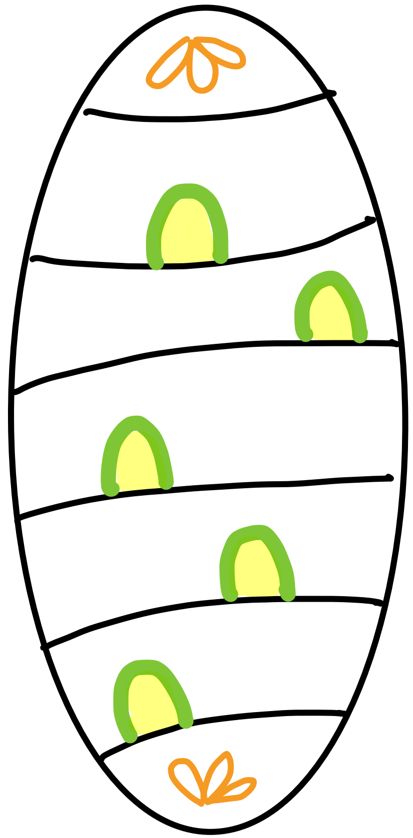
SRNI - LECTURE 2

DESCRIBING CONTACT STRUCTURES

- CONTACT CELL DECOMPOSITIONS
- CONVEX SURFACE THEORY - BYPASSES
- CONTACT HEEGAARD SPLITTINGS (PROOF OF EXISTENCE)
- OPEN BOOK DECOMPOSITIONS
- OPEN BOOK DECOMPOSITIONS & CONTACT HEEGAARD SPLITTINGS

THE GIROUX CORRESPONDENCE VIA CONVEX SURFACES

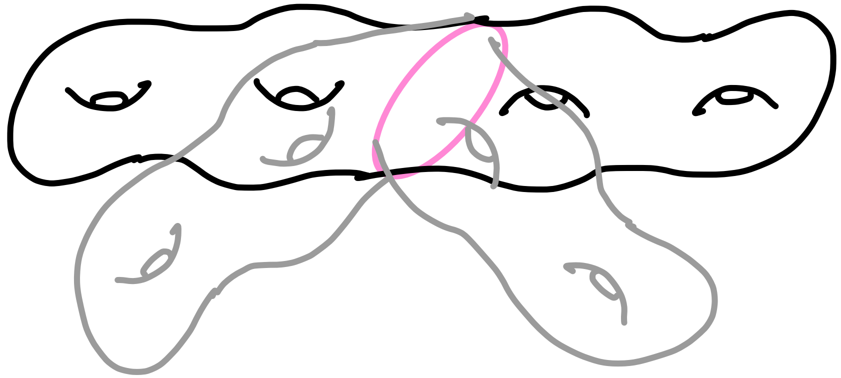


VERA VÉRTESI

JOINT WORK WITH JOAN LICATA

UNIVERSITY OF VIENNA

LECTURE 2



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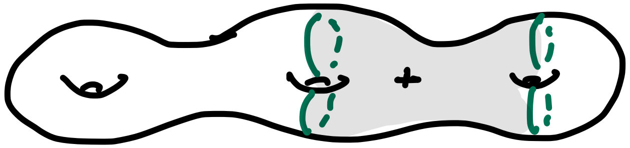
LAST TIME (GIROUX)

- $\Sigma \hookrightarrow M$ IS **CONVEX** IF $\exists$ CONTACT VECTORFIELD X : $\Sigma \pitchfork X$

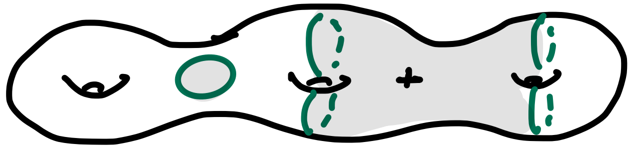


$$\Sigma = \underbrace{\Sigma_+}_{\alpha(X) > 0} \cup_{\underbrace{\Gamma}_{\alpha(X) = 0}} \underbrace{\Sigma_-}_{\alpha(X) < 0}$$

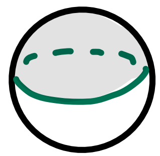
→ GENERIC $\Sigma \hookrightarrow M$ IS CONVEX

