MSE	Total variation flow
$f(\xi) = \sqrt{1 + \lvert\xi\rvert^2}$	$f(\xi) = \lvert\xi\rvert$
f is differentiable	f is not differentiable at 0
f is "1-homogeneous at infinity"	f is 1-homogeneous
$a = \nabla f$	$\mathbf{z}$ is not explicit (and nonunique)
separate conditions on the absolutely continuous and singular parts	a joint condition

Can we make a joint framework to study both problems?

Working assumptions

For simplicity, we consider the Neumann case.

$$\begin{cases} u_t(t,x) = \mathrm{div}(\partial_\xi f(x, Du(t,x))) & \text{in } (0,T)\times\Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

We assume the following two (quite general) conditions on the integrand.

(A1) $f \in C(\overline{\Omega} \times \mathbb{R}^N)$ is convex in the second variable and has linear growth, i.e. there exists $M > 0$ such that

$$|f(x,\xi)| \leq M(1+|\xi|) \qquad \text{for all } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N;$$

(A2) The following limit exists:

$$f^0(x,\xi) = \lim_{t\to 0^+} tf(x, \xi/t)$$

and it defines a *recession function* which is jointly continuous in (x,ξ).

Memo: function of a measure

For a function f with linear growth, one may define its action on a Radon measure, which is itself a Radon measure. In the particular case $\mu = Du$, where $u \in BV(\Omega)$, we define the measure $f(x, Du)$ by

$$\int_B f(x, Du) := \int_B f(x, \nabla u(x))\, dx + \int_B f^0\left(x, \frac{dD^s u}{d|D^s u|}\right) d|D^s u|$$

for all Borel sets $B \subset \Omega$ (and similarly for $\mu = D^s u$).

Under the assumptions (A1)-(A2), the functional

$$\mathcal{F}(u) = \int_\Omega f(x, Du)$$

is lower semicontinuous with respect to convergence in $L^2(\Omega)$.

Memo: Anzellotti pairings

Definition

For $\mathbf{z} \in X_2(\Omega)$ and $u \in BV(\Omega) \cap L^2(\Omega)$, define the functional $(\mathbf{z}, Du) : C_c^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (\mathbf{z}, Du), \varphi \rangle := -\int_\Omega u\, \varphi\, \mathrm{div}(\mathbf{z})\, dx - \int_\Omega u\, \mathbf{z} \cdot \nabla \varphi\, dx.$$

The distribution $(\mathbf{z}, Du)$ is a Radon measure, $(\mathbf{z}, Du) \ll |Du|$ and

$$|(\mathbf{z}, Du)| \leq \|\mathbf{z}\|_\infty |Du|.$$

Memo: Anzellotti pairings

One can verify that

$$\int_\Omega (\mathbf{z}, Du) = \int_\Omega \mathbf{z} \cdot \nabla u \, dx \qquad \text{for all } u \in W^{1,1}(\Omega),$$

so $(\mathbf{z}, Du)$ agrees on Sobolev functions with the dot product of $\mathbf{z}$ and ∇u.

Moreover, if we set

$$\mathbf{z} \cdot D^s u := (\mathbf{z}, Du) - (\mathbf{z} \cdot \nabla u) \, d\mathcal{L}^N,$$

we have that $\mathbf{z} \cdot D^s u$ is a bounded measure, $\mathbf{z} \cdot D^s u \ll |D^s u|$ and

$$|\mathbf{z} \cdot D^s u| \leq \|\mathbf{z}\|_\infty |D^s u|.$$

Our goal: characterisation of solutions

Theorem (G.-Mazón, Publ. Mat., to appear)

Given $u_0 \in L^2(\Omega)$, there exists a *weak solution* u of the Neumann problem in $[0, T]$, i.e. $u \in C([0, T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0, T; L^2(\Omega))$, $u(0, \cdot) = u_0$, and for a.e. $t \in (0, T)$ we have $u(t) \in BV(\Omega)$ and there exist vector fields $\mathbf{z}(t) \in X_2(\Omega)$ such that:

$$u_t(t) = \mathrm{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$\mathbf{z}(t) \in \partial_\xi f(x, \nabla u(t)) \quad \mathcal{L}^N - \text{a.e. in } \Omega;$$

$$\mathbf{z}(t) \cdot D^s u(t) = f^0(x, D^s u(t)) \quad \text{as measures};$$

$$[\mathbf{z}(t), \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Outline of proof

