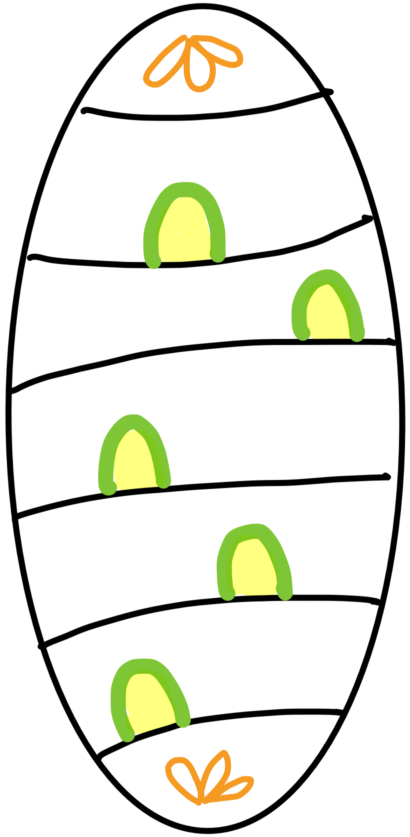
SRNI - LECTURE 3

PROOF OF GIROUX CORRESPONDENCE

- STABILISATION
- STATEMENT OF GIROUX CORRESPONDENCE
- IDEA OF PROOF
- FURTHER DIRECTIONS

THE GIROUX CORRESPONDENCE VIA CONVEX SURFACES



VERA VÉRTESI

JOINT WORK WITH JOAN LICATA

UNIVERSITY OF VIENNA

LECTURE 3

PROOF OF GIROUX'S CORRESPONDENCE

- STABILISATION
- STATEMENT OF GIROUX CORRESPONDENCE
- IDEA OF PROOF
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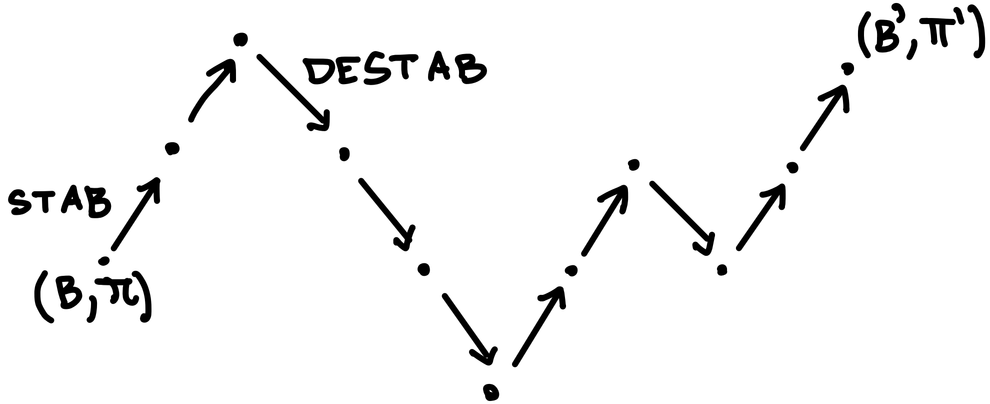
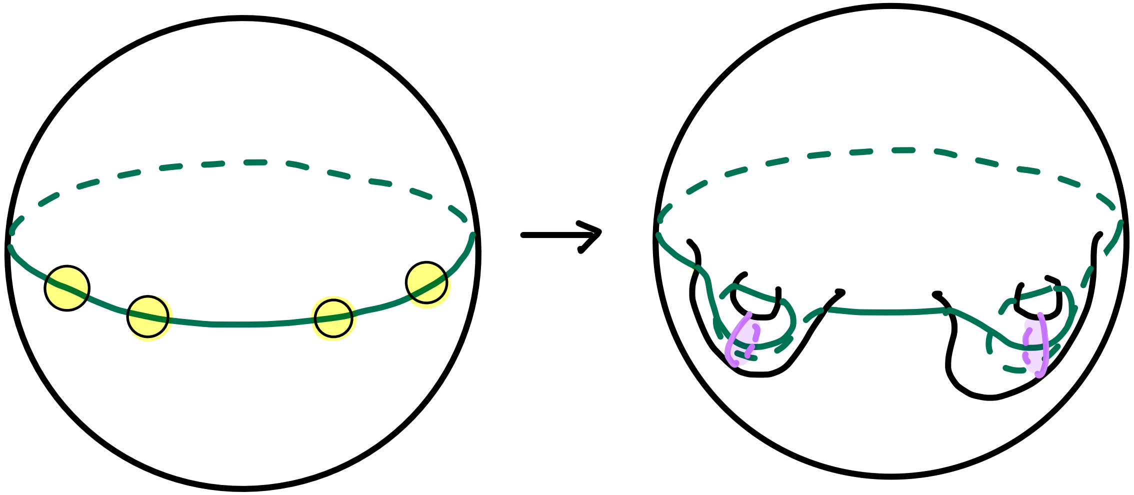


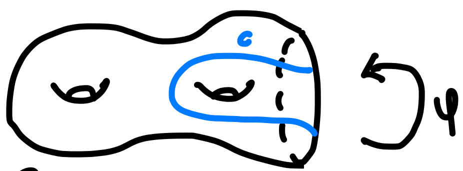
LAST TIME



- **CONTACT HANDLEBODY:**
  - → CTCT 0-h ∪ 1-h's
  - → N(LEGENDRIAN GRAPH)
  - → TIGHT + PRODUCT DISC DECOMPOSABLE

- **CONTACT HEEGAARD DECOMPOSITION** $M = U \cup_\Sigma V$
  - → Σ CONVEX
  - → U & V CONTACT HANDLEBODY

- **OPEN BOOK DECOMPOSITION**



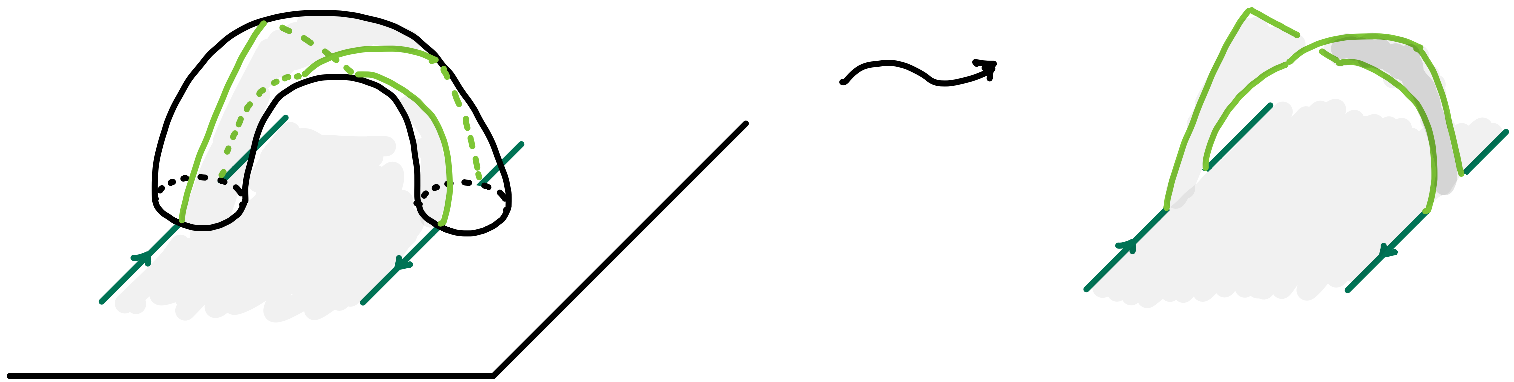
S

- $\boxed{\text{OPEN BOOK DECOMPOSITIONS OF } (M,\xi) / \text{ISOT} \longleftrightarrow \text{CONTACT HEEGAARD DECOMPOSITIONS OF } (M,\xi) / \text{ISOT}}$

STABILISATION OF

HEEGAARD DECOMPOSITIONS
- SMOOTH
- CONTACT

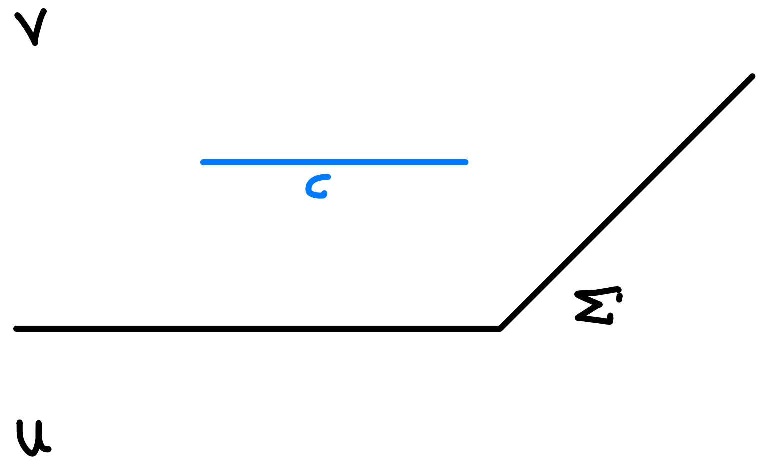
OPEN BOOK DECOMPOSITIONS



Stabilisation of Heegaard Decompositions

$M = U \cup_{\Sigma} V$ Heegaard decomposition

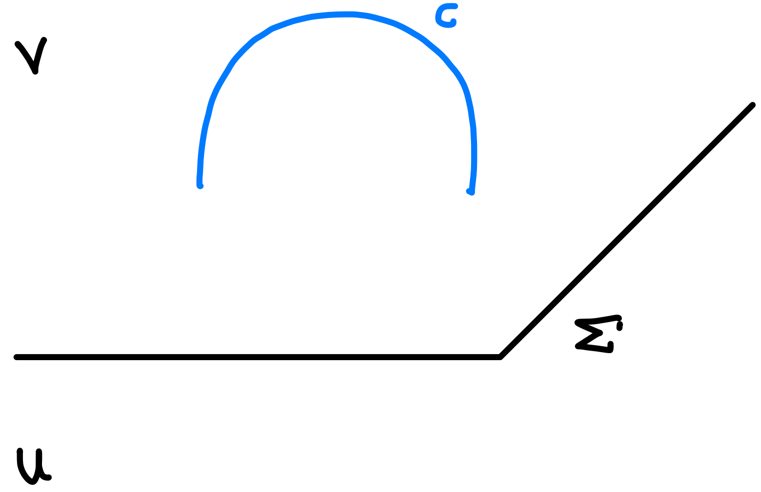
- c arc on Σ



STABILISATION OF HEEGAARD DECOMPOSITIONS

$M = U \cup_\Sigma V$ HEEGAARD DECOMPOSITION

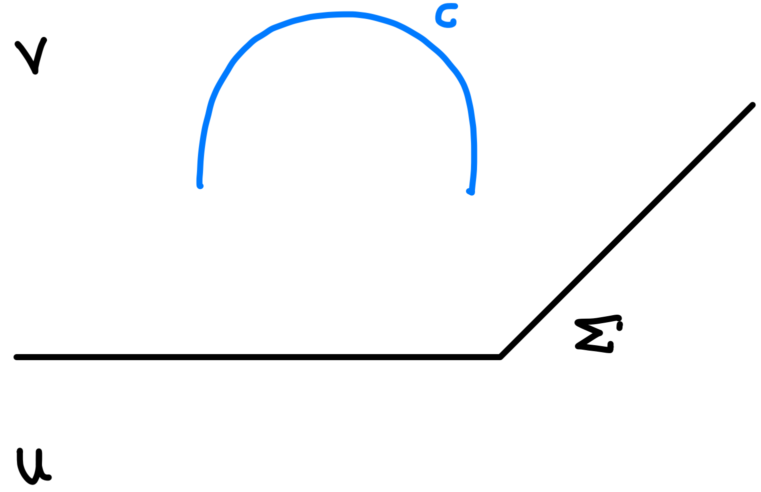
- c ARC ON Σ
- ISOTOPE $\mathring{c}$ INTO V



Stabilisation of Heegaard Decompositions

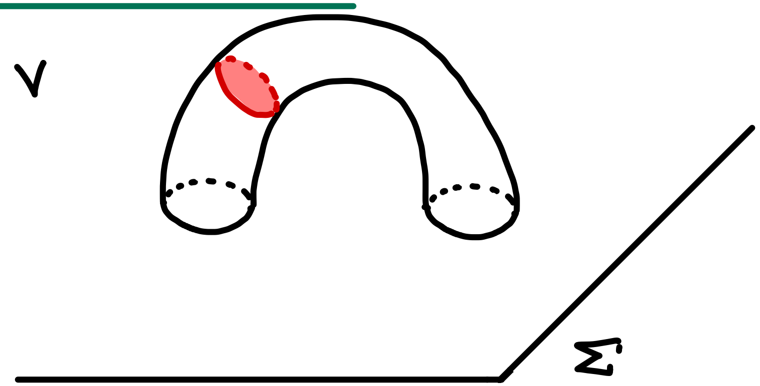
$M = U \cup_\Sigma V$ Heegaard decomposition

- c arc on Σ
- isotope $\mathring{c}$ into V



Stabilisation of Heegaard Decompositions

$M = U \cup_\Sigma V$ HEEGAARD DECOMPOSITION

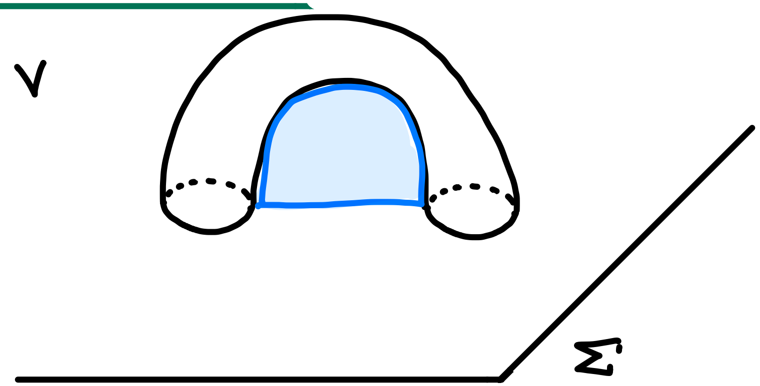


- c ARC ON Σ
- ISOTOPE $\mathring{c}$ INTO V

$\rightsquigarrow$ - $U' := U \cup N(c)$ IS A HANDLEBODY U

Stabilisation of Heegaard Decompositions

$M = U \cup_\Sigma V$ Heegaard decomposition



- c arc on Σ
- isotope $\mathring{c}$ into V

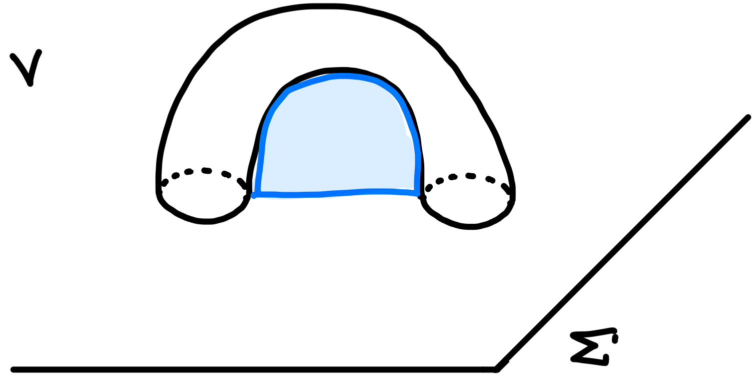
$\rightsquigarrow$ - $U' := U \cup N(c)$ is a handlebody U

- $V' := V \setminus N(c)$ is also a handlebody

- $\Sigma' := \partial U'$

STABILISATION OF HEEGAARD DECOMPOSITIONS

$M = U \cup_\Sigma V$ HEEGAARD DECOMPOSITION



- c ARC ON Σ
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$\rightsquigarrow$
- $U' := U \cup N(c)$ IS A HANDLEBODY
- $V' := V \setminus N(c)$ IS ALSO A HANDLEBODY
- $\Sigma' := \partial U'$

DEF:

$$M = U \cup_{(\Sigma,\Gamma)} V \underset{\text{DESTABILISATION}}{\overset{\text{STABILISATION}}{\rightleftarrows}} M = U' \cup_{(\Sigma',\Gamma')} V'$$

REIDEMEISTER-SINGER THEOREM: ANY TWO HD-S OF M^3 ARE RELATED BY STABILISATIONS & DESTABILISATIONS

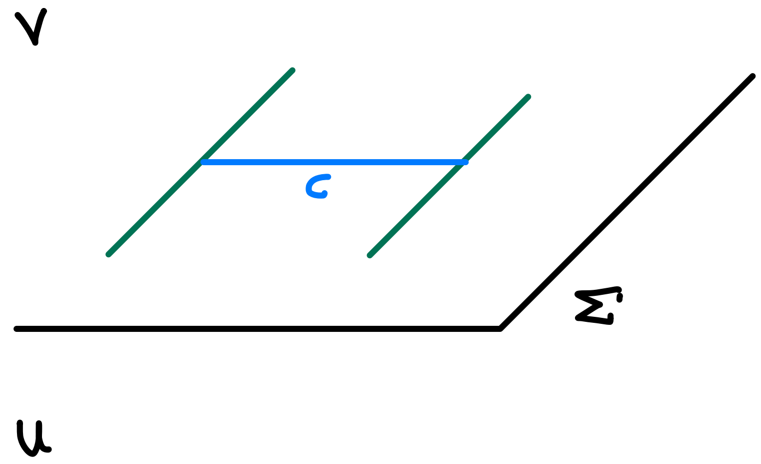
Stabilisation of Contact Heegaard Decompositions

