Optimal transport techniques in geometric problems

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Motivation 1: The problem of antiplane shear

Take $\Omega \subset \mathbb{R}^2$. Consider a cylinder $\Omega \times \mathbb{R} \subset \mathbb{R}^3$ made of perfectly plastic material. It is subject to an external surface traction g in the z-direction and independent on z. The functions

$$\sigma_{xx}, \quad \sigma_{yy}, \quad \sigma_{zz}, \quad \sigma_{xy}, \quad \sigma_{xz}, \quad \sigma_{yz}$$

model stress inside the material. For such g, it is enough to consider

$$\sigma_1 := \sigma_{xz}(x,y) \quad \text{and} \quad \sigma_2 := \sigma_{yz}(x,y)$$

and letting $\sigma = (\sigma_1, \sigma_2) \in L^\infty(\Omega; \mathbb{R}^2)$, the equilibrium equation becomes

$$\mathrm{div}(\sigma) = \frac{\partial \sigma_1}{\partial x} + \frac{\partial \sigma_2}{\partial y} = 0 \quad \text{in } \Omega; \qquad \sigma \cdot \nu^\Omega = g \quad \text{on } \partial\Omega$$

with the plasticity constraint

$$|\sigma| = (\sigma_1^2 + \sigma_2^2)^{1/2} \leqslant 1 \quad \text{in } \Omega.$$

R.V. Kohn, G. Strang, *Optimal design of cylinders in shear* (1982).

Motivation 1: The problem of antiplane shear

We aim to minimise the amount of material used to withstand this external force. No material needs to be used where $|\sigma| = 0$; thus, a lower bound is given by

$$\int_\Omega |\sigma| \, dx \, dy.$$

(In practice, usually $|\sigma| = 1$ on the solid part of the material, and $0 < |\sigma| < 1$ on one-dimensional fibers).

We thus aim to solve the minimisation problem

$$\min \left\{ \int_\Omega |\sigma| \, dx \, dy : \quad \mathrm{div}(\sigma) = 0, \quad |\sigma| \leqslant 1, \quad \sigma \cdot \nu^\Omega = g \right\}.$$

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We need to reformulate this problem. Whenever $\mathrm{div}(\sigma) = 0$, there exists a scalar potential $u : \Omega \to \mathbb{R}$ such that

$$\sigma = R_{-\frac{\pi}{2}} \nabla u = (u_y, -u_x).$$

Then, we automatically have

$$|\nabla u| \leqslant 1 \quad \text{in } \Omega$$

and

$$g = \sigma \cdot \nu^{\Omega} = (u_y, -u_x) \cdot (\nu_1, \nu_2) = (u_x, u_y) \cdot (-\nu_2, \nu_1) = \nabla u \cdot \tau = \frac{\partial u}{\partial \tau},$$

so integrating this over $\partial\Omega$ transforms the boundary condition for σ to a Dirichlet boundary condition for u:

$$u|_{\partial\Omega}(x) = f(x) = \int_{x_0}^{x} g(s)\, ds.$$

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Therefore, the minimisation problem

$$\min \left\{ \int_\Omega |\sigma| \, dx \, dy : \quad \mathrm{div}(\sigma) = 0, \quad |\sigma| \leqslant 1, \quad \sigma \cdot \nu^\Omega = g \right\}$$

is transformed into

$$\min \left\{ \int_\Omega |\nabla u| \, dx \, dy : \quad u \in C^{0,1}(\overline{\Omega}), \quad |\nabla u| \leqslant 1, \quad u|_{\partial\Omega} = f \right\}$$

with $g = \partial_\tau f$. This is the *constrained least gradient problem*.

P. Sternberg, G. Williams, W.P. Ziemer, Trans. AMS **339** (1993).

Least gradient problem

This led the same authors to consider the following unconstrained minimisation problem

$$\min \left\{ \int_\Omega |\nabla u| : \quad u \in W^{1,1}(\Omega), \quad u|_{\partial\Omega} = f \right\}$$

called the *least gradient problem*. Here, $f \in L^1(\partial\Omega)$.

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$$u \in BV(\Omega) \Leftrightarrow u \in L^1(\Omega) \text{ and } Du \text{ is a finite Radon measure.}$$

The main trouble is that the subspace

$$\left\{ u \in BV(\Omega) : \quad u|_{\partial\Omega} = f \right\}$$

is not weakly* closed and it is difficult to enforce the boundary condition.