We first introduce a multivalued operator $\mathcal{A} \subset L^2(\Omega) \times L^2(\Omega)$ which describes the desired characterisation (details on the next slide). We will prove that it coincides with the subdifferential of $\mathcal{F}$, where

$$\mathcal{F}(u) = \int_\Omega f(x, Du).$$

To this end, we check that we have the inclusion

$$\mathcal{A} \subset \partial \mathcal{F},$$

and in particular $\mathcal{A}$ is monotone. We then show that the range condition holds, i.e.,

$$\text{For every } g \in L^2(\Omega),\ \exists\, u \in D(\mathcal{A})\ s.t.\ \ g \in u + \mathcal{A}(u),$$

so by the Minty theorem $\mathcal{A}$ is maximal monotone. Hence, $\mathcal{A} = \partial \mathcal{F}$. Applying the Brezis-Komura semigroup theory, we get the desired result.

Auxiliary operator

We first introduce the following operator.

Definition

We say that $(u, v) \in \mathcal{A}$ if and only if $u, v \in L^2(\Omega)$, $u \in BV(\Omega)$ and there exists a vector field $\mathbf{z} \in X_2(\Omega)$ such that:

$$-\mathrm{div}(\mathbf{z}) = v \quad \text{in } \mathcal{D}'(\Omega);$$

$$\mathbf{z} \in \partial_\xi f(x, \nabla u) \quad \mathcal{L}^N - \text{a.e. in } \Omega;$$

$$\mathbf{z} \cdot D^s u = f^0(x, D^s u) \quad \text{as measures};$$

$$[\mathbf{z}, \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

We show that it coincides with $\partial\mathcal{F}$. The proof of the inclusion $\mathcal{A} \subset \partial\mathcal{F}$ is simple and follows using Green's formula; we focus on the range condition.

Range condition

The range condition states that

$$\text{For every } \; g \in L^2(\Omega), \; \exists\, u \in D(\mathcal{A}) \; s.t. \; \; g \in u + \mathcal{A}(u),$$

or equivalently there exists a bounded vector field $\mathbf{z} \in X_2(\Omega)$ such that

$$-\mathrm{div}(\mathbf{z}) = g - u \quad \text{in } \Omega;$$

$$\mathbf{z} \in \partial_\xi f(x, \nabla u) \quad \mathcal{L}^N - \text{a.e. in } \Omega;$$

$$\mathbf{z} \cdot D^s u = f^0(x, D^s u) \quad \text{as measures};$$

$$[\mathbf{z}, \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

We find such u and $\mathbf{z}$ using the Fenchel-Rockafellar duality theorem for a suitably defined functional.

Memo: Fenchel-Rockafellar theorem

Let us recall the notation in the Fenchel-Rockafellar duality theorem.

- U, V are two Banach spaces;
- $A : U \to V$ is a continuous linear operator;
- $E : V \to \mathbb{R}$ and $G : U \to \mathbb{R}$ be two convex functionals;
- The primal minimisation problem is

$$\inf_{u \in U} \Big\{ E(Au) + G(u) \Big\} \tag{P}$$

and its dual is

$$\sup_{p^* \in V^*} \Big\{ -E^*(-p^*) - G^*(A^* p^*) \Big\}. \tag{P*}$$

Memo: Fenchel-Rockafellar theorem

The Fenchel-Rockafellar duality theorem states that if for some $u_0 \in U$ we have $E(Au_0), G(u_0) < \infty$ and E is continuous at Au_0:

- There is no duality gap, i.e., $\inf(\mathrm{P}) = \sup(\mathrm{P}^*)$;
- The dual problem admits at least one solution;
- For any minimising sequence u_n for (P) and a maximiser p^* of (P*), we have

$$0 \leq E(Au_n) + E^*(-p^*) - \langle Au_n, -p^* \rangle_{V,V^*} \leq \varepsilon_n$$

$$0 \leq G(u_n) + G^*(A^* p^*) - \langle u_n, A^* p^* \rangle_{U,U^*} \leq \varepsilon_n$$

with $\varepsilon_n \to 0$.