(M, ξ) contact 3-manifold

$M = U \cup_{(\Sigma, \Gamma)} V$ contact HD

- c Legendrian arc on Σ

  with $tw_c(\xi, T\Sigma) = -1/2$



Stabilisation of Contact Heegaard Decompositions

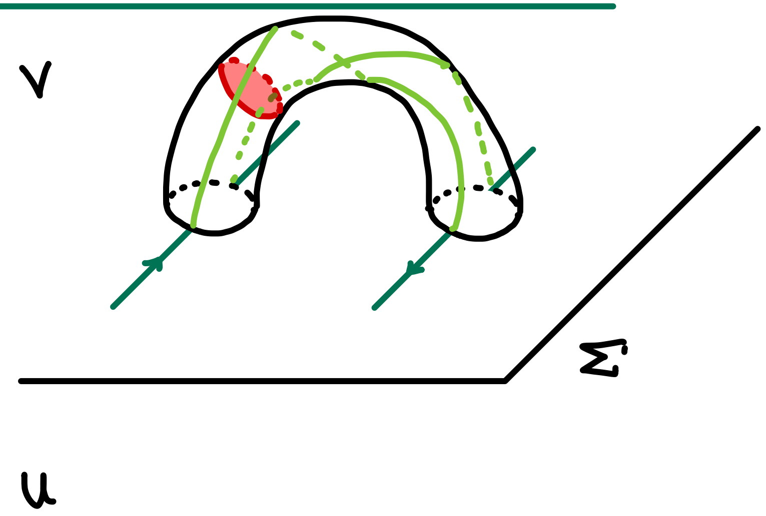
(M,ξ) contact 3-manifold

$M = U \cup_{(\Sigma,\Gamma)} V$ contact HD

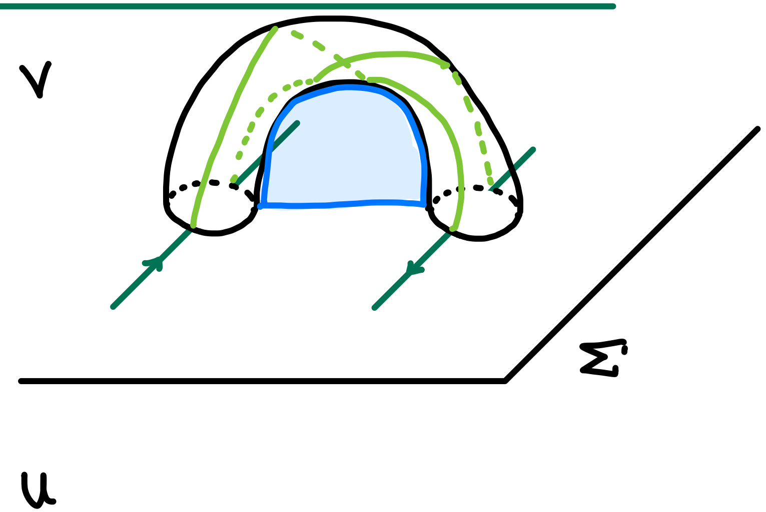
- c Legendrian arc on Σ
  with $tw_c(\xi, T\Sigma) = -\frac{1}{2}$

- Legendrian isotope $\mathring{c}$ into V

$\rightsquigarrow$ - $U' := U \cup N(c)$ is a contact handlebody



Stabilisation of Contact Heegaard Decompositions



(M, ξ) contact 3-manifold

$M = U \cup_{(\Sigma, \Gamma)} V$ contact HD

- c Legendrian arc on Σ
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- $U' := U \cup N(c)$ is a contact handlebody
- $V' := V \setminus N(c)$ is also a contact handlebody
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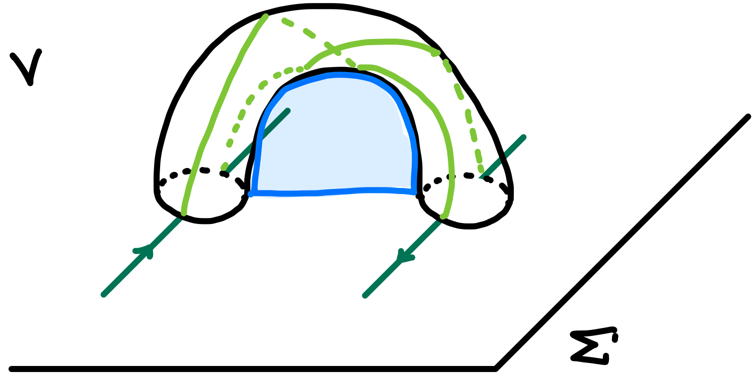
STABILISATION OF CONTACT HEEGAARD DECOMPOSITIONS

(M, ξ) CONTACT 3-MANIFOLD



$M = U \cup_{(\Sigma,\Gamma)} V$ CONTACT HD

- c LEGENDRIAN ARC ON Σ WITH $tw_c(\xi, T\Sigma) = -1/2$
- LEGENDRIAN ISOTOPE $\mathring{c}$ INTO V

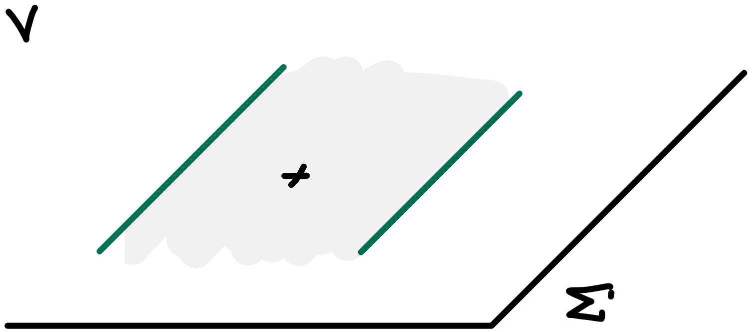
$\rightsquigarrow$ - $U' := U \cup N(c)$ IS A CONTACT HANDLEBODY
- $V' := V \setminus N(c)$ IS ALSO A CONTACT HANDLEBODY
- $\Sigma' := \partial U'$

DEF: $$M = U \cup_{(\Sigma,\Gamma)} V \underset{\text{CONTACT DESTABILISATION}}{\overset{\text{CONTACT STABILISATION}}{\rightleftarrows}} M = U' \cup_{(\Sigma',\Gamma')} V'$$

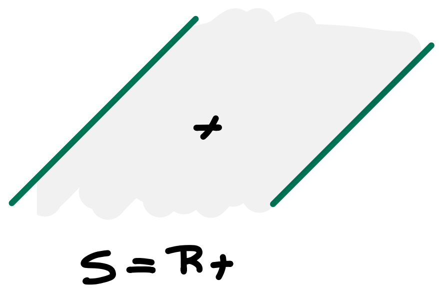
Stabilisation of Open Books

Recall: Contact Heegaard decomposition $\longleftrightarrow$ Open book decomposition

$$M = U \cup_{(\Sigma,\Gamma)} V \rightsquigarrow (S = R_+, h)$$



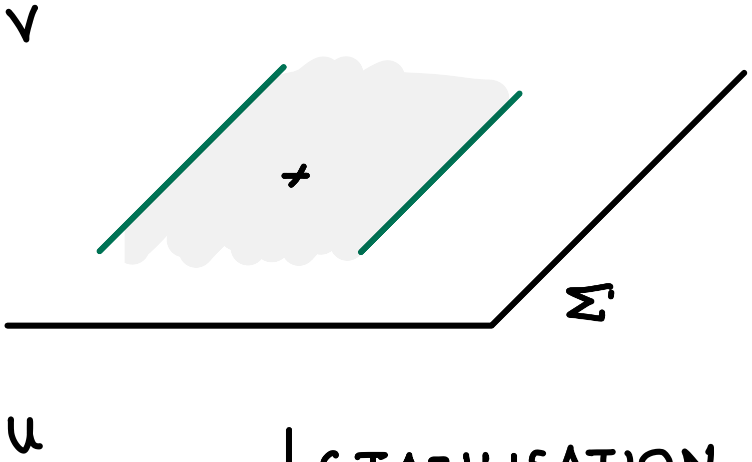
$\rightsquigarrow$



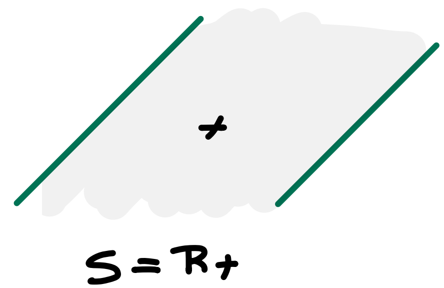
Stabilisation of Open Books

RECALL: CONTACT HEEGAARD DECOMPOSITION $\longleftrightarrow$ OPEN BOOK DECOMPOSITION

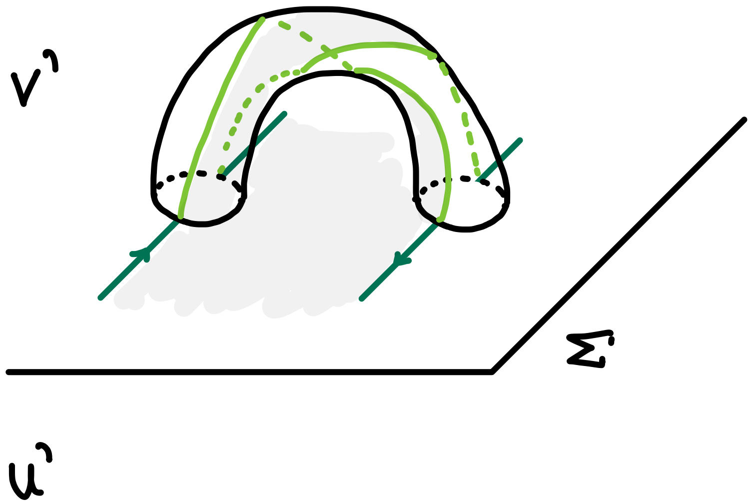
$$M = U \cup_{(\Sigma,\Gamma)} V \rightsquigarrow (S = R_+ , h)$$



$\rightsquigarrow$



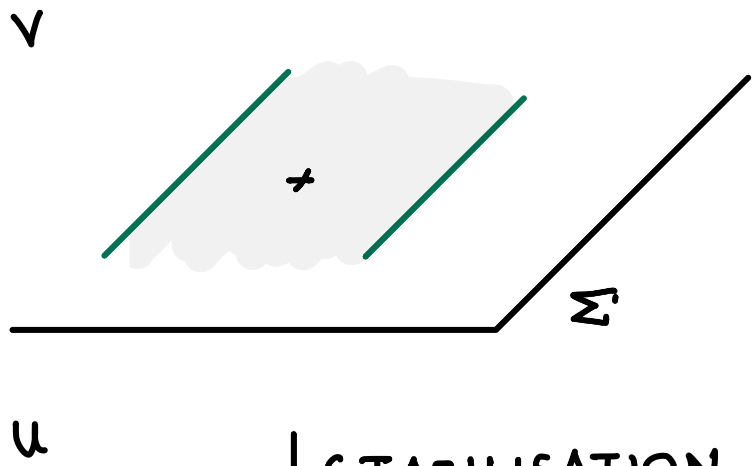
$\downarrow$ STABILISATION



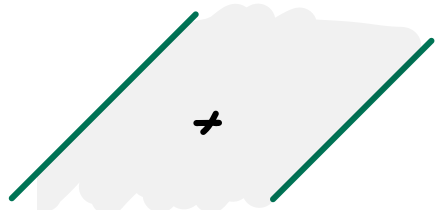
STABILISATION OF OPEN BOOKS

RECALL: CONTACT HEEGAARD DECOMPOSITION $\longleftrightarrow$ OPEN BOOK DECOMPOSITION

$$M = U \cup_{(\Sigma,\Gamma)} V \quad \rightsquigarrow \quad (S = R_+, \varphi)$$

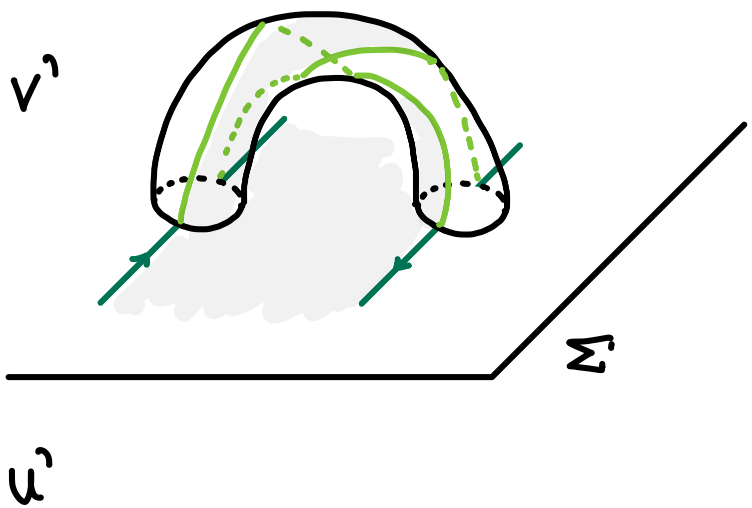


$\rightsquigarrow$



$S = R_+$

$\Big\downarrow$ STABILISATION

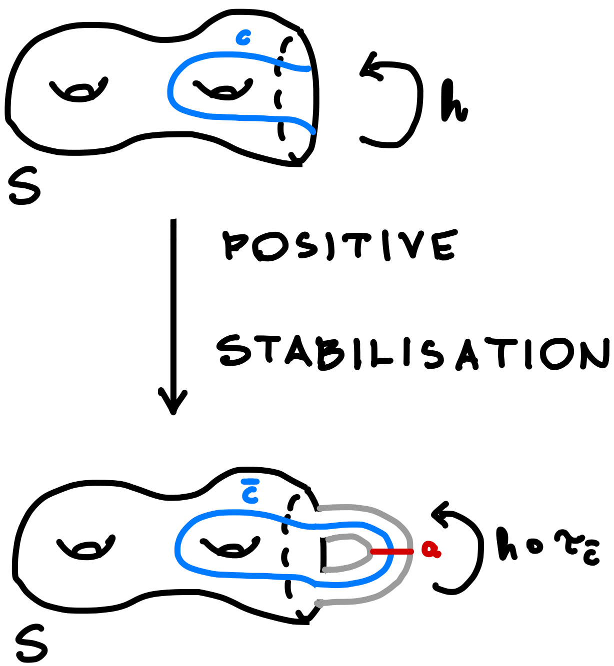


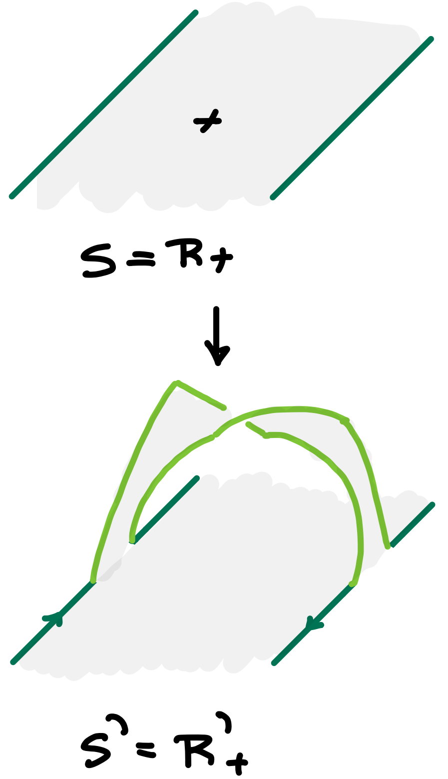
$\rightsquigarrow$

$S' = R'_+$

Stabilisation of Open Books

THE MONODROMIES CAN ALSO BE READ OFF & WE GET



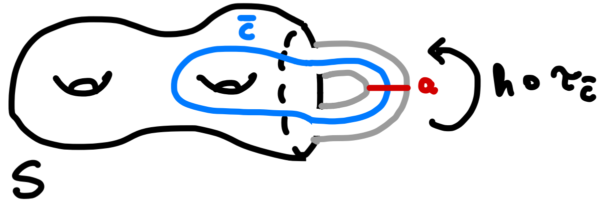


SO UNDER THE CORRESPONDENCE

CONTACT HEEGAARD DECOMPOSITION $\longleftrightarrow$ OPEN BOOK DECOMPOSITION

STABILISATION $\longleftrightarrow$ POSITIVE STABILISATION

(RE)STATEMENT OF GIROUX CORRESPONDENCE



Giroux Correspondence - Rephrased

GIROUX CORRESPONDANCE

OPEN BOOKS / POSITIVE STABILISATION $\longleftrightarrow$ CONTACT 3-MANIFOLDS / CONTACTOMORPHISM

GIROUX CORRESPONDANCE (REPHRASED)

CONTACT HEEGAARD DECOMPOSITIONS / CONTACT STABILISATION $\longleftrightarrow$ CONTACT 3-MANIFOLDS / CONTACTOMORPHISM

Giroux Correspondence – Rephrased

GIROUX CORRESPONDANCE

OPEN BOOKS / POSITIVE STABILISATION ↔ CONTACT 3-MANIFOLDS / CONTACTOMORPHISM

GIROUX CORRESPONDANCE (REPHRASED)

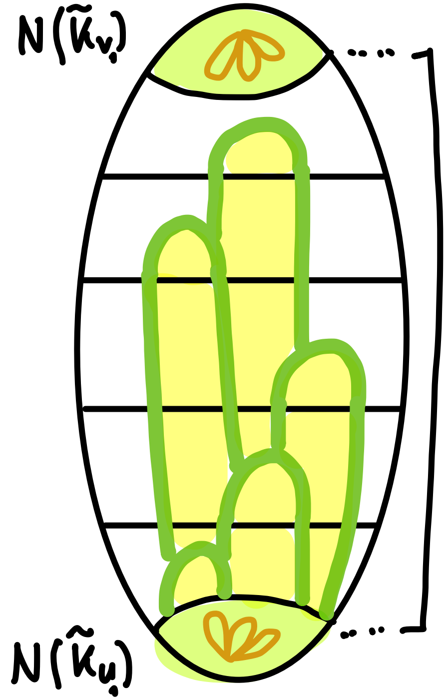
CONTACT HEEGAARD DECOMPOSITIONS / CONTACT STABILISATION ↔ CONTACT 3-MANIFOLDS / CONTACTOMORPHISM

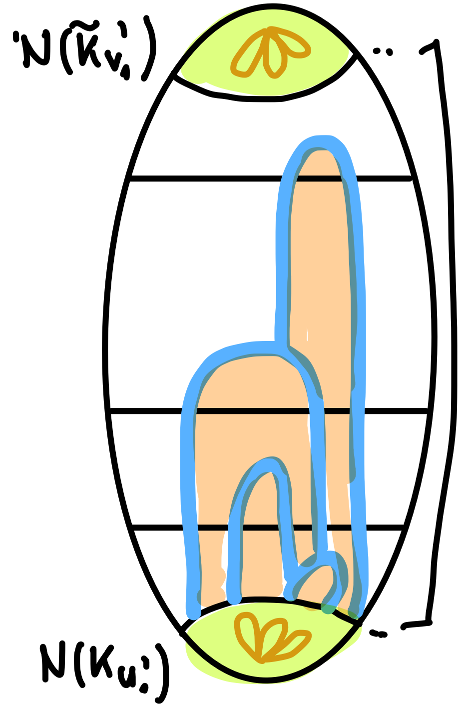
FROM SMOOTH TOPOLOGY:

REIDEMEISTER-SINGER THEOREM

HEEGAARD DECOMPOSITIONS / STABILISATION ↔ SMOOTH 3-MANIFOLDS / DIFFEOMORPHISM

IDEA OF PROOF





What do we Want to Prove?