P. Sternberg, G. Williams, W.P. Ziemer, J. Reine Angew. Math. **430** (1992).

Comment: linear growth functionals

More generally, the same phenomenon appears in minimisation problems for functionals of linear growth

$$\min \left\{ \int_\Omega g(x, Du) : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \right\},$$

where

$$|g(x, \xi)| \leqslant M(1 + |\xi|).$$

A classical example is the area functional, i.e. $g(x, \xi) = \sqrt{1 + |\xi|^2}$. The corresponding Euler-Lagrange equation is

$$-\mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0.$$

Then, the natural space for solutions is again $BV(\Omega)$ and the boundary condition needs to be treated separately.

Least gradient problem

The Euler-Lagrange equation corresponding to the least gradient problem

$$\min \left\{ \int_\Omega |Du| : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \right\}$$

is formally the 1-Laplace equation

$$\begin{cases} -\mathrm{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

It appears in relation to the study of minimal surfaces: for $u = \chi_E$, where ∂E is smooth, the expression on the left-hand side is the (minus) mean curvature of ∂E.

The level set formulation

The following result is due to Bombieri, de Giorgi and Giusti (1969).

u is a solution of the least gradient problem

$\Rightarrow$ for every $t \in \mathbb{R}$ the function $\chi_{\{u>t\}}$ solves

the least gradient problem for its own boundary data

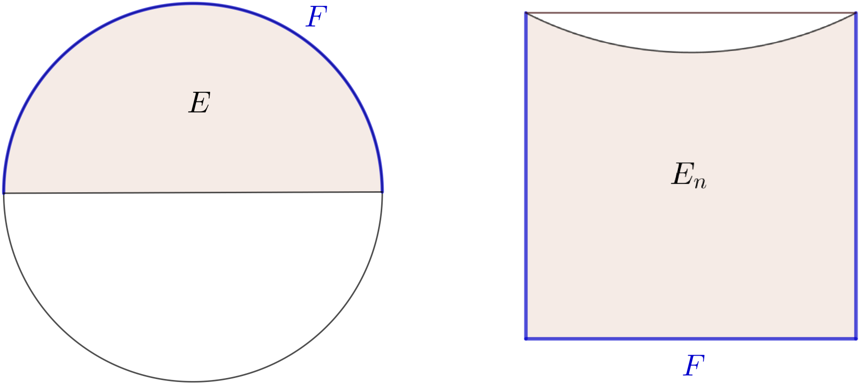
Since $\mathrm{div}(\frac{D\chi_E}{|D\chi_E|})$ is the mean curvature of E if its boundary is smooth, this result and the regularity theory for area-minimising surfaces imply that in 2D every superlevel set $\{u > t\}$ of a solution u has a boundary which is a locally finite union of line segments.

E. Bombieri, E. De Giorgi, E. Giusti, Invent. Math. **7** (1969).

Geometric meaning of the problem

$$\min \left\{ \int_\Omega |Du| : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \right\}$$

If $u = \chi_E$ and $f = \chi_F$, the problem has a simple geometric meaning:



Existence and properties of solutions depend on the shape of the domain!

Classical results

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If Ω is uniformly convex, then

- $f \in C^{0,\alpha}(\partial\Omega) \Rightarrow u \in C^{0,\alpha/2}(\overline{\Omega})$.

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Out of range of classical methods:

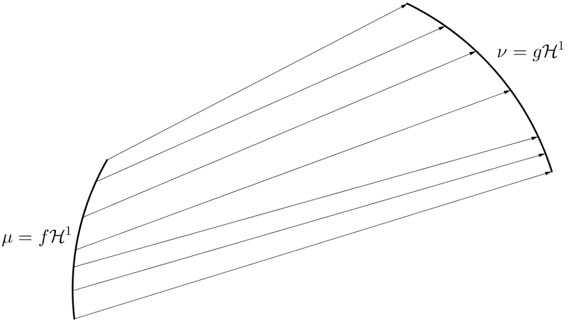
- Weak differentiability of solutions.

Monge-Kantorovich problem

The Monge mass transportation problem

$$\min\left\{ \int_{\overline{\Omega}} |x - T(x)| \, d\mu : \ T : \overline{\Omega} \to \overline{\Omega}, \ T_{\#}\mu = \nu \right\}.$$

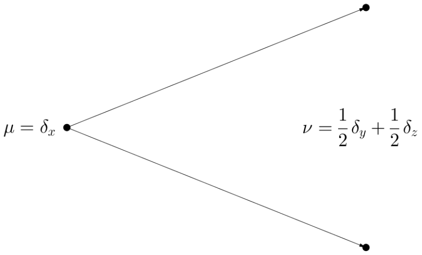
consists of finding an optimal map T which transports the measure μ onto the measure ν with minimal cost (induced by a distance).