→ $\xi|_{N(\Sigma)}$ IS DETERMINED BY $\Gamma \subset \Sigma$

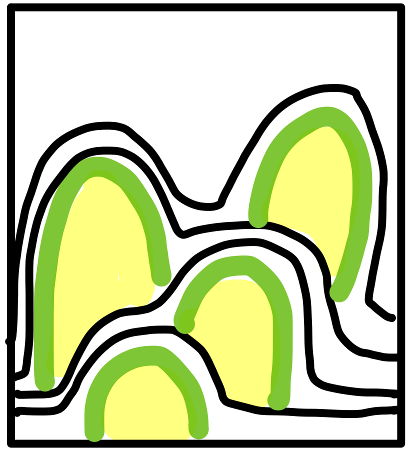
- (M, ξ) **OVERTWISTED** $\Leftrightarrow$ $\exists$ $\Sigma \hookrightarrow M$ CONVEX W/ Σ_+ OR Σ_- CONTAINING A DISC



TIGHT $\neq$ OVERTWISTED

- $\exists !$ TIGHT CONTACT STRUCTURE ON D^3 WITH CONVEX BDRY: 

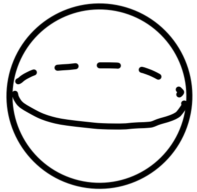
CONTACT CELL--DECOMPOSITIONS



Handle Decompositions of Smooth 3-Manifolds

0-HANDLE

$h^0 = D^0 \times D^3$

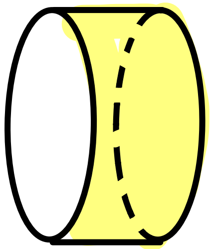


1-HANDLE

$h^1 = D^1 \times D^2$

2-HANDLE

$h^2 = D^2 \times D^1$



3-HANDLE

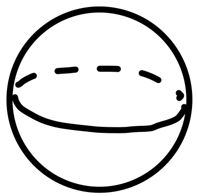
$h^3 = D^3 \times D^0$

THM (MORSE): ANY 3-MANIFOLD CAN BE OBTAINED VIA SUCCESIVELY ATTACHING HANDLES ONTO EACH OTHER ALONG $\partial D^i \times D^{3-i}$ (HANDLE DECOMPOSITION)

HANDLE DECOMPOSITIONS OF SMOOTH 3-MANIFOLDS

0-HANDLE

$h^0 = D^0 \times D^3$



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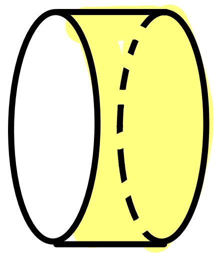
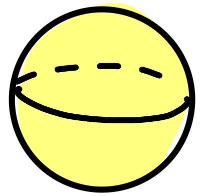


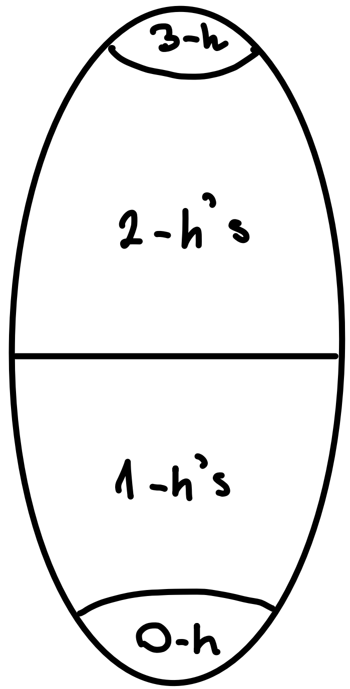
3-HANDLE

$h^3 = D^3 \times D^0$



THM (MORSE): ANY 3-MANIFOLD CAN BE OBTAINED VIA SUCCESIVELY ATTACHING HANDLES ONTO EACH OTHER ALONG $\partial D^i \times D^{3-i}$ (**HANDLE DECOMPOSITION**)

MOREOVER THE INDICES OF HANDLES CAN BE ASSUMED TO BE IN INCREASING ORDER



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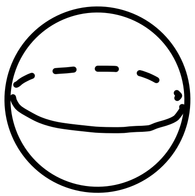

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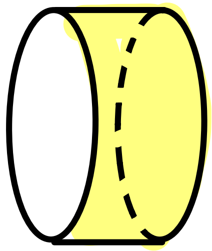

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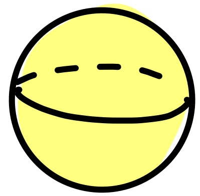