Giroux Correspondance (rephrased)

$$\text{Contact Heegaard decompositions} \,/\, \text{contact stabilisation} \longleftrightarrow \text{Contact 3-manifolds} \,/\, \text{contactomorphism}$$

WHAT DO WE WANT TO PROVE?

GIROUX CORRESPONDANCE (REPHRASED)

CONTACT HEEGAARD DECOMPOSITIONS / CONTACT STABILISATION $\longleftrightarrow$ CONTACT 3-MANIFOLDS / CONTACTOMORPHISM

MORE PRECISELY : GIVEN TWO CONTACT HEEGAARD DECOMPOSITIONS

$\mathcal{H}^{2}$

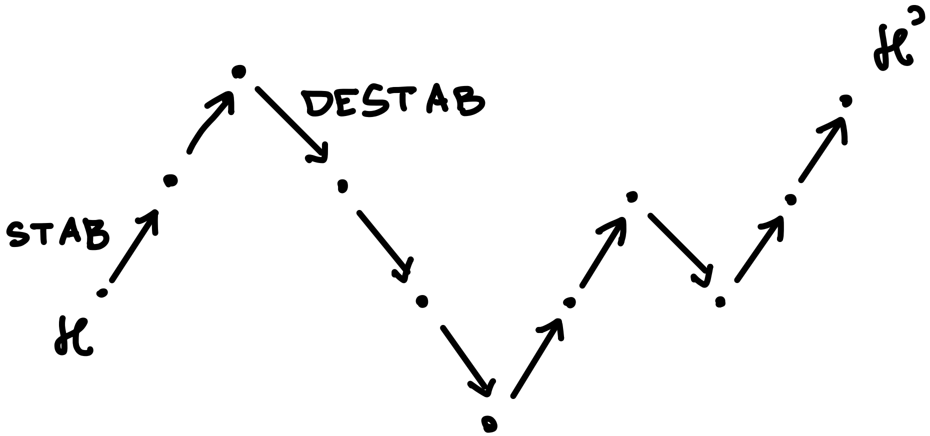
$\mathcal{H}$

WHAT DO WE WANT TO PROVE?

GIROUX CORRESPONDANCE (REPHRASED)

CONTACT HEEGAARD DECOMPOSITIONS / CONTACT STABILISATION $\longleftrightarrow$ CONTACT 3-MANIFOLDS / CONTACTOMORPHISM

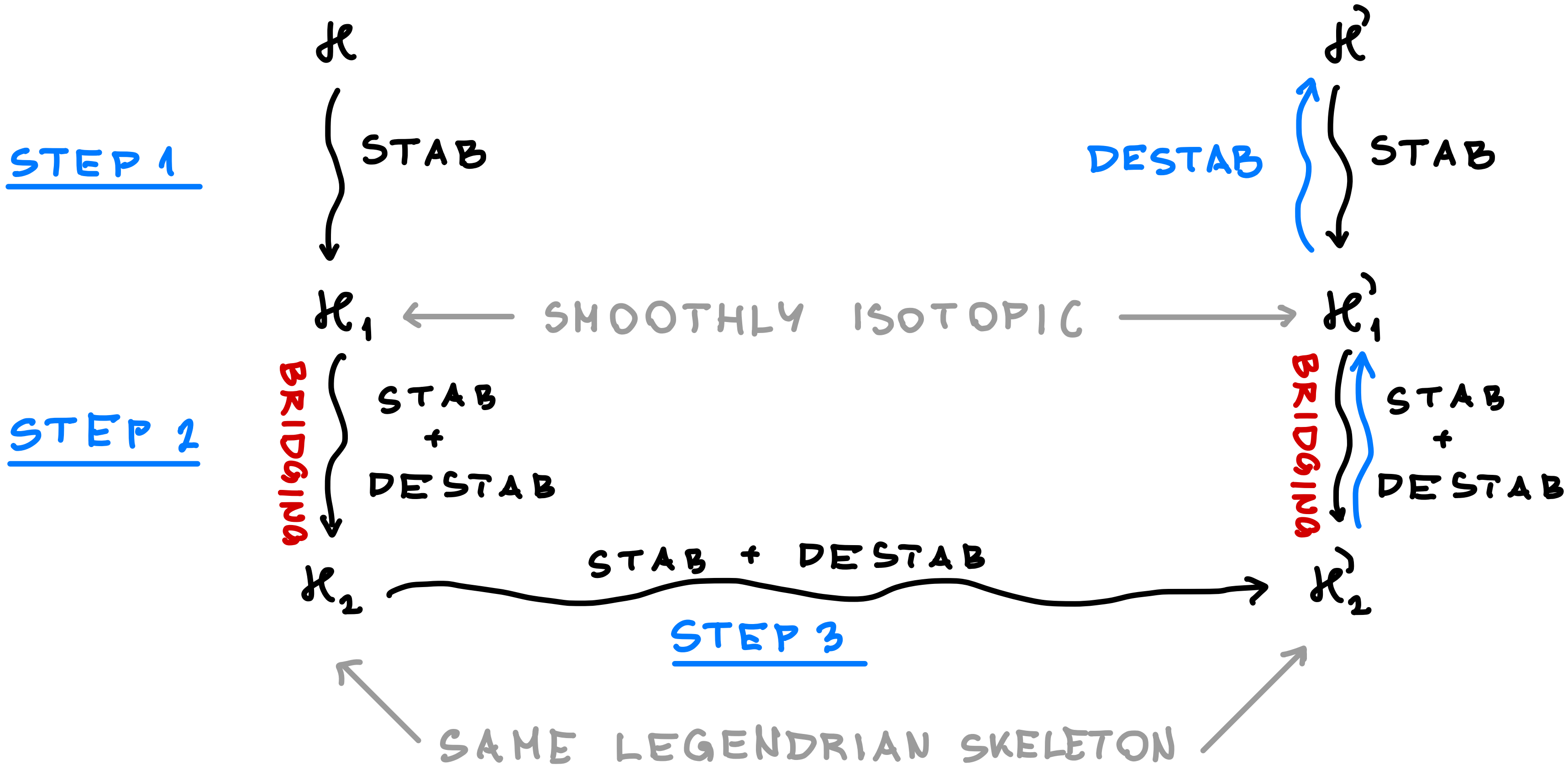
MORE PRECISELY: GIVEN TWO CONTACT HEEGAARD DECOMPOSITIONS



THEN THERE IS A SEQUENCE OF CONTACT STABILISATIONS & CONTACT DESTABILISATIONS CONNECTING THEM

Idea of Proof

In three steps we make them more & more similar

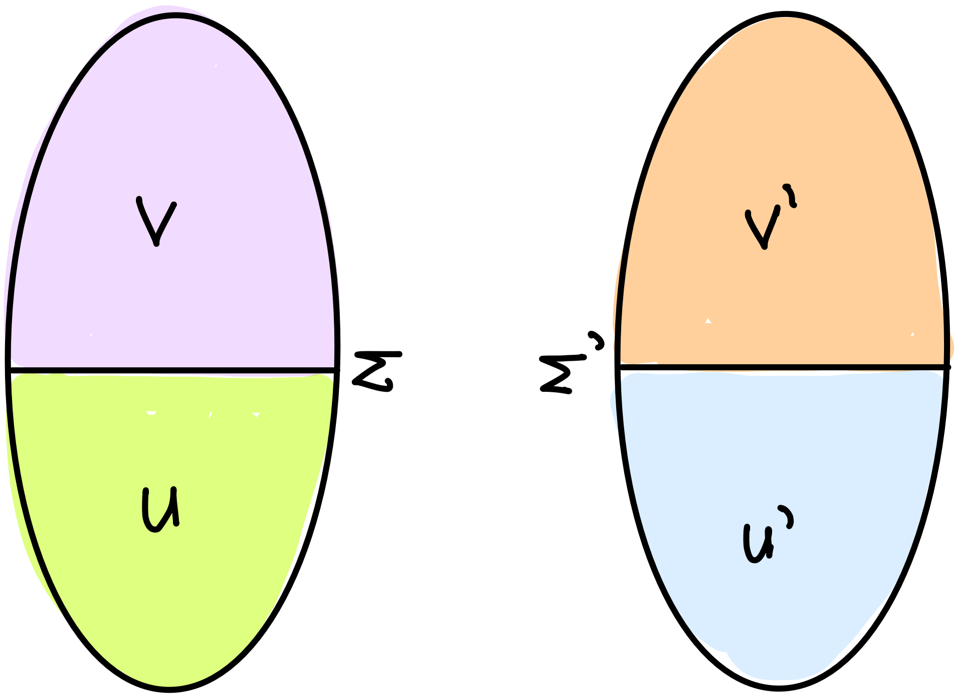


STEP 1

GIVEN TWO CTCT HD'S

$\underbrace{M = U \cup_{\Sigma} V}_{\mathcal{H}}$ & $\underbrace{M' = U' \cup_{\Sigma'} V'}_{\mathcal{H}'}$

$\xRightarrow[\text{SINGER}]{\text{REIDEMEISTER}}$ $\mathcal{H}$ & $\mathcal{H}'$ ADMITS A COMMON SMOOTH STABILISATION



STEP 1

GIVEN TWO CTCT HD'S

$\underbrace{M = U \cup_{\Sigma} V}_{\mathcal{H}}$ & $\underbrace{M' = U' \cup_{\Sigma'} V'}_{\mathcal{H}'}$

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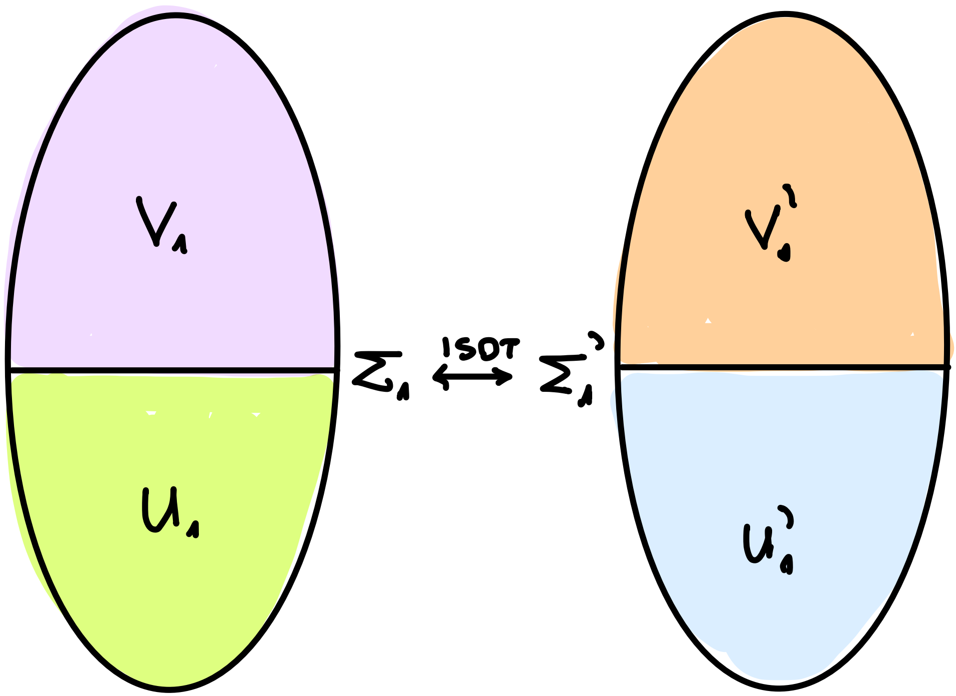


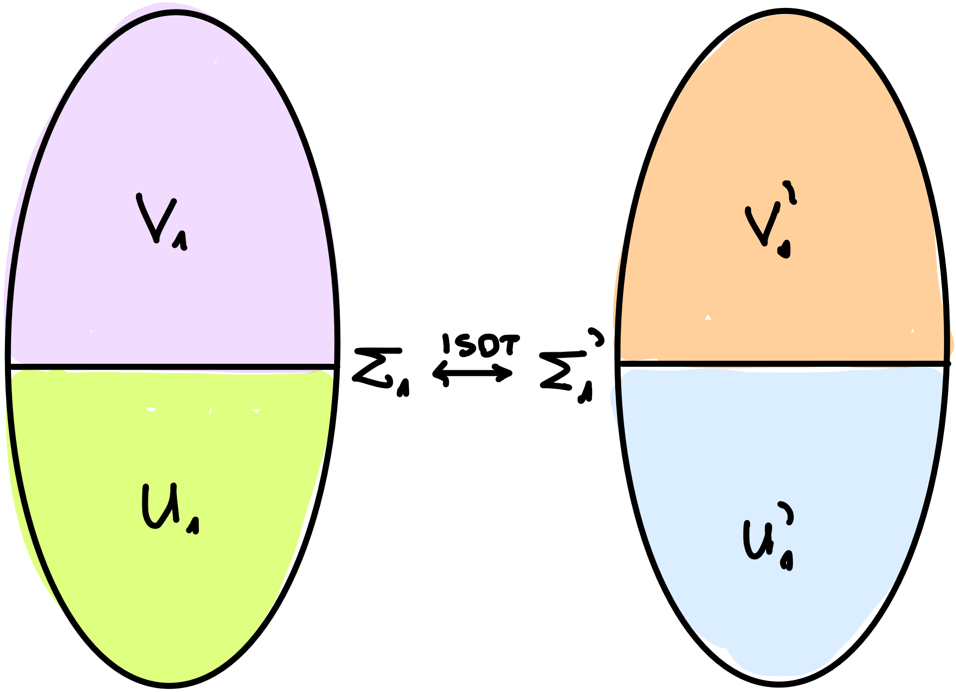
REALISE THESE STABILISATIONS WITH CTCT STABILISATIONS ON EACH

$\rightsquigarrow$ GET $\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1}$ & $\overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$, WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC

STEP 2

$$\underbrace{M = U_1 \cup_{\Sigma_1} V_1}_{\mathcal{H}_1} \quad \& \quad \underbrace{M = U_1' \cup_{\Sigma_1'} V_1'}_{\mathcal{H}_2}$$

WHERE Σ_1 & Σ_1' ARE **SMOOTHLY** ISOTOPIC

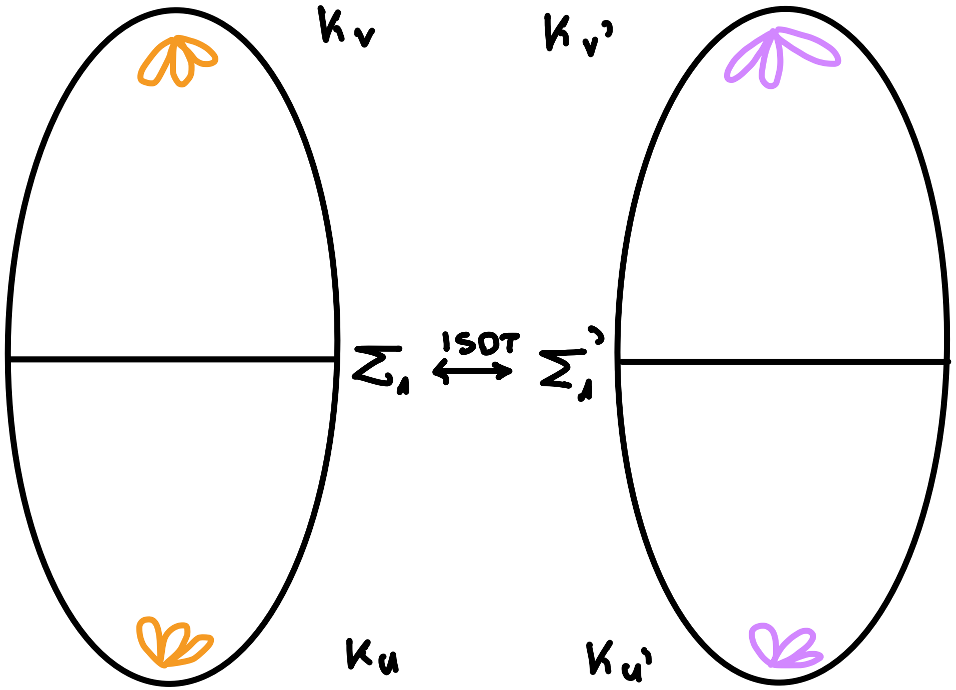


APPLY THE EXISTENCE PROOF

STEP 2

$$\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1} \quad \& \quad \overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



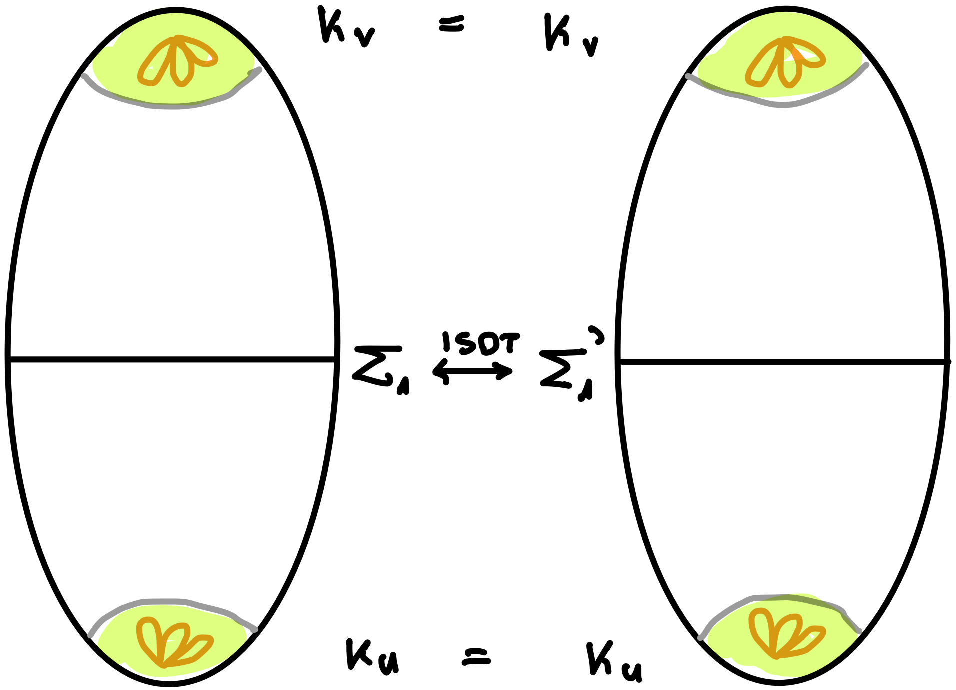
APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