Monge-Kantorovich problem

The Monge problem may have no solutions if μ is not absolutely continuous; its relaxation is the Monge-Kantorovich problem

$$\min\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\gamma : \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\ (\Pi_x)_\#\gamma = \mu, (\Pi_y)_\#\gamma = \nu \right\}.$$



One can define a measure σ_γ, called the transport density, which encodes how much of the transport takes place in a given subset of Ω.

Motivation 2: Related problems in mechanics

The least gradient problem is not suitable as an elastic counterpart to plastic optimal design - let us highlight two other models.

$\longrightarrow$ One may consider vector-valued functions and minimise the *elastic compliance* which depends on the whole tensor σ.

G. Bouchitté, G. Buttazzo, P. Seppecher, C. R. Acad. Sci. Paris Sér I. Math **324** (1997).

G. Bouchitté, G. Buttazzo, J. Eur. Math. Soc. **3** (2001).

$\longrightarrow$ One may consider *Michell trusses* which are limits of finite structures consisting of one-dimensional elastic bars.

G. Bouchitté, W. Gangbo, P. Seppecher, Math. Models Meth. Appl. Sci. **18** (2008).

Each problem is equivalent to a suitable optimal transport problem.

Optimal transport interpretation

Suppose that $\Omega \subset \mathbb{R}^2$ is strictly convex. Then, the least gradient problem is equivalent to the *Beckmann problem*:

$$\min \left\{ \int_{\overline{\Omega}} |v| : \quad v \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^2), \quad \mathrm{div}\, v = 0, \quad v \cdot \nu|_{\partial\Omega} = g \right\},$$

where $g = \frac{\partial f}{\partial \tau}$.

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where $g = \frac{\partial f}{\partial \tau}$.

The equivalence is formally given by $v = R_{-\frac{\pi}{2}} Du$.

W. Górny, P. Rybka, A. Sabra, Nonlinear Anal. **151** (2017).

S. Dweik, F. Santambrogio, Calc. Var. PDE **58** (2019).

W. Górny and J.M. Mazón, Functions of Least Gradient, Monographs in Mathematics **110** (2024).

Optimal transport interpretation

Again on convex domains, the Beckmann problem is equivalent to the Monge-Kantorovich optimal transport problem with source and target measures on $\partial\Omega$:

$$\min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \, d\gamma : \, \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}), \, (\Pi_x)_\# \gamma = g^+, (\Pi_y)_\# \gamma = g^- \right\}.$$

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From every solution v to the Beckmann problem we can construct a solution to the OTP with transport density σ_γ (and vice versa) and

$$\sigma_\gamma = |v|.$$

Optimal transport interpretation

Least gradient problem	Monge-Kantorovich problem
$f \in BV(\partial\Omega)$	$(\partial_\tau f)^\pm \in \mathcal{M}^+(\partial\Omega)$
level lines	transport rays
$u\|_{\partial\Omega} = f$	$\sigma_\gamma(\partial\Omega) = 0$
$f \in C(\partial\Omega)$	$(\partial_\tau f)^\pm$ is atomless
$f \in W^{1,p}(\partial\Omega)$	$(\partial_\tau f)^\pm \in L^p(\partial\Omega)$
$u \in W^{1,p}(\Omega)$	$\sigma_\gamma \in L^p(\Omega)$

Main result

S. Dweik, W. Górny, SIAM J. Math. Anal. **55** (2023).

Theorem 1

For geodesically convex $\Omega \subset \mathbb{R}^2$, the weighted least gradient problem

$$\min \left\{ \int_\Omega k(x)|Du| : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \right\}$$

with $k \in C^{1,1}(\Omega)$ is equivalent to the Monge-Kantorovich optimal transport problem with Riemannian cost, i.e.,

$$\min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} d_k(x,y)\, d\gamma : \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\, (\Pi_x)_\# \gamma = g^+, (\Pi_y)_\# \gamma = g^- \right\}.$$

$\longrightarrow$ Sobolev regularity of solutions!

Proof strategy

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- We sum up these estimates and get

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Remarks

• $C^{1,1}$ regularity of the weight plays a crucial role - for the blue estimate, we need uniqueness of a geodesic in a given direction. Namely, if $\alpha(s)$ is a parametrisation of $\partial\Omega$ and $y = \gamma_s(t)$, we have

$$\sigma(y) = \frac{\alpha'(s)\, d_k(\alpha(s), x_i)}{\det[D_{(s,t)}\gamma_s(t)]} \cdot g^+(\alpha(s)).$$

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• Also *the proof of the equivalence* requires uniqueness of the geodesics.

Results

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This is optimal in terms of the exponent. For $p > 2$, we need

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For less regular boundary data, we get

- $f \in SBV(\partial\Omega) \Rightarrow u \in SBV(\Omega)$.