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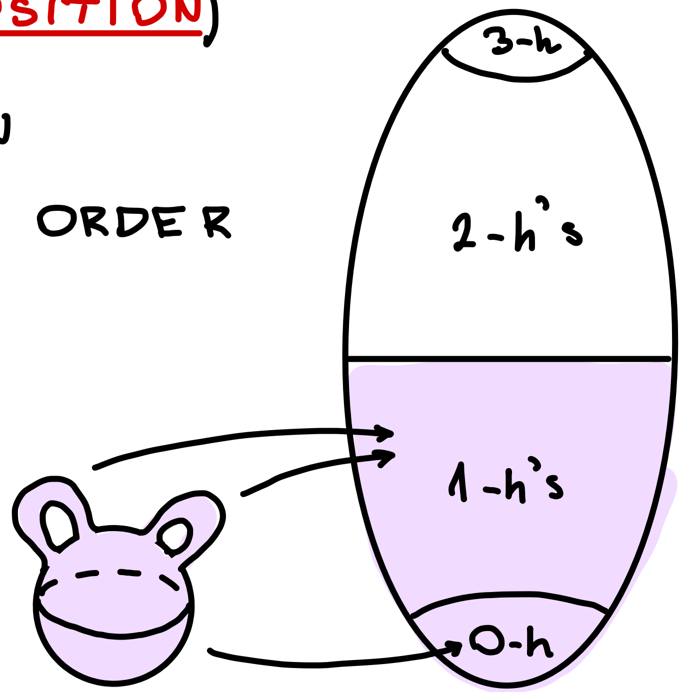
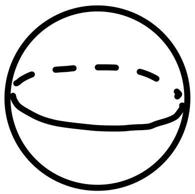


HANDLEBODY

HANDLE DECOMPOSITIONS OF SMOOTH 3-MANIFOLDS

0-HANDLE

$h^0 = D^0 \times D^3$



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$h^1 = D^1 \times D^2$

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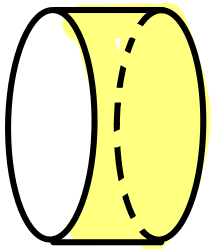

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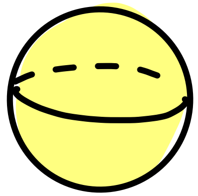

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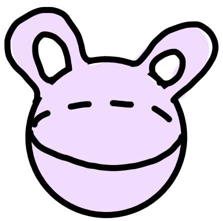
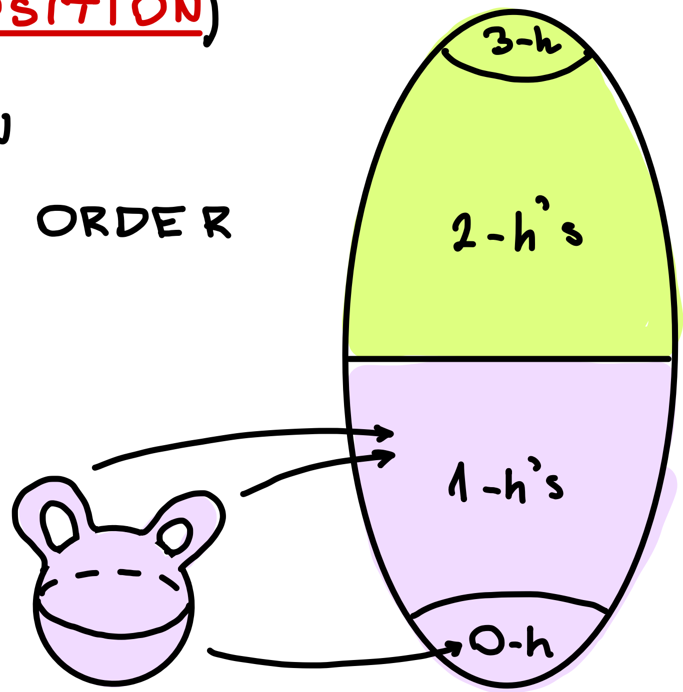
COR: $M =$  $\cup_\Sigma$ 



HANDLEBODY

HANDLE DECOMPOSITIONS OF SMOOTH 3-MANIFOLDS

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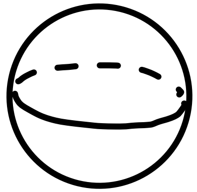