STEP 2

$$M = \underbrace{U_1 \cup_{\Sigma_1} V_1}_{\mathcal{H}_1} \quad \& \quad M = \underbrace{U_1' \cup_{\Sigma_1'} V_1'}_{\mathcal{H}_2}$$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

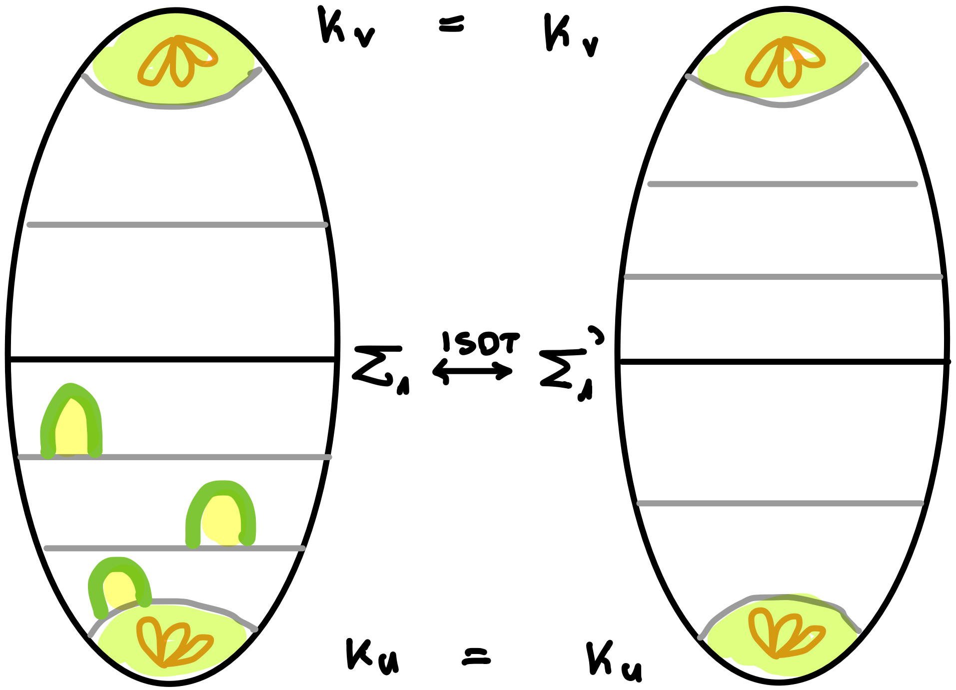
FUCHS-TABACHNIKOV: AFTER SUFFICIENTLY MANY LEGENDRIAN STAB. WE CAN ASSUME $K_u = K_{u'}$ & $K_v = K_{v'}$

- TAKE THE STANDARD NEIGHBOURHOOD $N(K_u), N(K_v)$

STEP 2

$$\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1} \quad \& \quad \overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

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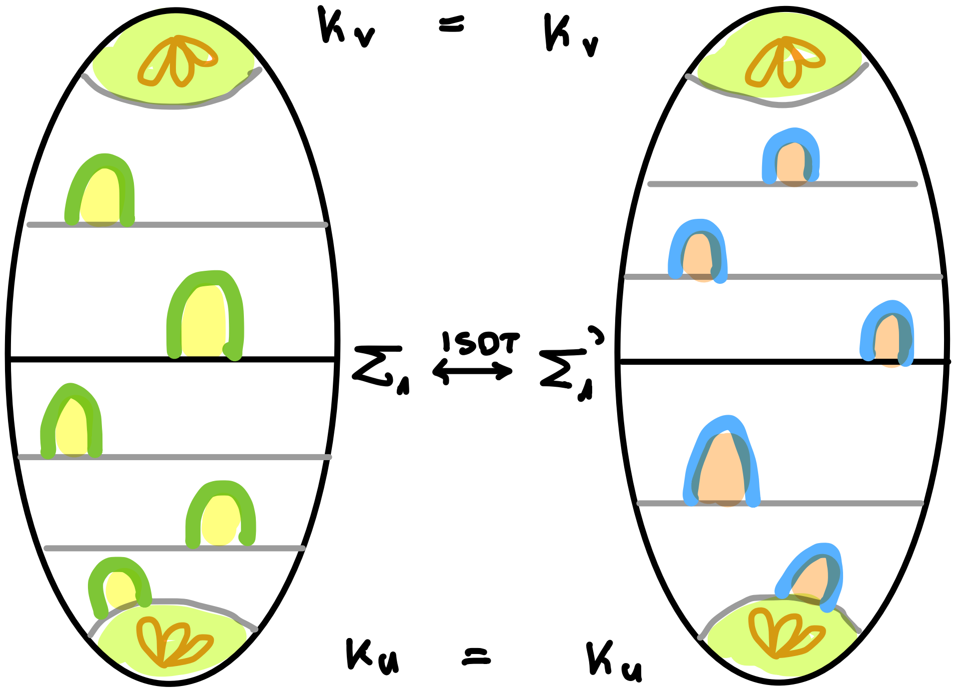
- TAKE THE STANDARD NEIGHBOURHOOD $N(K_U)$, $N(K_V)$
- $U' \setminus N(K_U)$ & $V' \setminus N(K_V)$ ARE $\cong \Sigma_1 \times I \overset{\text{HONDA}}{\Longrightarrow}$ = STACKS OF BYPASSES

STEP 2

$$\underbrace{M = U_1 \cup_{\Sigma_1} V_1}_{\mathcal{H}_1} \quad \& \quad \underbrace{M = U_1' \cup_{\Sigma_1'} V_1'}_{\mathcal{H}_2}$$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



APPLY THE EXISTENCE PROOF

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- TAKE THE STANDARD NEIGHBOURHOOD $N(K_U), N(K_V)$
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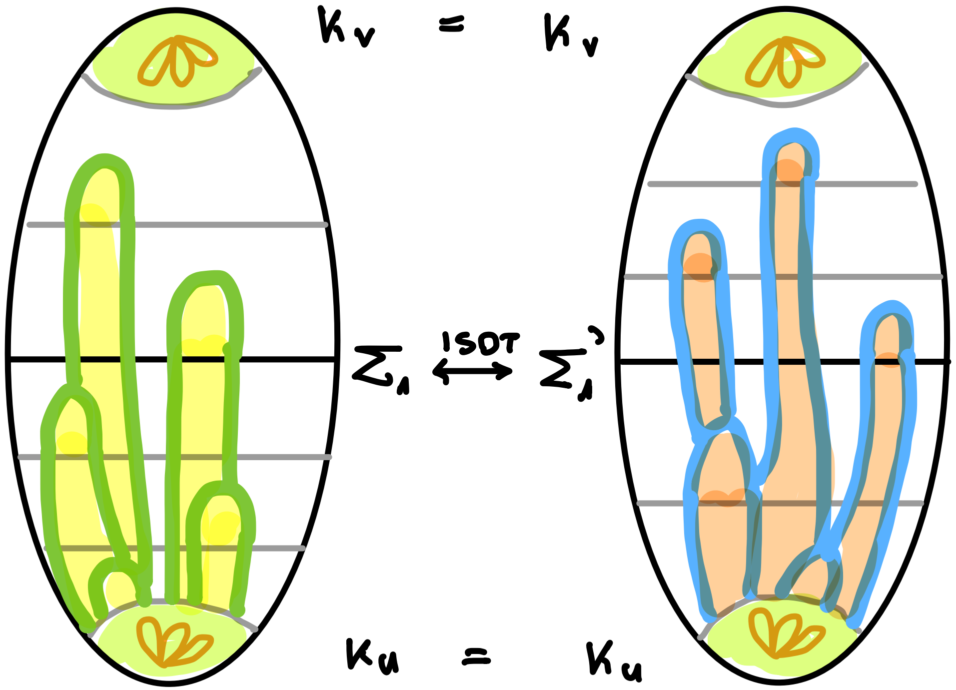
STEP 2

$\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1}$ & $\overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

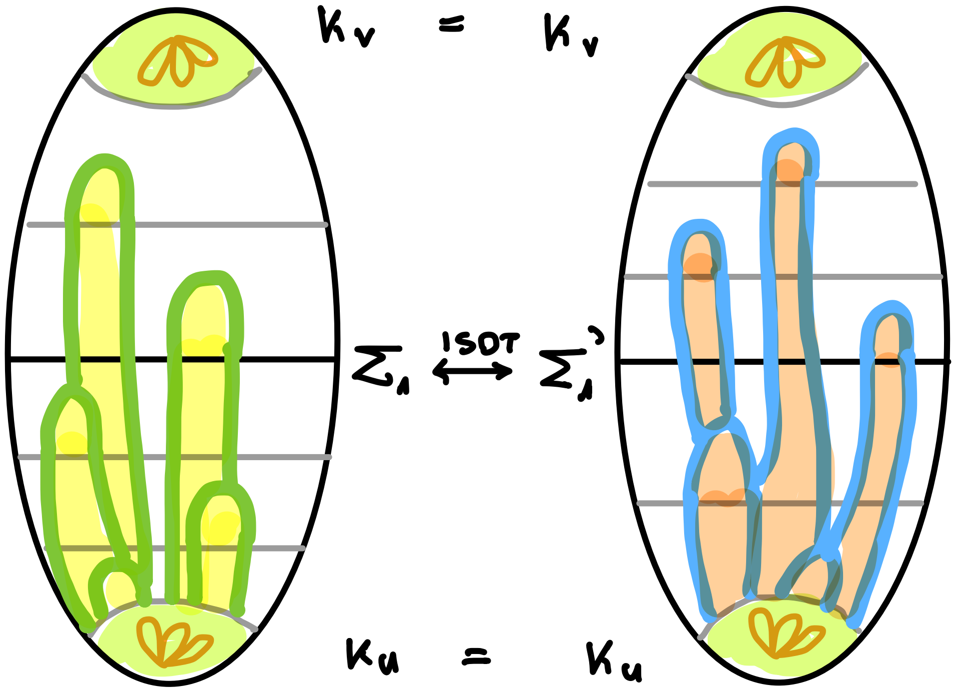
FUCHS - TABACHNIKOV: AFTER SUFFICIENTLY MANY LEGENDRIAN STAB. WE CAN ASSUME $K_U = K_{U'}$ & $K_V = K_{V'}$

- TAKE THE STANDARD NEIGHBOURHOOD $N(K_U)$, $N(K_V)$
- $U' \setminus N(K_U)$ & $V' \setminus N(K_V)$ ARE $\cong \Sigma_1 \times I \xRightarrow{\text{HONDA}}$ = STACKS OF BYPASSES
- EXTEND THE HANDLES

STEP 2

$$\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1} \quad \& \quad \overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

> FUCHS - TABACHNIKOV: AFTER SUFFICIENTLY MANY LEGENDRIAN STAB. WE CAN ASSUME $K_U = K_{U'}$ & $K_V = K_{V'}$

- TAKE THE STANDARD NEIGHBOURHOOD $N(K_U), N(K_V)$
- $U' \setminus N(K_U)$ & $V' \setminus N(K_V)$ ARE $\cong \Sigma_1 \times I \overset{\text{HONDA}}{\Longrightarrow}$ = STACKS OF BYPASSES
- EXTEND THE HANDLES

$$U_2' := N(U_K) \cup (1\text{-}h's) \qquad V_2' = M \setminus U_2'$$

$$\rightsquigarrow \quad \underbrace{M = U_2 \cup V_2}_{\mathcal{H}_2} \quad \& \quad \underbrace{M = U_2' \cup V_2'}_{\mathcal{H}_2'}$$

STEP 2

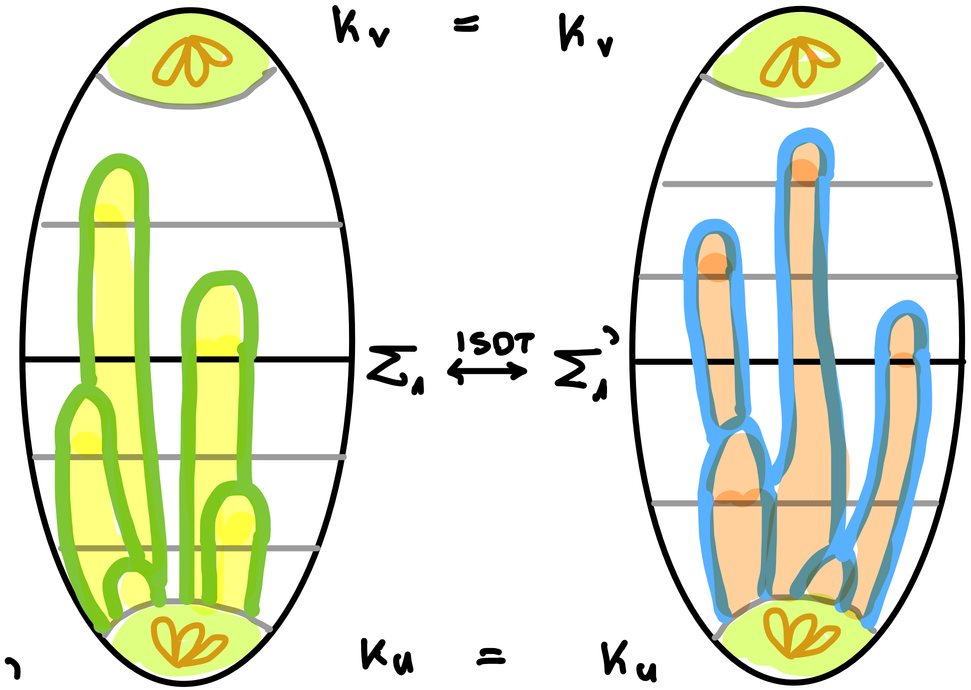
- $U' \setminus N(K_u)$ & $V' \setminus N(K_v)$ ARE $\cong \Sigma_1 \times I$

$\xRightarrow{\text{HONDA}}$ = STACKS OF BYPASSES

- EXTEND THE HANDLES

$U'_2 := N(U_K) \cup (1\text{-}h'_s)$ $\qquad V'_2 = M \setminus U'_2$

$\rightsquigarrow \quad \underbrace{M = U_2 \cup V_2}_{\mathcal{H}_2} \quad \& \quad \underbrace{M = U'_2 \cup V'_2}_{\mathcal{H}'_2}$



PROP (L-V) $\mathcal{H}'_2$ CAN BE OBTAINED FROM $\mathcal{H}'_1$ VIA A SEQUENCE OF CONTACT STAB & DESTAB

STEP 2

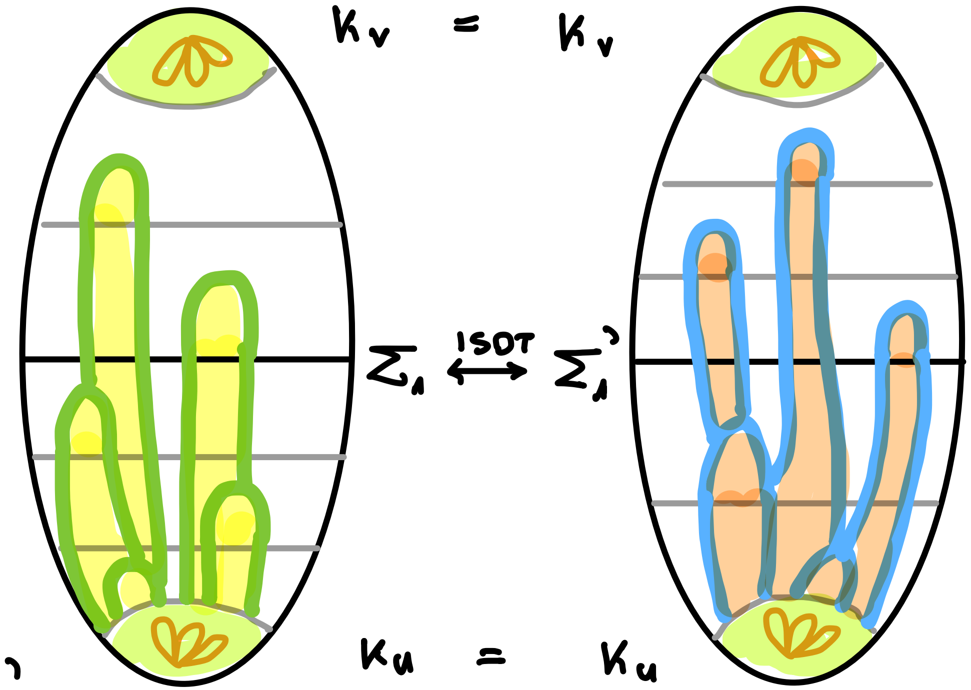
• $U'\setminus N(K_u)$ & $V'\setminus N(K_v)$ ARE $\cong \Sigma_1 \times I$

$\xRightarrow{\text{HONDA}}$ = STACKS OF BYPASSES

• EXTEND THE HANDLES

$U_2' := N(U_K) \cup (1\text{-}h's)$ $\quad V_2' = M \setminus U_2'$

$\rightsquigarrow \underbrace{M = U_2 \cup V_2}_{\mathcal{H}_2}$ & $\underbrace{M = U_2' \cup V_2'}_{\mathcal{H}_2'}$



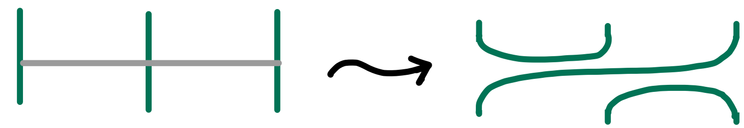
$$\boxed{\text{PROP (L-V) } \mathcal{H}_2' \text{ CAN BE OBTAINED FROM } \mathcal{H}_1' \text{ VIA A SEQUENCE OF CONTACT STAB \& DESTAB}}$$

BEFORE NEXT STEP NOTICE

$$\mathcal{H}_2 = \hat{\mathcal{H}}(U_K, V_K, \mathcal{B}) \qquad \mathcal{H}_2' = \hat{\mathcal{H}}(U_K, V_K, \mathcal{B}')$$