Sketch of proof

We set $U = W^{1,1}(\Omega) \cap L^2(\Omega)$, $V = L^1(\partial\Omega, \mathcal{H}^{N-1}) \times L^1(\Omega; \mathbb{R}^N)$, and

$$Au = (u|_{\partial\Omega}, \nabla u).$$

Clearly, $A : U \to V$ is linear and continuous. We denote $p = (p_0, \overline{p}) \in V$ and define $E : L^1(\partial\Omega, \mathcal{H}^{N-1}) \times L^1(\Omega; \mathbb{R}^N) \to \mathbb{R}$ as

$$E(p_0, \overline{p}) = E_0(p_0) + E_1(\overline{p}), \quad E_0(p_0) = 0, \quad E_1(\overline{p}) = \int_\Omega f(x, \overline{p})\, dx.$$

We also define $G : W^{1,1}(\Omega) \cap L^2(\Omega) \to \mathbb{R}$ as

$$G(u) := \frac{1}{2} \int_\Omega u^2\, dx - \int_\Omega ug\, dx.$$

Sketch of proof

We compute the objects needed to apply the Fenchel-Rockafellar theorem. First, we show that

$$p^* \text{ admissible in } (\mathrm{P}^*) \quad \Rightarrow \quad G^*(A^* p^*) < \infty \quad \Rightarrow \quad A^* p^* \in L^2(\Omega).$$

Since the adjoint of the gradient is minus divergence (in a distributional sense), we get

$$A^* p^* = -\mathrm{div}(\overline{p}^*).$$

Consequently, $\mathrm{div}(\overline{p}^*) \in L^2(\Omega)$, or in other words $\overline{p}^* \in X_2(\Omega)$.

Sketch of proof

Since E has separated variables, we have

$$E^*(p_0, \overline{p}^*) = E_0^*(p_0^*) + E_1^*(\overline{p}^*).$$

The functional $E_0^* : L^\infty(\partial\Omega, \mathcal{H}^{N-1}) \to [0, +\infty]$ is given by

$$E_0^*(p_0^*) = \begin{cases} 0 & \text{if } \; p_0^* = 0; \\ +\infty & \text{if } \; p_0^* \neq 0 \end{cases}$$

and $E_1^* : L^\infty(\Omega; \mathbb{R}^N) \to (-\infty, +\infty]$ is given by

$$E_1^*(\overline{p}^*) = \int_\Omega f^*(x, \overline{p}^*)\, dx.$$

Sketch of proof

Consider the energy functional $\mathcal{G} : L^2(\Omega) \to (-\infty, +\infty]$ defined by

$$\mathcal{G}(u) := \begin{cases} \mathcal{F}(u) + G(u) & \text{if } u \in BV(\Omega) \cap L^2(\Omega); \\ \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases}$$

This is an extension of the functional $E \circ A + G$, which is well-defined for functions in $W^{1,1}(\Omega) \cap L^2(\Omega)$, to the whole $L^2(\Omega)$. Since $\mathcal{G}$ is coercive, convex and lower semicontinuous, the primal minimisation problem

$$\min_{u \in L^2(\Omega)} \mathcal{G}(u) = \inf_{u \in W^{1,1}(\Omega) \cap L^2(\Omega)} \Big\{ E(Au) + G(u) \Big\} \tag{P}$$

admits a solution u.