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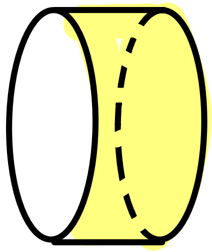

3-HANDLE

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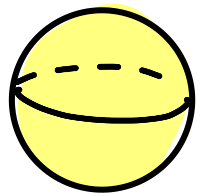

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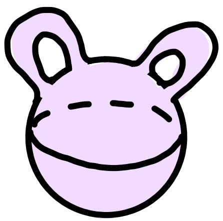
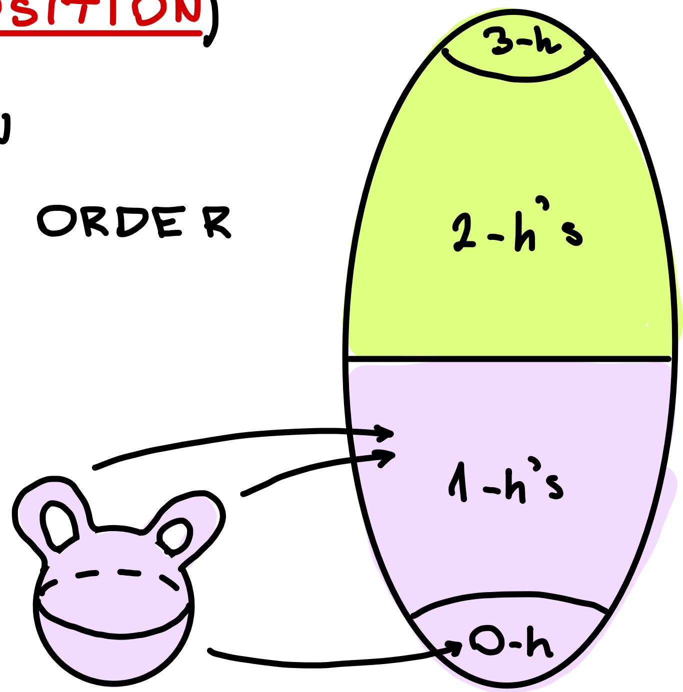
COR: M =  $\cup_\Sigma$ 



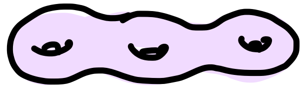
HANDLEBODY

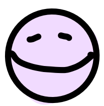
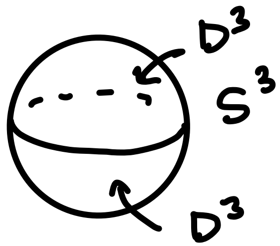
> EVERY 3-MANIFOLD ADMITS A **HEEGAARD DECOMPOSITION**

HEEGAARD DECOMPOSITIONS

> **THM** (ALEXANDER) EVERY 3-MANIFOLD ADMITS A
> **HEEGAARD DECOMPOSITION**

$M = U \cup V$

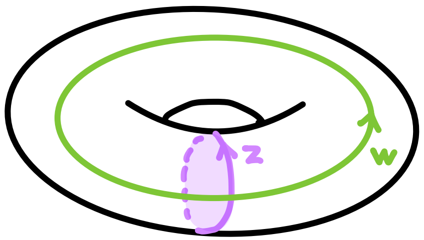
HANDLEBODIES: 

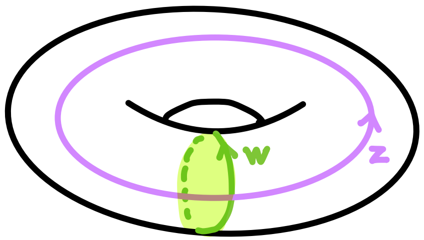
E.G.:
- $S^3 =$  $\cup$  (SCHEMATIC 

)

- $S^3 = \{|z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$

$$= \{|z|^2 \leq 1/2\} \cup \{|w|^2 \leq 1/2\}, \qquad \Sigma = \{|z| = |w| = 1/2\} \cong S^1 \times S^1$$



$\cup$


ORIGIN - MORSE FUNCTIONS : M SMOOTH (3-) MANIFOLD

DEF.: $f: M \longrightarrow \mathbb{R}$ SMOOTH FUNCTION IS MORSE IF ALL CRITICAL POINTS OF f ARE NONDEGENERATE