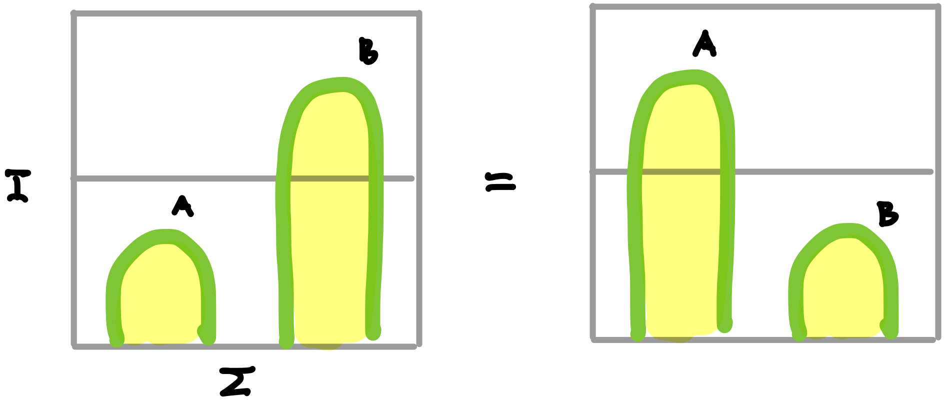
WHERE $\mathcal{B}$ & $\mathcal{B}'$ ARE DIFFERENT DECOMPOSITIONS OF $\xi|_{M-(N(U_K)\cup N(V_K))}$ AS BYPASS STACKS

Interlude - Stacking Bypasses



THE FOLLOWING MOVES DO NOT CHANGE ξ

COMMUTATION



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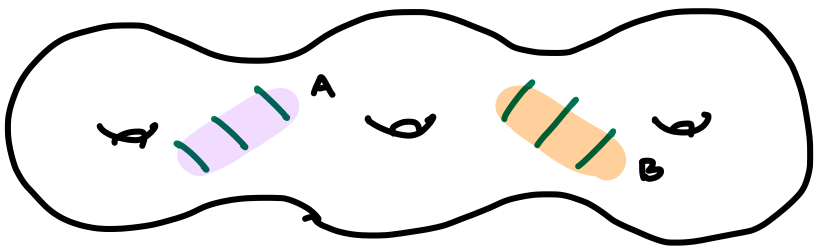
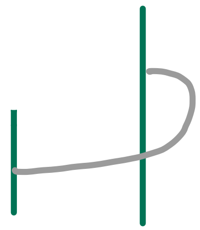
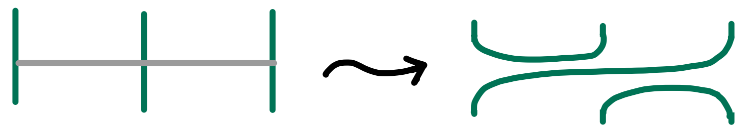


TRIVIAL BYPASS



Interlude - Stacking Bypasses



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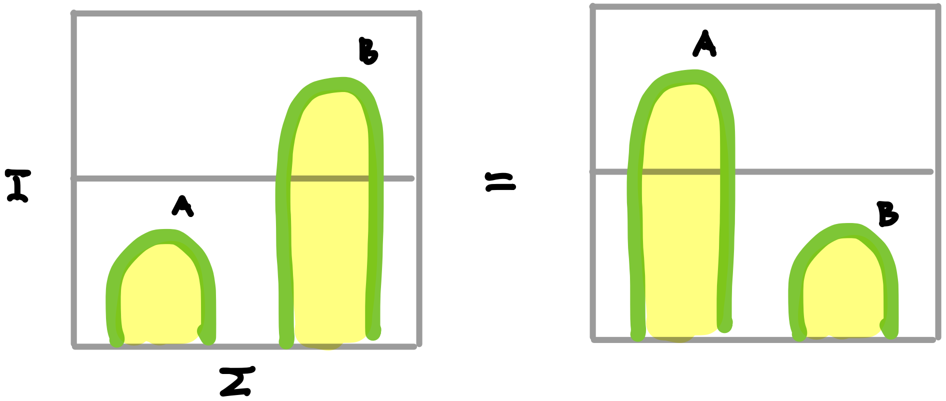

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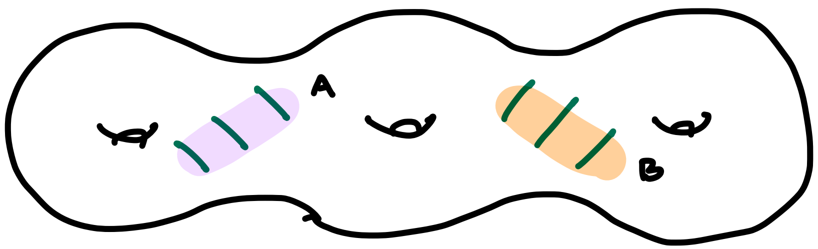

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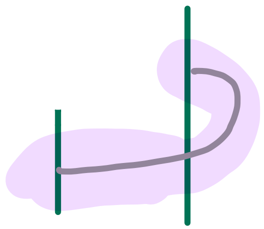

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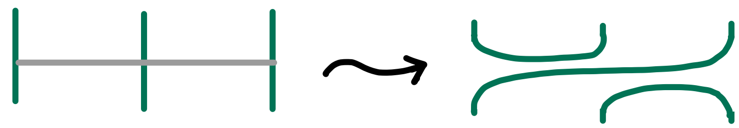

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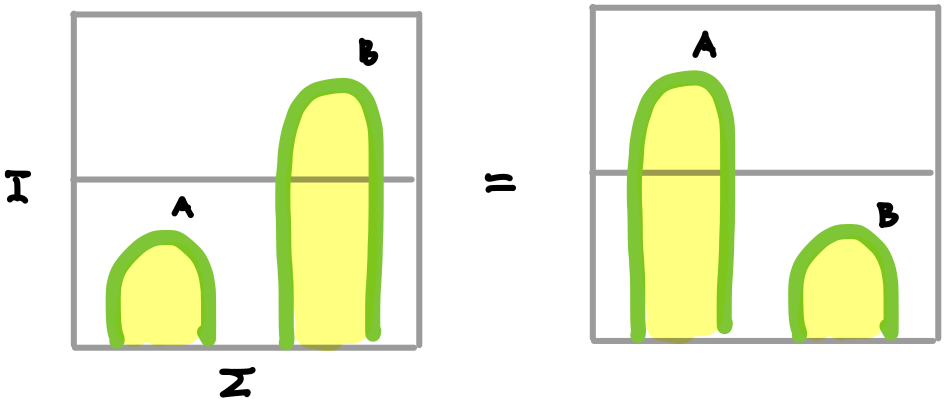

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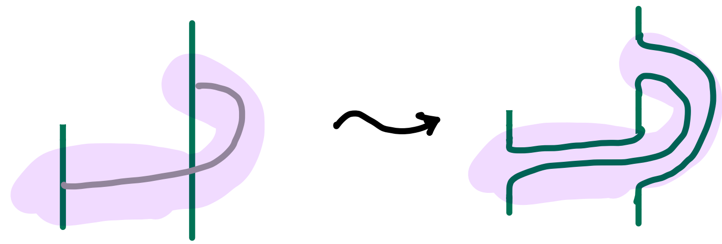

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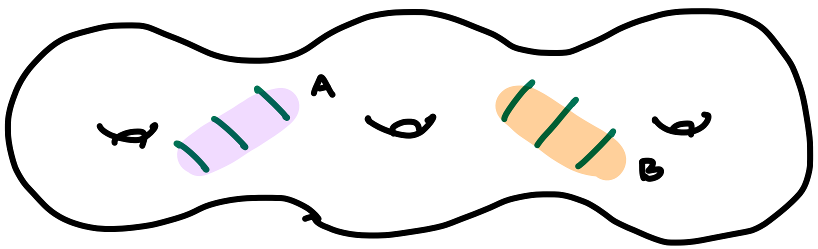

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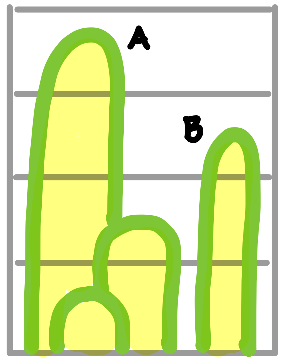
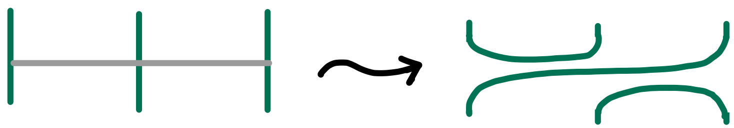


Interlude - Stacking Bypasses



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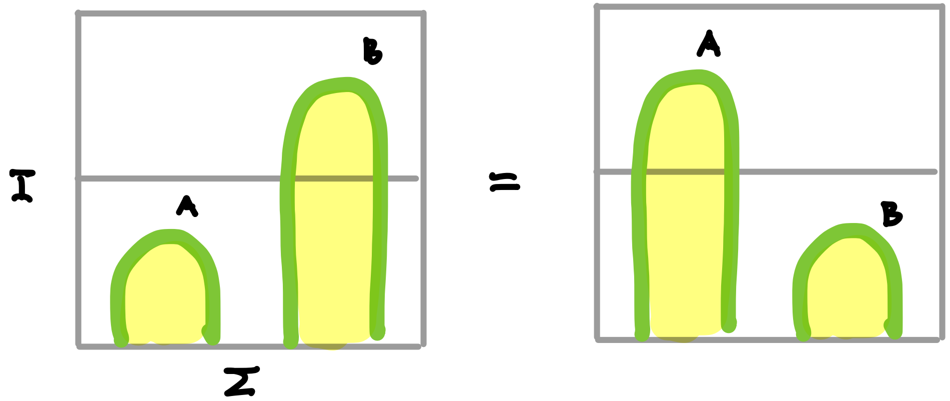

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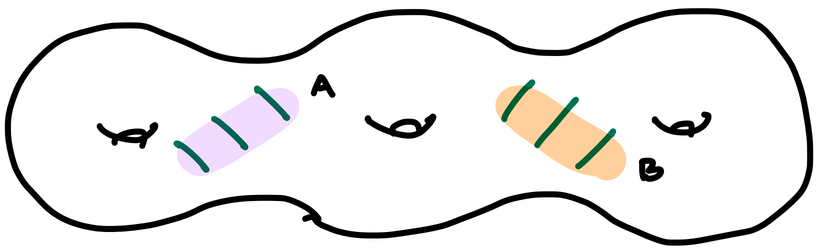

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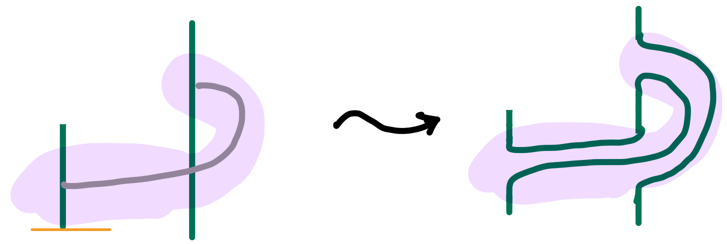

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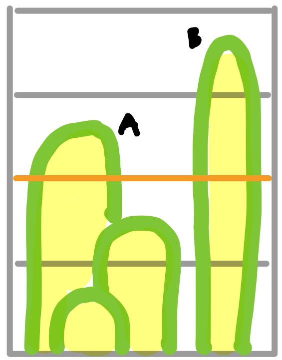
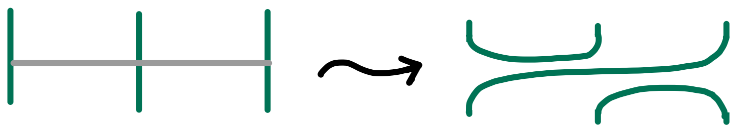


INTERLUDE - STACKING BYPASSES



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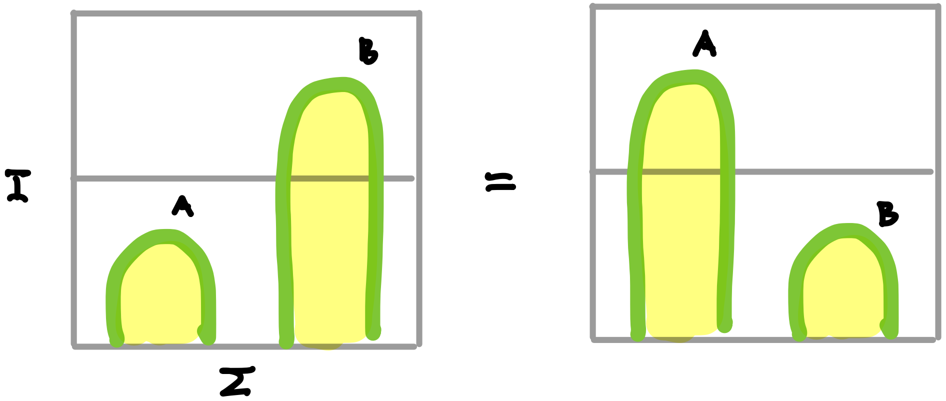
COMMUTATION



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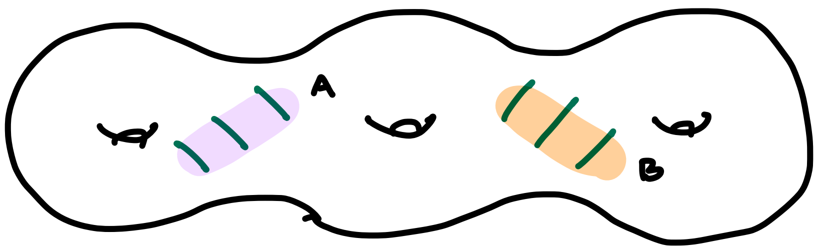

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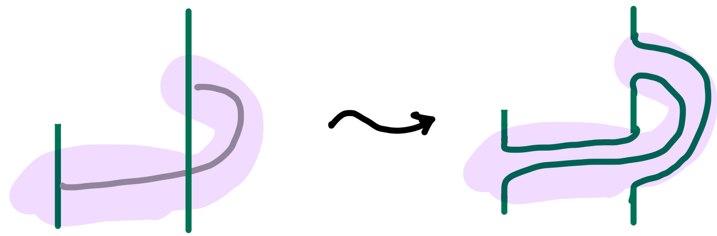

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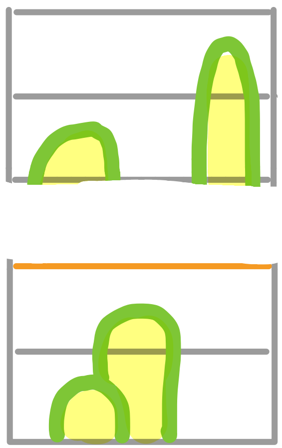

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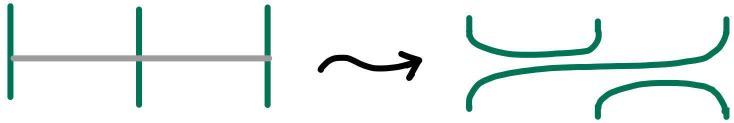

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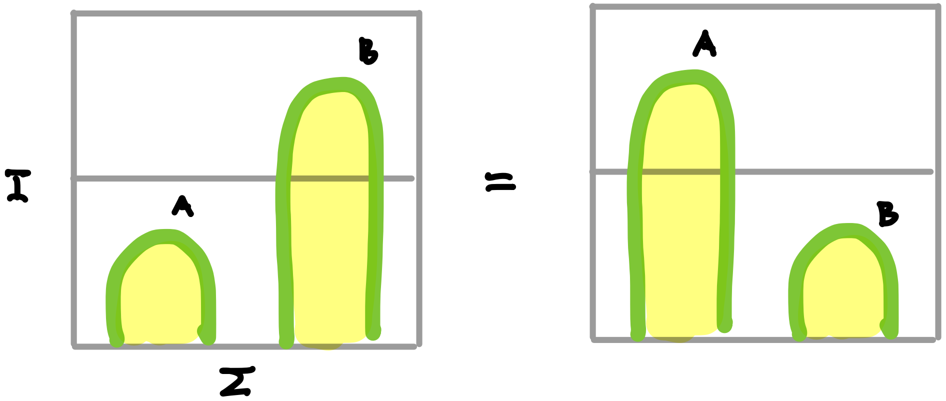

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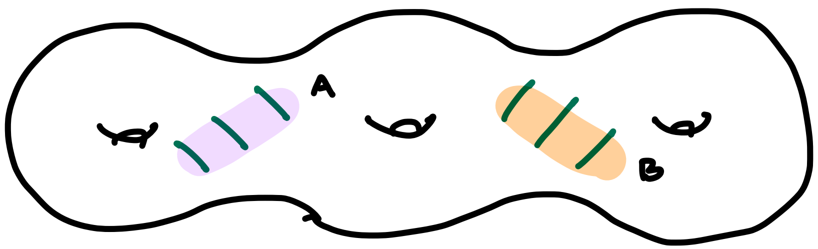

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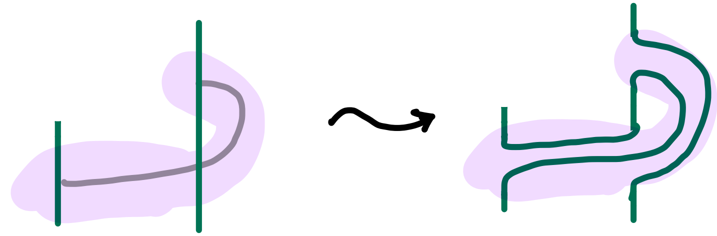

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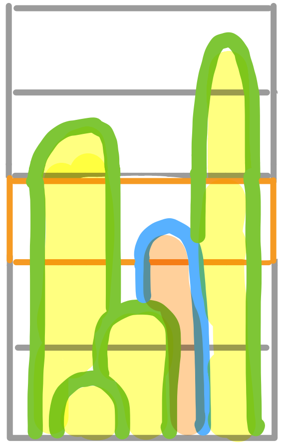

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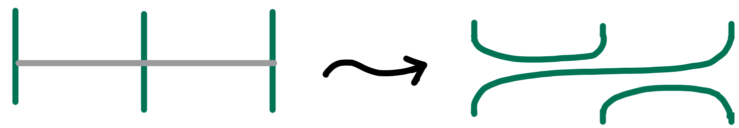
INTERLUDE - STACKING BYPASSES