Sketch of proof

For $u_0 \equiv 0$ we have $E(Au_0) = G(u_0) = 0 < \infty$ and E is continuous at Au_0, so by the Fenchel-Rockafellar duality theorem there is no duality gap and the dual problem

$$\sup_{p^* \in L^\infty(\partial\Omega, \mathcal{H}^{N-1}) \times L^\infty(\Omega;\mathbb{R}^N)} \left\{ -E_0^*(-p_0^*) - E_1^*(-\overline{p}^*) - G^*(A^* p^*) \right\} \quad \text{(P*)}$$

admits at least one solution. Moreover, for any minimising sequence u_n for (P) and a maximiser p^* of (P*), we have

$$0 \leq E(Au_n) + E^*(-p^*) - \langle Au_n, -p^* \rangle_{V,V^*} \leq \varepsilon_n$$

$$0 \leq G(u_n) + G^*(A^* p^*) - \langle u_n, A^* p^* \rangle_{U,U^*} \leq \varepsilon_n$$

with $\varepsilon_n \to 0$.

Back to the range condition

Since E_0^* takes only values 0 and $+\infty$, for the maximiser we have $E_0^*(-p_0^*) = 0$, from which we infer that

$$p_0^* = [-\overline{p}^*, \nu_\Omega] = 0.$$

Moreover, the condition

$$0 \leq G(u_n) + G^*(A^* p^*) - \langle u_n, A^* p^* \rangle_{U,U^*} \leq \varepsilon_n$$

on the minimising sequences implies that

$$-\mathrm{div}(\overline{p}^*) = A^* p^* \in \partial G(u) = \{u - g\}.$$

Back to the range condition

Finally, the condition

$$0 \leq E(Au_n) + E^*(-p^*) - \langle Au_n, -p^* \rangle_{V,V^*} \leq \varepsilon_n$$

coupled with the Reshetnyak continuity theorem yields that

$$\int_\Omega f(x, \nabla u)\, dx + \int_\Omega f^*(x, \overline{p}^*)\, dx = \int_\Omega -\overline{p}^* \cdot \nabla u\, dx,$$

so $-\overline{p}^* \in \partial_\xi f(x, \nabla u)$, and

$$\int_\Omega f^0\left(\cdot, \frac{dD^s u}{d|D^s u|}\right) d|D^s u| = \int_\Omega (-\overline{p}^*, Du)^s,$$

so $-\overline{p}^* \cdot D^s u = f^0(x, D^s u)$. Therefore, the range condition is satisfied for the pair $(u, -\overline{p}^*)$, where u is a minimiser of $\mathcal{G}$ and p^* is a solution of the dual problem.

Definition of solutions

Definition

Given $u_0 \in L^2(\Omega)$, we say that u is a *weak solution* of the Neumann problem in $[0,T]$, i.e. $u \in C([0,T];L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T;L^2(\Omega))$, $u(0,\cdot) = u_0$, and for a.e. $t \in (0,T)$ we have

$$0 \in u_t(t,\cdot) + \mathcal{A}u(t,\cdot).$$

In other words, we have $u(t) \in BV(\Omega)$ and there exist vector fields $\mathbf{z}(t) \in X_2(\Omega)$ such that:

$$u_t(t) = \mathrm{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$\mathbf{z}(t) \in \partial_\xi f(x, \nabla u(t)) \quad \mathcal{L}^N - \text{a.e. in } \Omega;$$

$$\mathbf{z}(t) \cdot D^s u(t) = f^0(x, D^s u(t)) \quad \text{as measures;}$$

$$[\mathbf{z}(t), \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Existence and uniqueness

Since $\mathcal{A} = \partial \mathcal{F}$, we can apply the classical theory of gradient flows of maximal monotone operators and get the following result.

Theorem (G.-Mazón, Publ. Mat., to appear)

For any $u_0 \in L^2(\Omega)$ and all $T > 0$ there exists a unique weak solution of the Neumann problem

$$\begin{cases} u_t(t,x) = \mathrm{div}(\partial_\xi f(x, Du(t,x))) & \text{in } (0,T) \times \Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } (0,T) \times \partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

A similar result holds for the Dirichlet and Cauchy problems.