$\boxed{\nabla f \neq 0}$

$\boxed{\text{Hess}(f) \text{ IS NONDEGENERATE}}$

Origin - Morse Functions : M smooth (3-)manifold

DEF.: $f: M \longrightarrow \mathbb{R}$ smooth function is **MORSE** if all **CRITICAL POINTS** of f are **NONDEGENERATE**

$\boxed{\nabla f \neq 0}$

$\boxed{\text{Hess}(f) \text{ is nondegenerate}}$

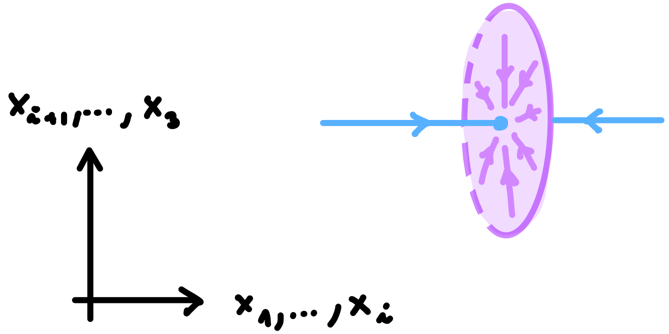
FACTS:

- Morse functions are C^∞-generic
- Local model near a Morse-critical point:

INDEX

$$-\sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{3} x_j^2$$

$\rightsquigarrow$ flow of ∇f



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FACTS:
- MORSE FUNCTIONS ARE C^∞-GENERIC
- LOCAL MODEL NEAR A MORSE-CRITICAL POINT:

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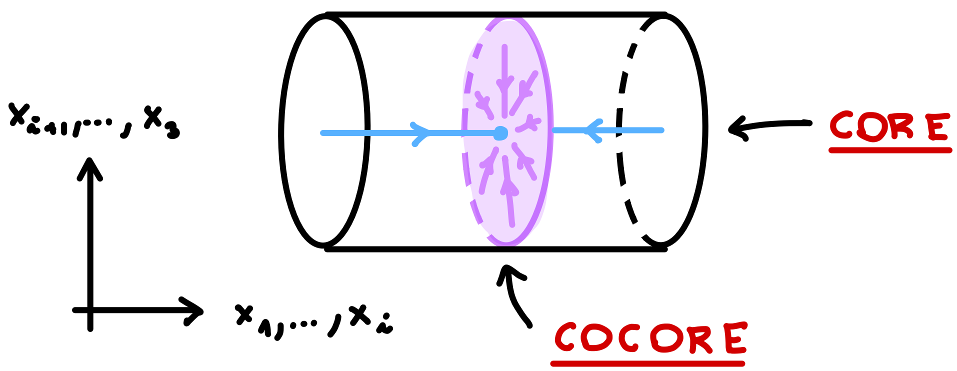

- INDEX-i CRITICAL POINTS $\rightsquigarrow$ $h^i = D^i \times D^{n-i}$ i-HANDLES

Origin – Morse Functions : M smooth (3-)manifold

Def.: $f: M \to \mathbb{R}$ smooth function is **Morse** if all **critical points** of f are **nondegenerate**

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Facts:

- Morse functions are C^∞-generic
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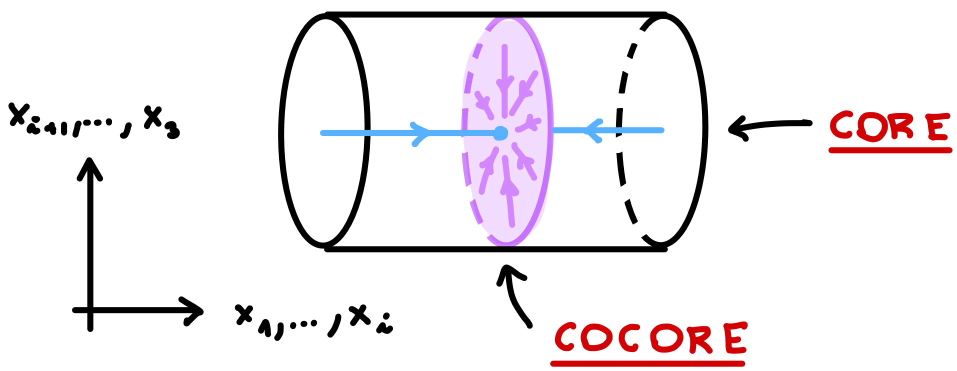