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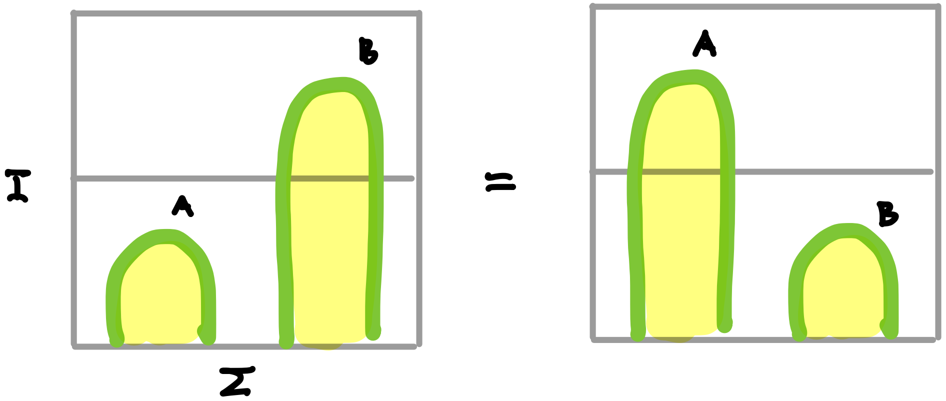

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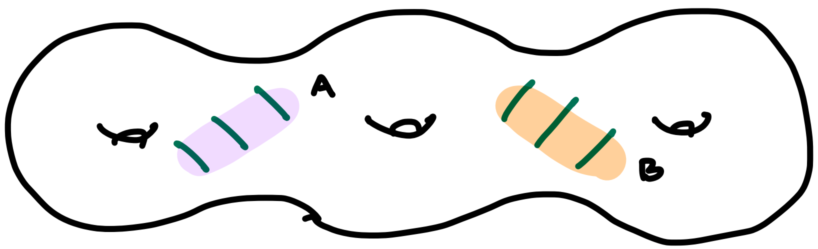

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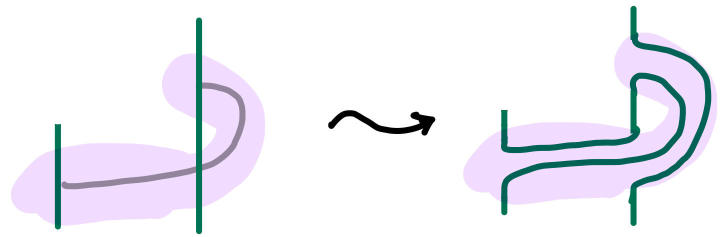

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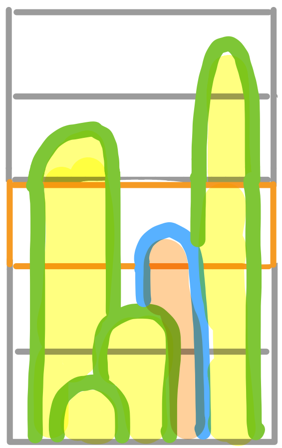

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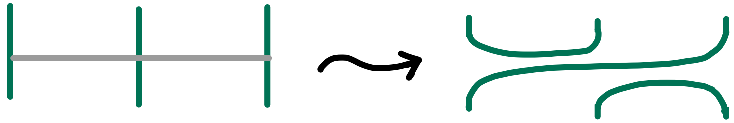
STILL GET SAME ξ

INTERLUDE - STACKING BYPASSES



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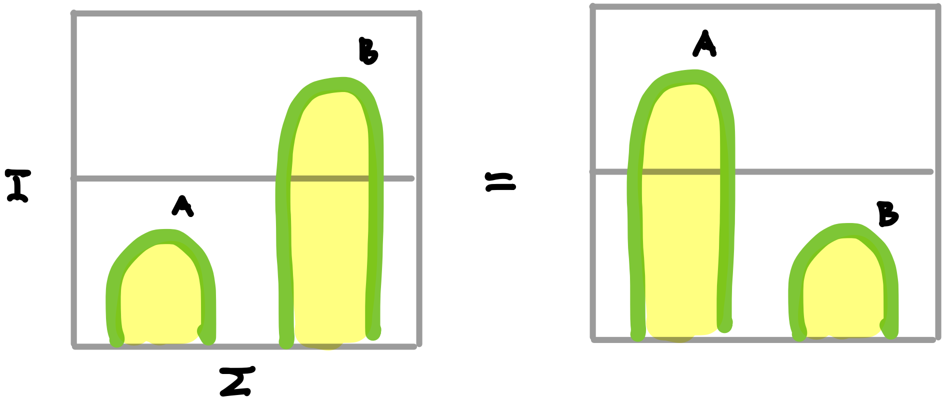
COMMUTATION



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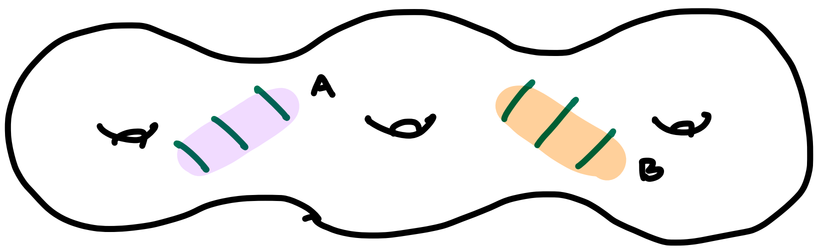
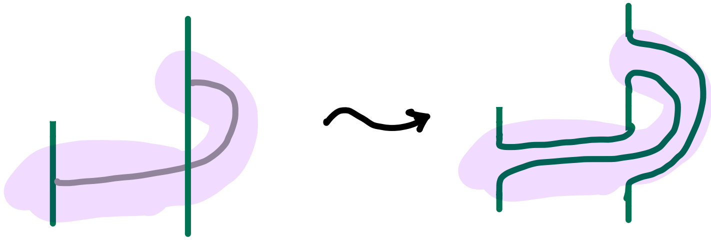


TRIVIAL BYPASS



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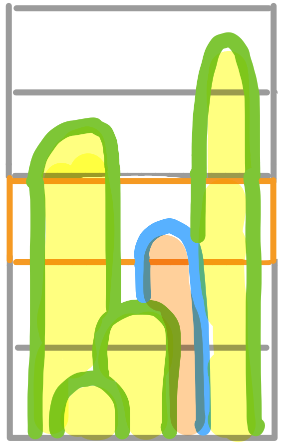


← TRIVIAL

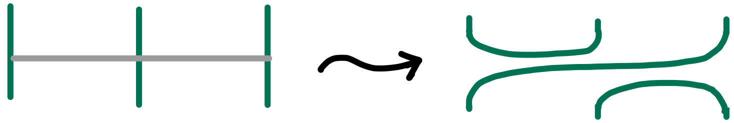
...

STILL GET SAME ξ

THM (TIAN, BREEN-HONDA-HUANG) ON $\Sigma \times I$ ANY TWO DECOMPOSITION OF ξ INTO STACKS OF BYPASS SLICES ARE RELATED VIA

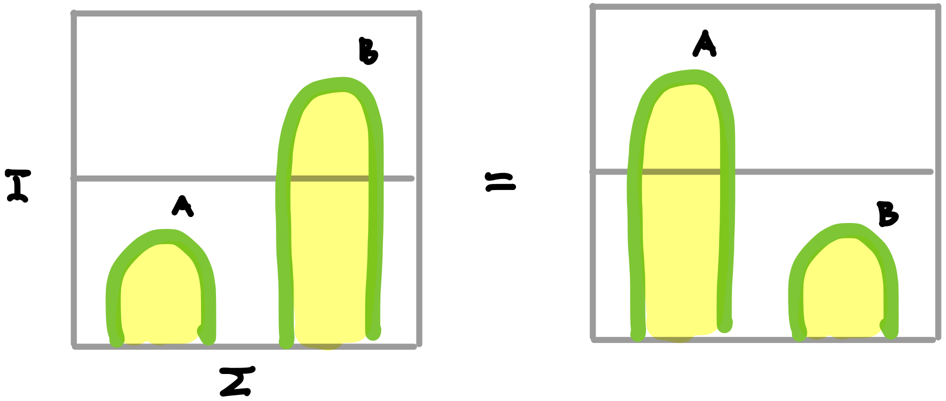
- ADDING A TRIVIAL BYPASS
- COMMUTING DISJOINT BYPASSES

INTERLUDE - STACKING BYPASSES



THE FOLLOWING MOVES DO NOT CHANGE ξ

COMMUTATION



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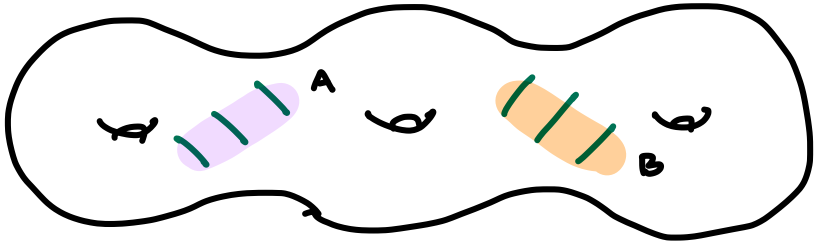
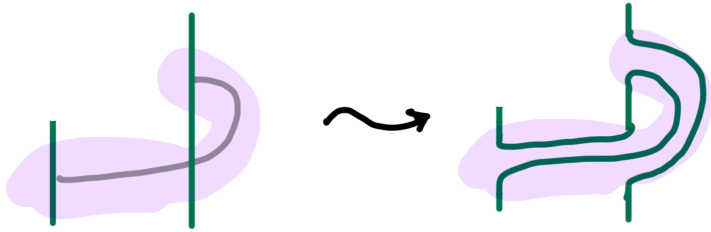


TRIVIAL BYPASS



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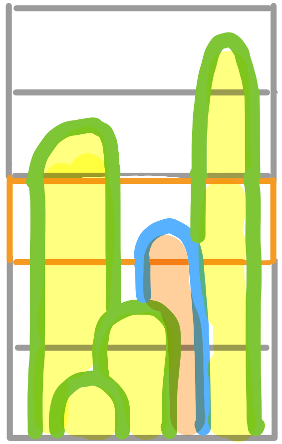

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...

STILL GET SAME ξ

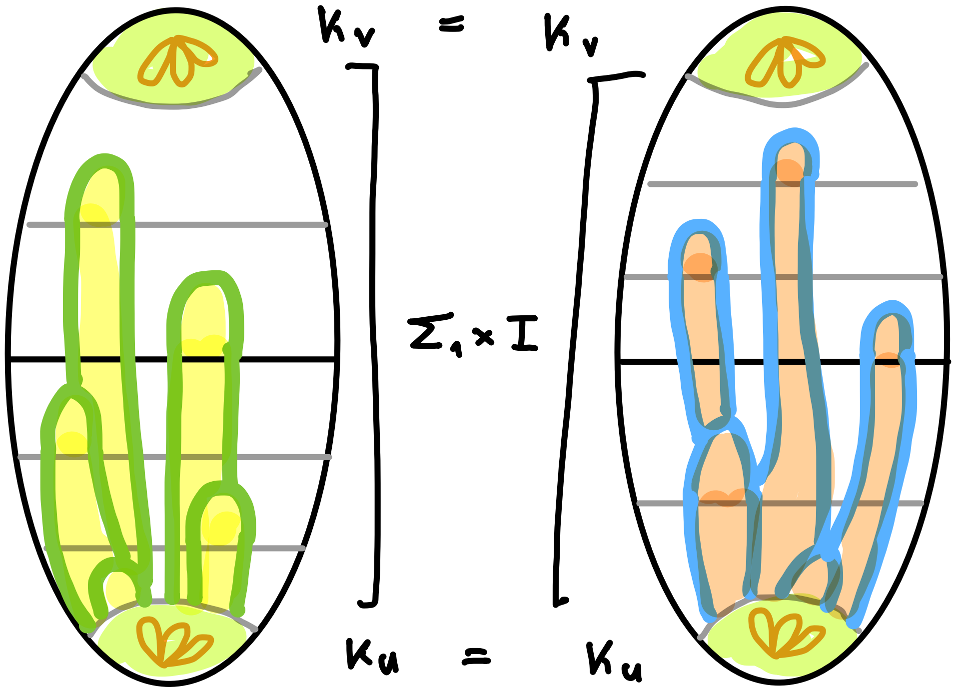
THM (TIAN, BREEN - HONDA - HUANG) ON $\Sigma \times I$ ANY TWO DECOMPOSITION OF ξ INTO STACKS OF BYPASS SLICES ARE RELATED VIA

- ADDING A TRIVIAL BYPASS
- COMMUTING DISJOINT BYPASSES

] BYPASS MOVES

STEP 3

$\mathcal{H}_2 = \hat{\mathcal{H}}(U_K, V_K, \mathcal{B})$ $\quad$ $\mathcal{H}_2' = \hat{\mathcal{H}}(U_K, V_K, \mathcal{B}')$



WHERE $\mathcal{B}$ & $\mathcal{B}'$ ARE TWO BYPASS - DECOMPOSITIONS OF $\xi|_{M-(N(U_K)\cup N(V_K))}$

$\xRightarrow[\text{B-H-H}]{\text{TIAN}}$ $\mathcal{B}$ & $\mathcal{B}'$ ARE RELATED VIA BYPASS MOVES

$$\mathcal{B} = \mathcal{B}_0 \rightsquigarrow \mathcal{B}_1 \rightsquigarrow \cdots \rightsquigarrow \mathcal{B}_k = \mathcal{B}'$$

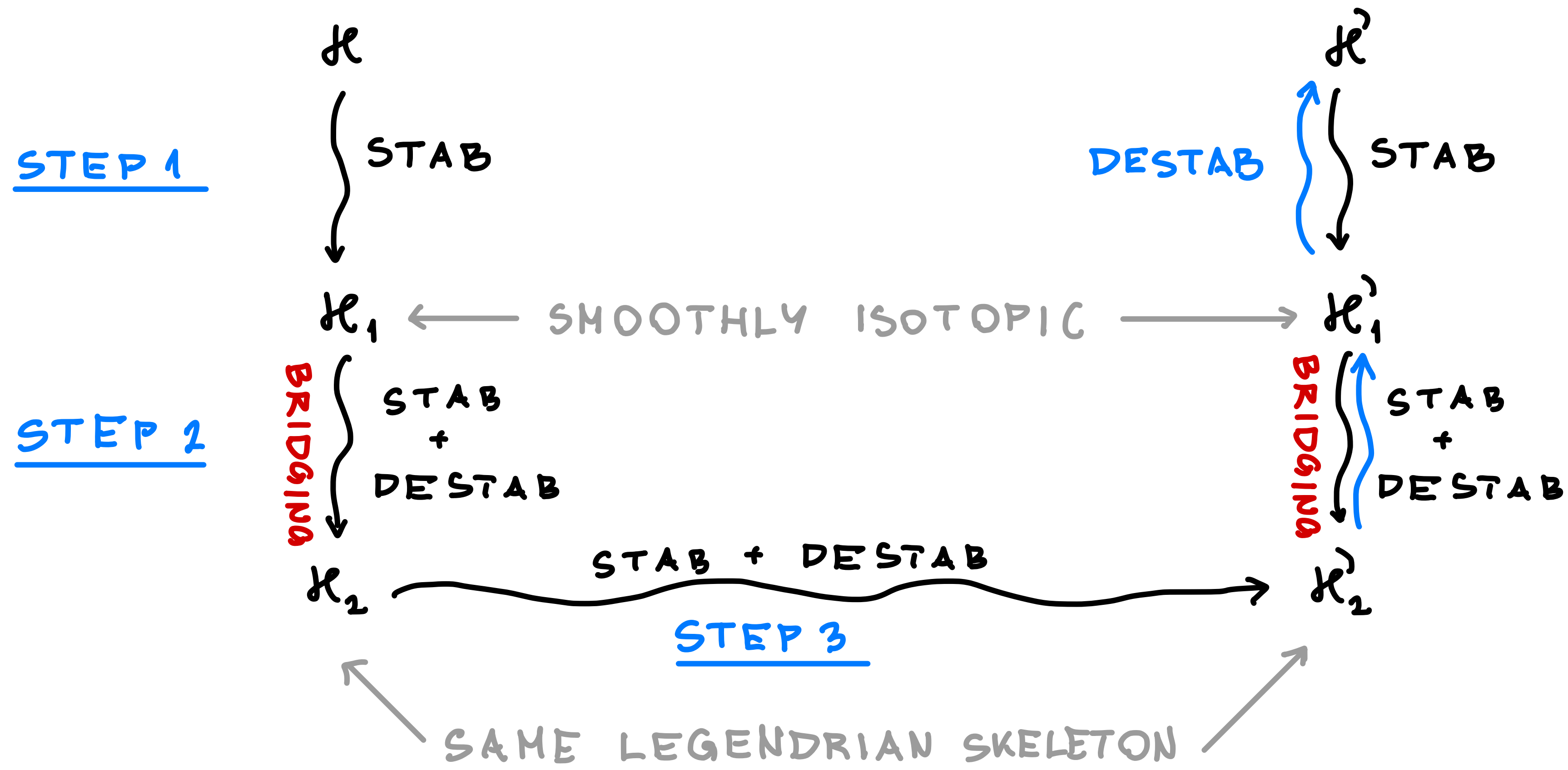
THM (L-V): BYPASS MOVES ON $\Sigma_1 \times I$ CORRESPOND TO CONTACT STAB & DESTAB OF $\hat{\mathcal{H}}$

$$\underset{\mathcal{H}_2}{\overset{\displaystyle \hat{\mathcal{H}}(U_K, V_K, \mathcal{B})}{\|}} \xrightarrow{\text{STAB+DESTAB}} \hat{\mathcal{H}}(U_K, V_K, \mathcal{B}_1) \rightsquigarrow \cdots \xrightarrow{\text{STAB+DESTAB}} \underset{\mathcal{H}_2'}{\overset{\displaystyle \hat{\mathcal{H}}(U_K, V_K, \mathcal{B}')}{\|}}$$

Summary

$$
\begin{tikzcd}
 & \mathcal{H} \arrow[d, squiggly, "\text{STAB}"] & & \mathcal{H}' \arrow[d, squiggly, "\text{STAB}"] \\
\text{STEP 1} & \mathcal{H}_1 \arrow[d, squiggly, "\text{STAB + DESTAB}", "\text{BRIDGING}"'] & \text{SMOOTHLY ISOTOPIC} \arrow[l] \arrow[r] & \mathcal{H}'_1 \arrow[d, squiggly, "\text{STAB + DESTAB}", "\text{BRIDGING}"'] \\
\text{STEP 2} & \mathcal{H}_2 \arrow[rr, squiggly, "\text{STAB + DESTAB}", "\text{STEP 3}"'] & & \mathcal{H}'_2 \\
 & & \text{SAME LEGENDRIAN SKELETON} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