Highlight of used techniques

1. The Green formula:

$$\int_\Omega u \, \mathrm{div}(\mathbf{z}) \, dx + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u \, [\mathbf{z}, \nu_\Omega] \, d\mathcal{H}^{N-1}.$$

2. Pointwise estimates for the normal trace: the formula

$$[\mathbf{z}, \nu_\Omega](x) = \lim_{\rho \to 0^+} \lim_{r \to 0^+} \frac{1}{2r\omega_{N-1}\rho^{N-1}} \int_{C_{r,\rho}(x, \nu_\Omega(x))} \mathbf{z}(y) \cdot \nu_\Omega(x) \, dy$$

holds for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, where

$$C_{r,\rho}(x, \alpha) := \{\xi \in \mathbb{R}^N \, : \, |(\xi - x) \cdot \alpha| < r, \; |(\xi - x) - [(\xi - x) \cdot \alpha]\alpha| < \rho\}.$$

A similar formula holds for the Radon-Nikodym derivative $\frac{d(\mathbf{z}, Du)}{d|Du|}$.

3. Reshetnyak continuity theorem.
4. Fenchel-Rockafellar duality theorem.

Extension: inhomogeneous growth

(Joint work in progress with M. Łasica and A. Matsoukas.)

Let Ω be a bounded Lipschitz domain. Take $\varphi : \Omega \times [0,\infty) \to [0,+\infty]$ such that for every $x \in \Omega$:

(i) $x \mapsto \varphi(x, |f|(x))$ is measurable for every measurable $f : \Omega \to \mathbb{R}$;

(ii) $t \mapsto \varphi(x,t)$ is non-decreasing, convex, and left-continuous;

(iii) $t \mapsto \varphi(x,t)$ has the following limits at 0 and $+\infty$:

$$\varphi(x,0) = \lim_{t\to 0^+} \varphi(x,t) = 0 \quad \text{and} \quad \lim_{t\to+\infty} \varphi(x,t) = +\infty.$$

(iv) $t \mapsto \frac{\varphi(x,t)}{t}$ is L-almost increasing on $(0,+\infty)$, i.e., there exists $L \geq 1$ independent of x such that

$$\frac{\varphi(x,s)}{s} \leq L \frac{\varphi(x,t)}{t} \quad \text{for all} \quad 0 < s \leq t$$

In short, φ is a *convex* Φ*-function*, and we denote $\varphi \in \Phi_c(\Omega)$.

Extension: inhomogeneous growth

For $\varphi \in \Phi_c(\Omega) \cap C(\Omega \times [0,\infty))$, one can define the variable-growth total variation $TV_\varphi \colon L^1(\Omega) \to [0,\infty]$ by

$$TV_\varphi(v) := \sup \left\{ \int v \operatorname{div} \xi - \varphi^*(x, |\xi|)\, dx \,\middle|\, \xi \in C_c^1(\Omega)^n \right\},$$

where $\varphi^*(x,\cdot)$ is the Legendre–Fenchel dual to $\varphi(x,\cdot)$ for $x \in \Omega$.

Our main goal is the characterisation of the subdifferential of $\mathcal{E}_\varphi$, which is a restriction of TV_φ to $L^2(\Omega)$; this is obtained by a (slightly different) dualisation argument.

Extension: inhomogeneous growth

Theorem (G.-Łasica-Matsoukas, coming soon)

Under reasonable conditions for φ, for $v \in L^2(\Omega)$ the following conditions are equivalent:

(a) $w \in \partial \mathcal{E}_\varphi(v)$;

(b) $v \in BV(\Omega)$ *and there exists* $\xi \in L^1(\Omega)^N$ *with* $\int_\Omega \varphi^*(x, |\xi|)\, dx < \infty$, $\operatorname{div} \xi = w$ *and* $\xi \cdot \nu = 0$ *such that*

$$\int_\Omega vw\, dx = \int_\Omega \varphi(x, |Dv|) + \int_\Omega \varphi^*(x, |\xi|)\, dx. \tag{1}$$

We also provide a 'pointwise' condition equivalent to (b), as well as a discussion of the meaning of this result in the specific cases of variable exponent and double phase, with applications to image processing.