SUMMARY



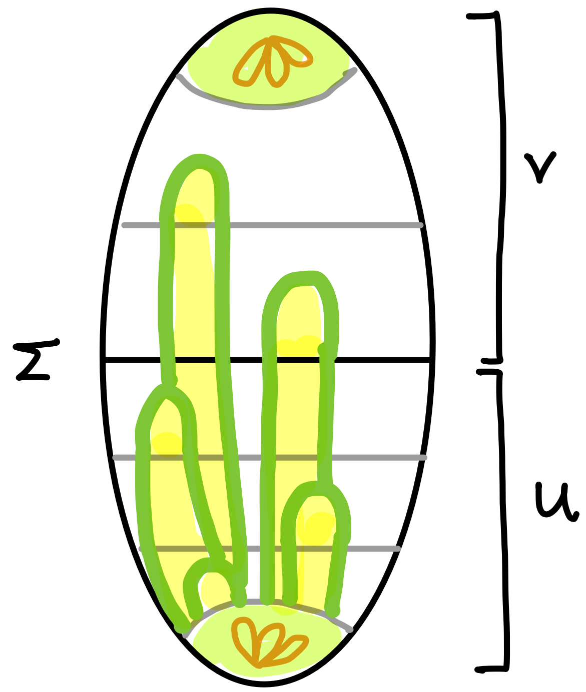
$\Longrightarrow$ $\mathcal{H}$ & $\mathcal{H}'$ ARE RELATED VIA A SEQUENCE OF CONTACT STAB. & DESTAB

$\blacksquare$

KEY POINT

IN STEP 2 WE NEEDED:

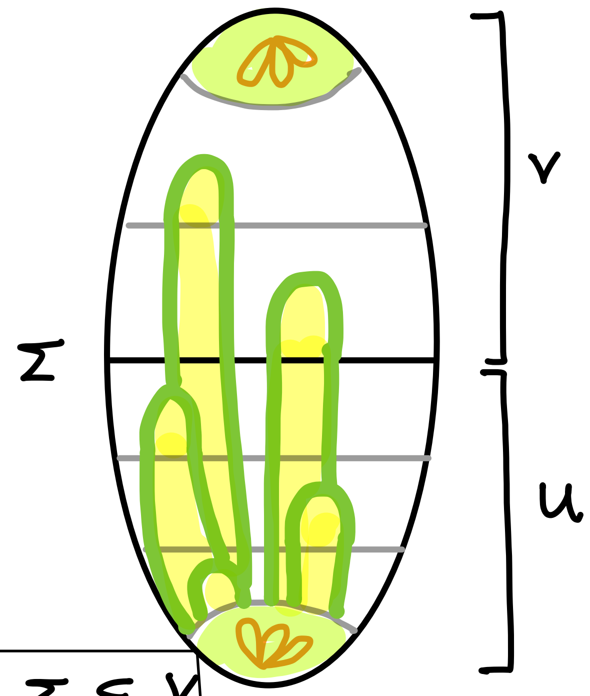
> **THM (L-V):** $\mathcal{H}$ CTCT HD $\Longrightarrow$ THE BRIDGE $\hat{\mathcal{H}}$ OF $\mathcal{H}$ IS CONNECTED TO $\mathcal{H}$ VIA CTCT STAB'S & DESTAB'S



KEY POINT

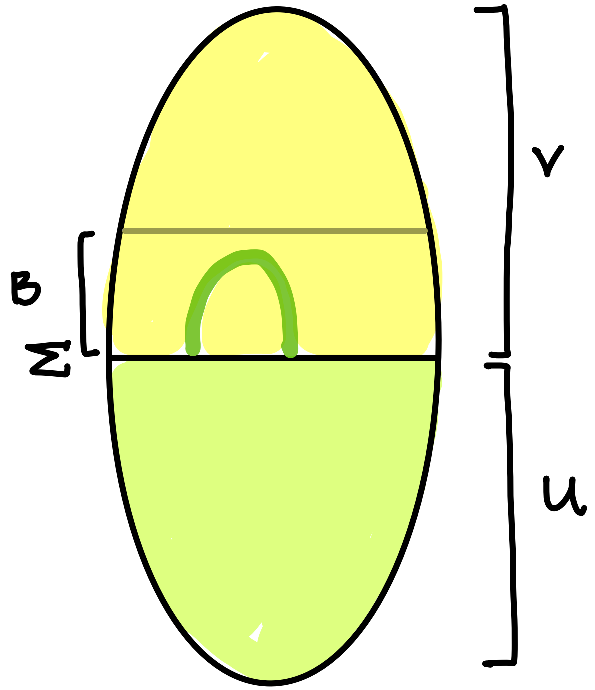
IN STEP 2 WE NEEDED:

THM (L-V): $\mathcal{H}$ CTCT HD $\Longrightarrow$ THE BRIDGE $\hat{\mathcal{H}}$ OF $\mathcal{H}$ IS CONNECTED TO $\mathcal{H}$ VIA CTCT STAB'S & DESTAB'S



AN ELEMENTARY STEP IN THE PROOF

PROP (L-V): $\mathcal{H}(M = U \cup V)$ CTCT HD, B BYPASS ONTO $\Sigma \subset V$ THEN FOR • $U' = U \cup 1\text{-}h$ • $V' = \overline{M - U'}$ THE CTCT HD $\mathcal{H}'(M = U' \cup V')$ IS CONNECTED TO $\mathcal{H}$ VIA CTCT STAB'S & DESTAB'S

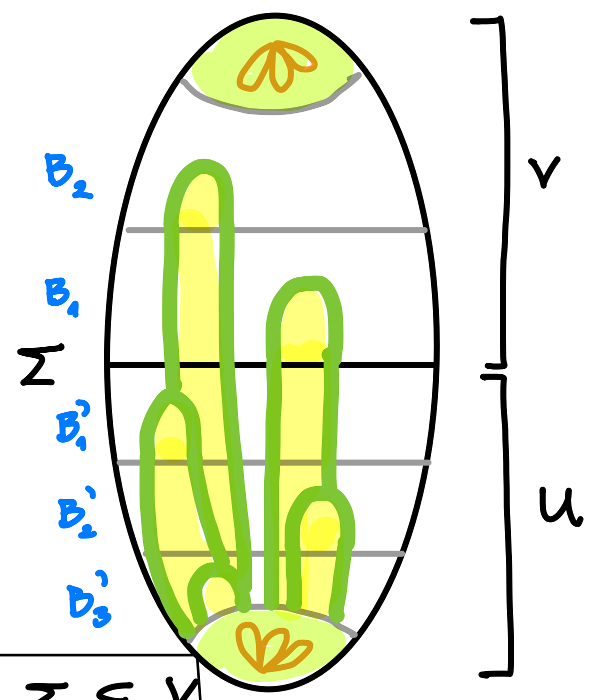


PROP $\Longrightarrow$ THM:

KEY POINT

IN STEP 2 WE NEEDED:

THM (L-V): $\mathcal{H}$ CTCT HD $\Longrightarrow$ THE BRIDGE $\hat{\mathcal{H}}$ OF $\mathcal{H}$ IS CONNECTED TO $\mathcal{H}$ VIA CTCT STAB'S & DESTAB'S

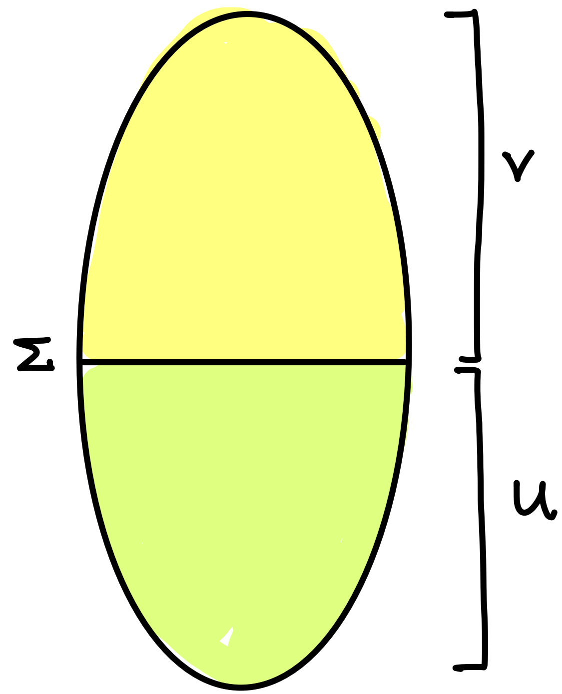


AN ELEMENTARY STEP IN THE PROOF

PROP (L-V): $\mathcal{H}(M = U \cup V)$ CTCT HD, B BYPASS ONTO $\Sigma \subset V$

THEN FOR
- $U' = U \cup 1\text{-}h$
- $V' = \overline{M - U'}$

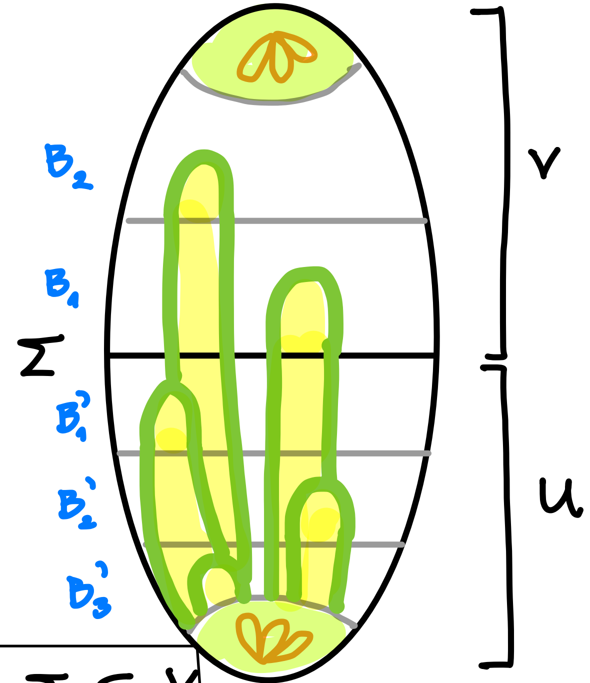
THE CTCT HD $\mathcal{H}'(M = U' \cup V')$ IS CONNECTED TO $\mathcal{H}$ VIA CTCT STAB'S & DESTAB'S



PROP $\Longrightarrow$ THM: START FROM $\mathcal{H}$

KEY POINT

IN STEP 2 WE NEEDED:



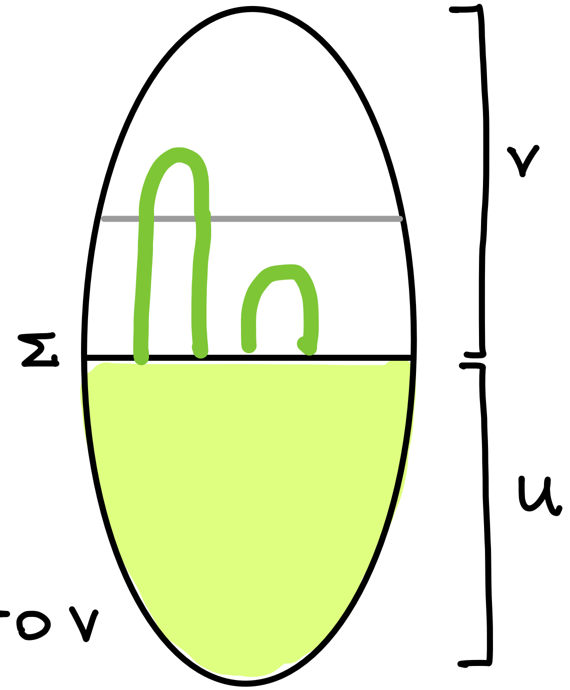
THM (L-V): $\mathcal{H}$ CTCT HD $\Longrightarrow$ THE BRIDGE $\hat{\mathcal{H}}$ OF $\mathcal{H}$ IS CONNECTED TO $\mathcal{H}$ VIA CTCT STAB'S & DESTAB'S

AN ELEMENTARY STEP IN THE PROOF

PROP (L-V): $\mathcal{H}(M=U\cup V)$ CTCT HD, B BYPASS ONTO $\Sigma\subset V$

THEN FOR
- $U' = U \cup 1\text{-h}$
- $V' = \overline{M - U'}$

THE CTCT HD $\mathcal{H}'(M=U'\cup V')$ IS CONNECTED TO $\mathcal{H}$ VIA CTCT STAB'S & DESTAB'S



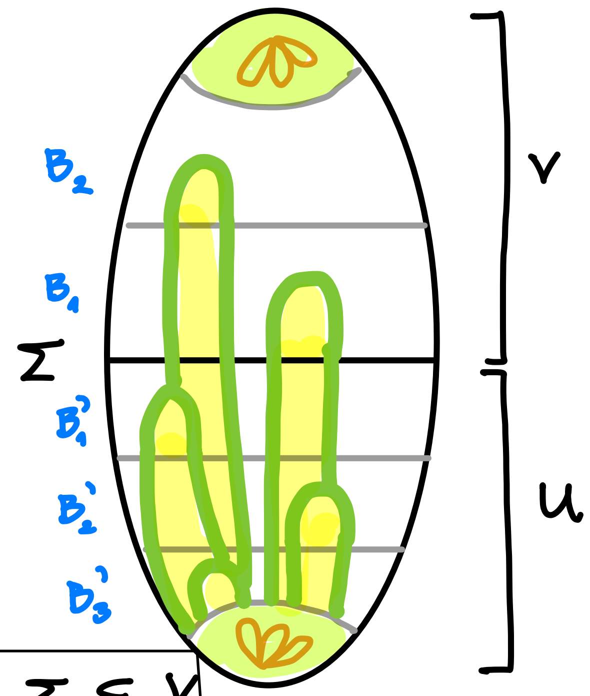
PROP $\Longrightarrow$ THM: START FROM $\mathcal{H}$

- ATTACH THE 1-h OF B_1 & B_2 TO U
- TURN UPSIDE DOWN & ATTACH THE 2-h OF B_i' TO V

KEY POINT

IN STEP 2 WE NEEDED:

THM (L-V): $\mathcal{H}$ CTCT HD $\Longrightarrow$ THE BRIDGE $\hat{\mathcal{H}}$ OF $\mathcal{H}$ IS CONNECTED TO $\mathcal{H}$ VIA CTCT STAB'S & DESTAB'S



AN ELEMENTARY STEP IN THE PROOF

PROP (L-V): $\mathcal{H}(M = U \cup V)$ CTCT HD, B BYPASS ONTO $\Sigma \subset V$

THEN FOR
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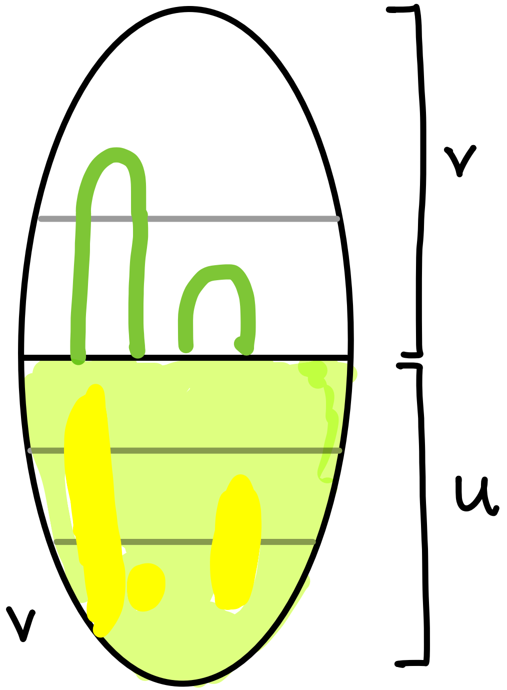


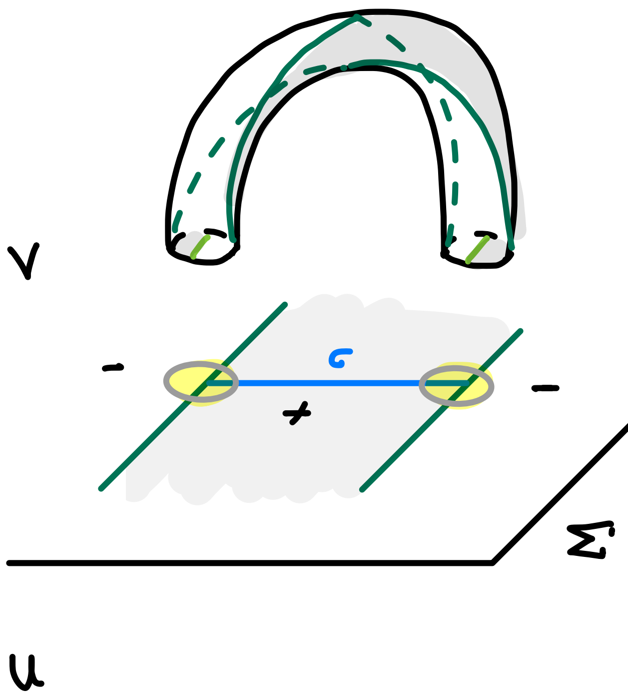
PROP $\Longrightarrow$ THM: START FROM $\mathcal{H}$

- ATTACH THE 1-h OF B_1 & B_2 TO U
- TURN UPSIDE DOWN & ATTACH THE 2-h OF B_i' TO V

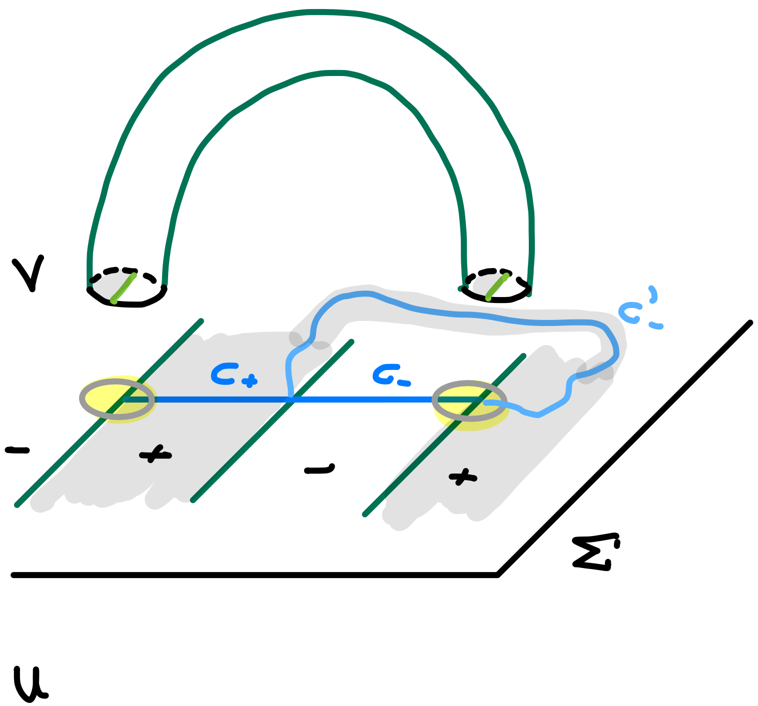
$\rightsquigarrow$ GET $\mathcal{H}'$ ■

Adding Bypass 1-handle

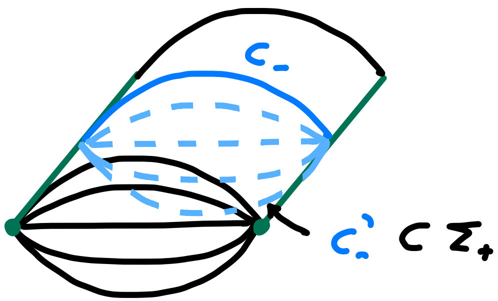
STABILISATION



BYPASS 1-HANDLE



RECALL: $U \cong \Sigma_- \times I/\sim$:



SO THE BYPASS 1-HANDLE ADDITION ALONG $c_+ \cup c_-$ = STAB ALONG

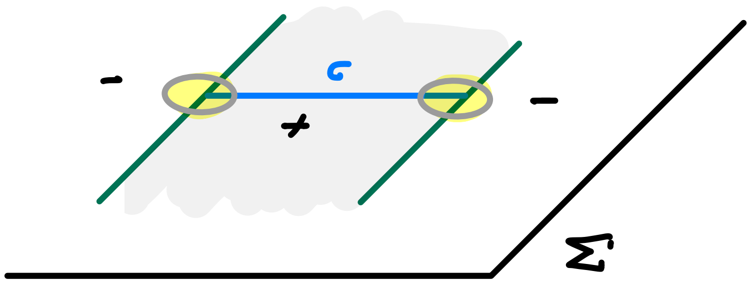
$\parallel$

STABILISATION ALONG $c_+ \cup c_-'$

Adding Bypass 1-handle

Stabilisation

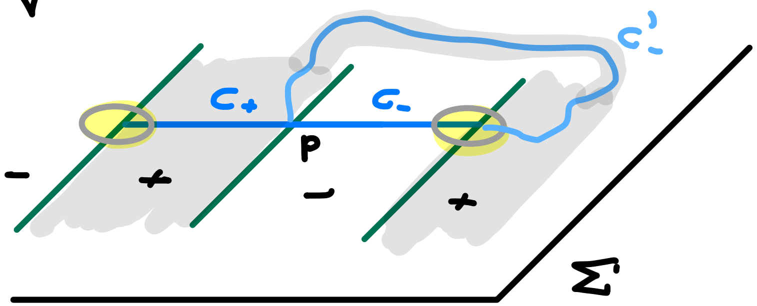
V



U

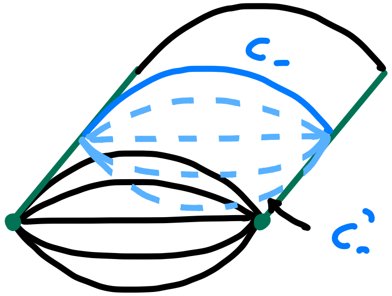
Bypass 1-handle

V



U

RECALL: $U \cong \Sigma_- \times I/\sim$:

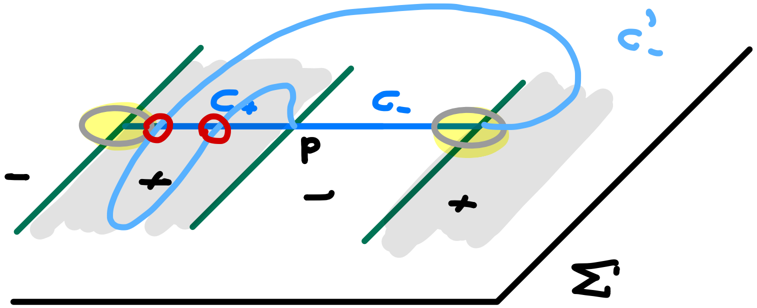


$c_-' \subset \Sigma_+$

SO THE BYPASS 1-HANDLE ADDITION ALONG $c_+ \cup c_-$ = STAB ALONG

$\|$

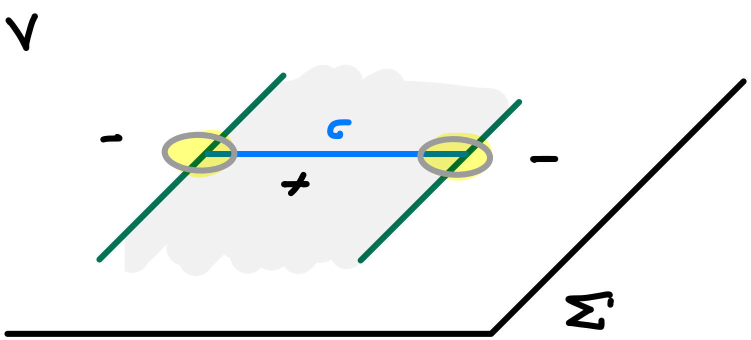
STABILISATION ALONG $c_+ \cup c_-'$



ONLY WORKS IF $c_-' \cap c_+ = p$

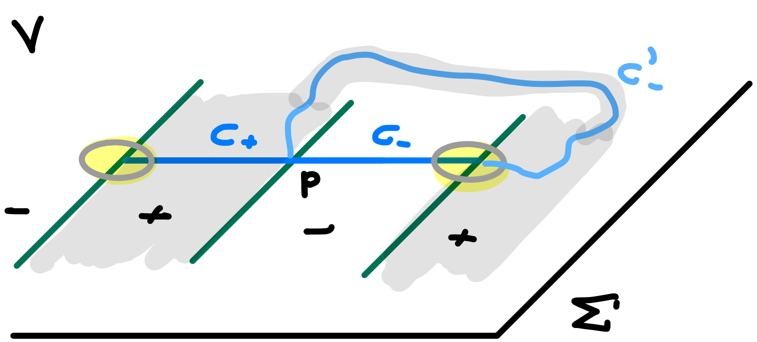
Adding Bypass 1-handle

Stabilisation



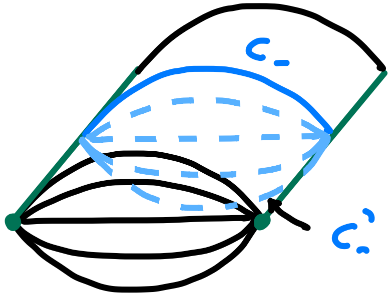
U

Bypass 1-handle



U

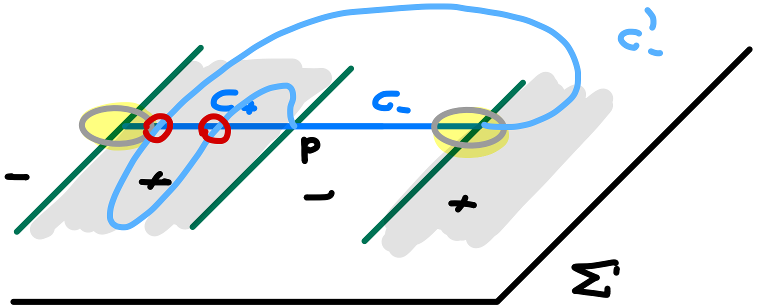
RECALL: $U \cong \Sigma_- \times I/\sim$:



$c'_- \subset \Sigma_+$

SO the bypass 1-handle addition along $c_+ \cup c_-$ = stab along
$\|$
stabilisation along $c_+ \cup c'_-$

ONLY WORKS IF $c'_- \cap c_+ = p$



WE SHOW: after more (de)stab we can make sure $c'_- \cap c_+ = p$

Adding Bypass 1-handle

WE GET

PROP (L-V): $\mathcal{H}(M = U \cup V)$ CTCT HD, B BYPASS ONTO $\Sigma \subset V$

THEN FOR • $U' = U \cup 1\text{-}h$

• $V' = \overline{M - U'}$

THE CTCT HD $\mathcal{H}'(M = U' \cup V')$ IS CONNECTED TO $\mathcal{H}$

VIA CTCT STAB'S & DESTAB'S

COR (L-V): $\mathcal{H}(M = U \cup V)$ CTCT HD, B BYPASS ONTO $\Sigma \subset V$

IF $U' = U \cup B$ IS TIGHT, THEN FOR $V' = \overline{M - U}$

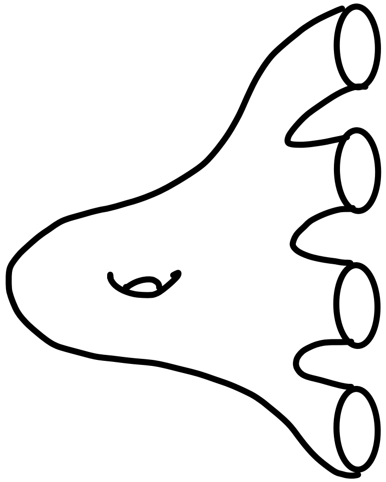
THE CTCT HD $\mathcal{H}'(M = U' \cup V')$ IS CONNECTED TO $\mathcal{H}$

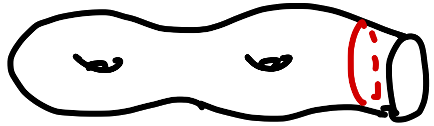
VIA CTCT STAB'S & DESTAB'S

THIS GIVES A SIMPLER PROOF FOR GIROUX CORRESPONDENCE

FOR TIGHT CONTACT STRUCTURES

FURTHER DIRECTIONS





WHAT'S NEXT?

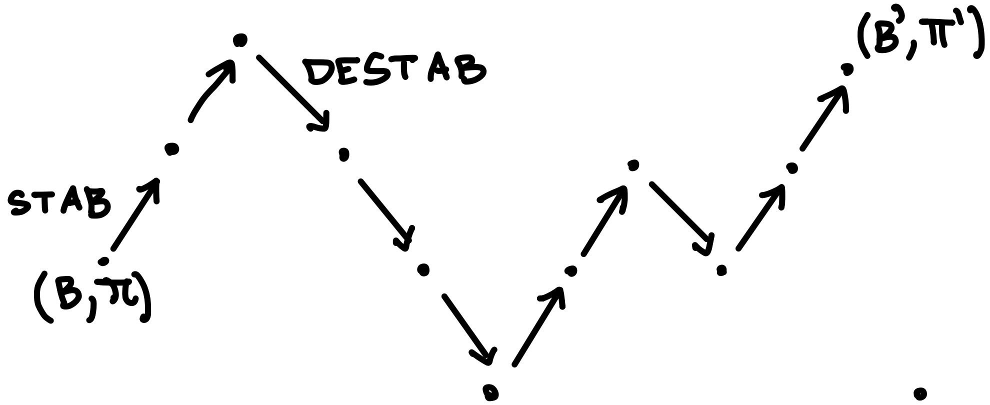
FEELS LIKE END OF A LONG STORY

BUT AS ALWAYS THERE IS A LOT TO DO:

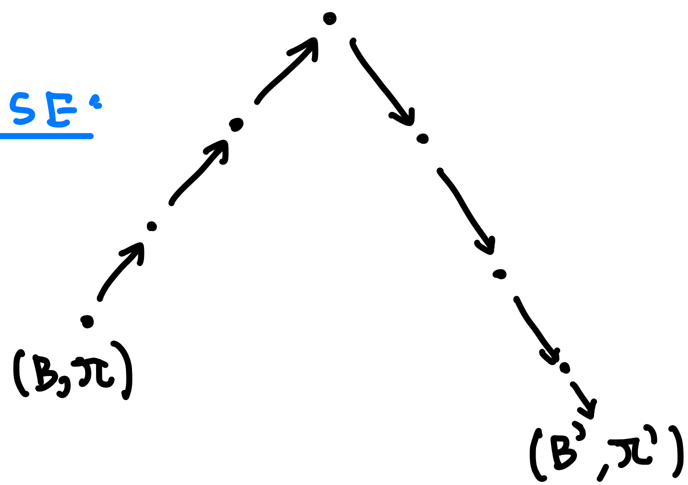
- (HOPEFULLY) MINOR ISSUE:

  COMMON STABILISATION VS. SEQUENCE OF (DE)STABILISATIONS

WHAT WE PROVED:



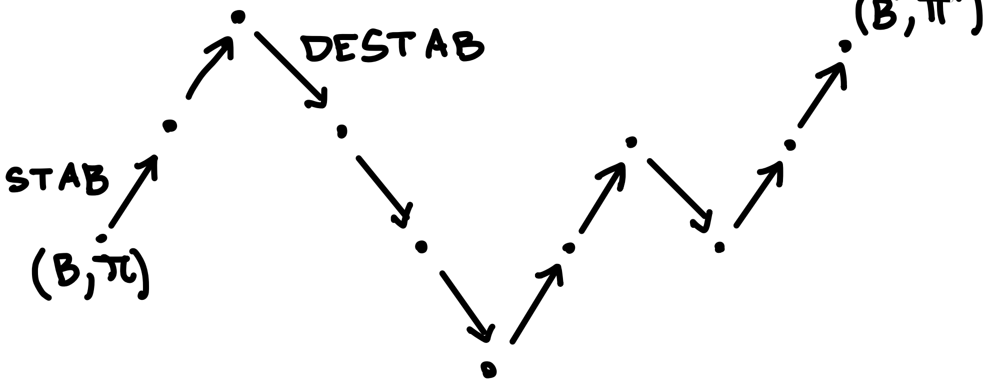
WHAT SOME OF THE APPLICATIONS USE:



ARE THEY EQUIVALENT?

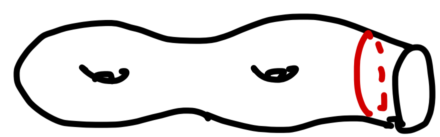
WHAT'S NEXT?

- **HOW MANY STABILISATIONS DO WE NEED?**



- ETNYRE: **DOES EVERY CONTACT STRUCTURE HAVE A GENUS 1 OPEN BOOK?**

  - OVERTWISTED CONTACT STRUCTURES HAVE PLANAR (= GENUS 0) OPEN BOOKS (ETNYRE)
  - THERE ARE CONTACT STRUCTURES THAT DO NOT HAVE PLANAR OPEN BOOKS (ETNYRE)
  - POSSIBLE COUNTEREXAMPLE (HASSOT)



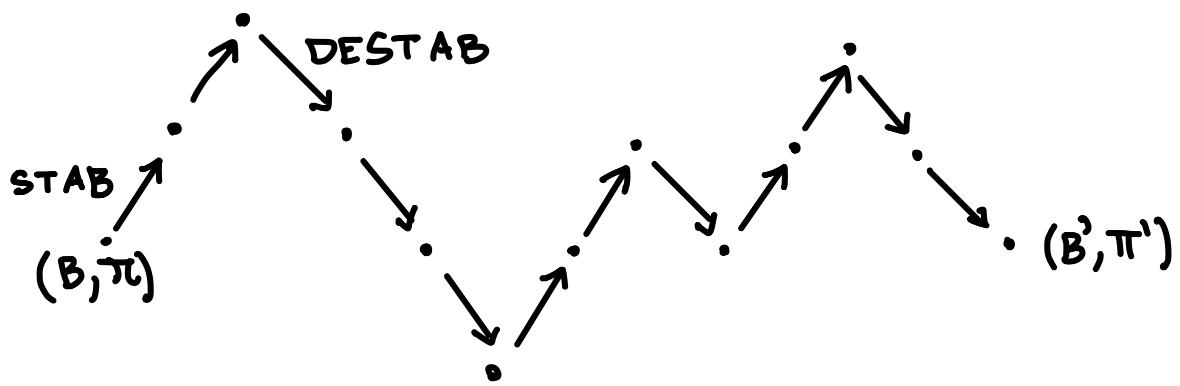
WHAT'S NEXT?

THM (WAND): LEGENDRIAN SURGERY PRESERVS TIGHTNESS

- THE PROOF RELIES ON AN EQUIVALENT CHARACTERISATION OF TIGHTNESS IN TERMS OF OPEN BOOKS
- THIS CHARACTERISATION IS GIVEN IN A SEQUENCE OF COMBINATORIAL DEFINITIONS THAT TAKE UP MULTIPLE PAGES

IS THERE A SIMPLER PROOF USING CONTACT HEEGAARD DECOMPOSITIONS?

- MOVES BETWEEN OPEN BOOKS OF THE SAME GENUS/ EULER CHARACTERISTIC



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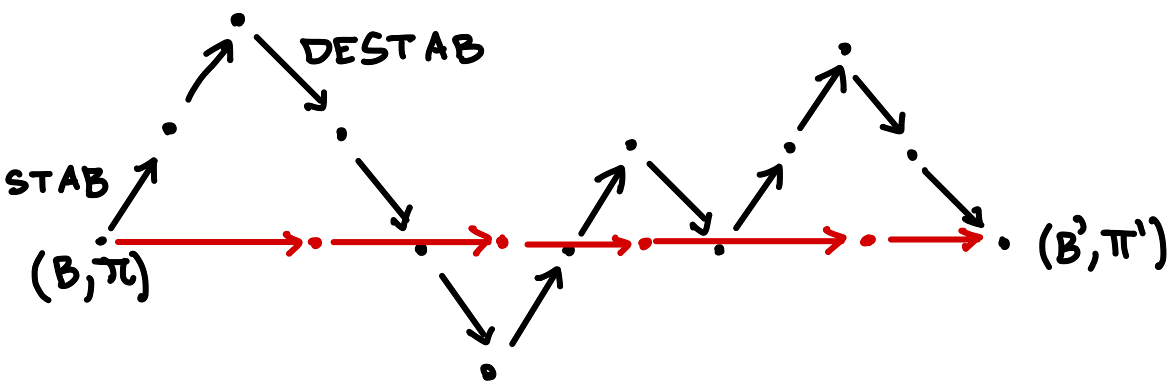

THANKS FOR YOUR ATTENTION!