- Index-i critical points $\rightsquigarrow$ $h^i = D^i \times D^{n-i}$ i-handles

Rearranging critical points: Moving index i-handles under index i'-handles

Uses transversality & that $\overset{i}{\text{Dim}(\text{core})} + \overset{3-i'}{\text{Dim}(\text{cocore})} < 3$

$$\Updownarrow$$

$\boxed{i < i'}$

CONTACT HANDLES

0-HANDLE

$D^0 \times D^3$

h^0

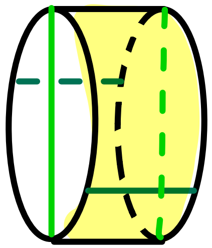
1-HANDLE

$D^1 \times D^2$

h^1

2-HANDLE

$D^2 \times D^1$



h^2

3-HANDLE

$D^3 \times D^0$

h^3

AFTER ROUNDING THE EDGES EACH BECOMES 

& BY ELIASHBERG'S THM THIS ADMITS A UNIQE TIGHT CONTACT STRUCTURE

CONTACT HANDLES

0-HANDLE

$D^0 \times D^3$

(h^0, ξ^0)

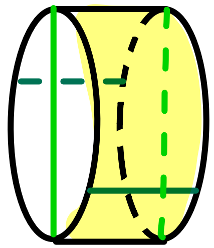
1-HANDLE

$D^1 \times D^2$

(h^1, ξ^1)

2-HANDLE

$D^2 \times D^1$



(h^2, ξ^2)

3-HANDLE

$D^3 \times D^0$

(h^3, ξ^3)

AFTER ROUNDING THE EDGES EACH BECOMES 

& BY ELIASHBERG'S THM THIS ADMITS A UNIQE TIGHT CONTACT STRUCTURE

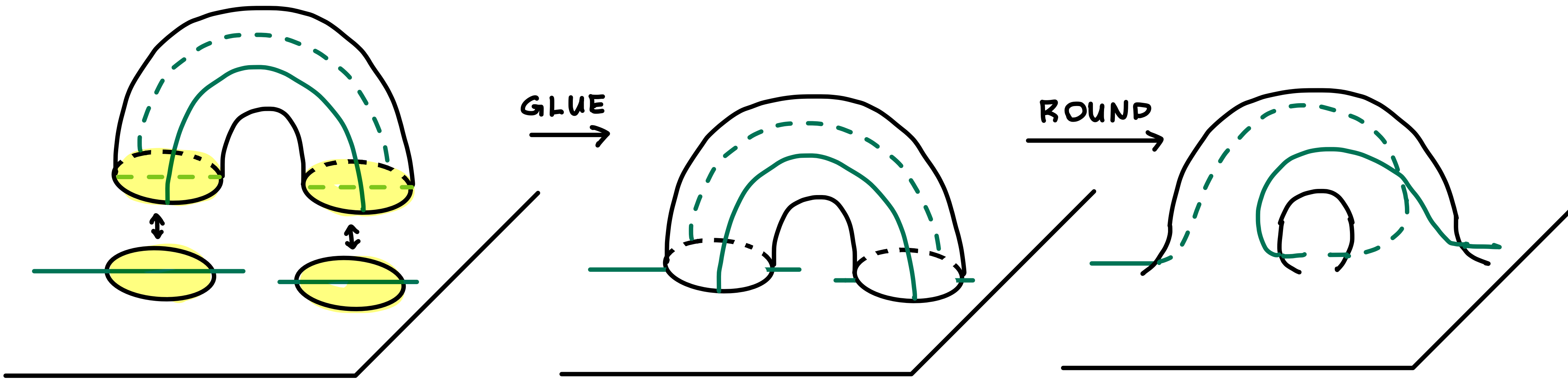
$\rightsquigarrow$ UP TO ISOTOPY WE GET WELL DEFINED CONTACT STRUCTURES ON THE h^i

ATTACHING CONTACT HANDLES

i-HANDLE $h^i = D^i \times D^{3-i}$

$$\boxed{\partial h^i = \partial D^i \times D^{3-i} \cup D^i \times \partial D^{-i}}$$

WE CAN CONSTRUCT CONTACT MANIFOLDS BY SUCCESIVELY GLUING HANDLES ALONG $\partial D^i \times D^{3-i}$:



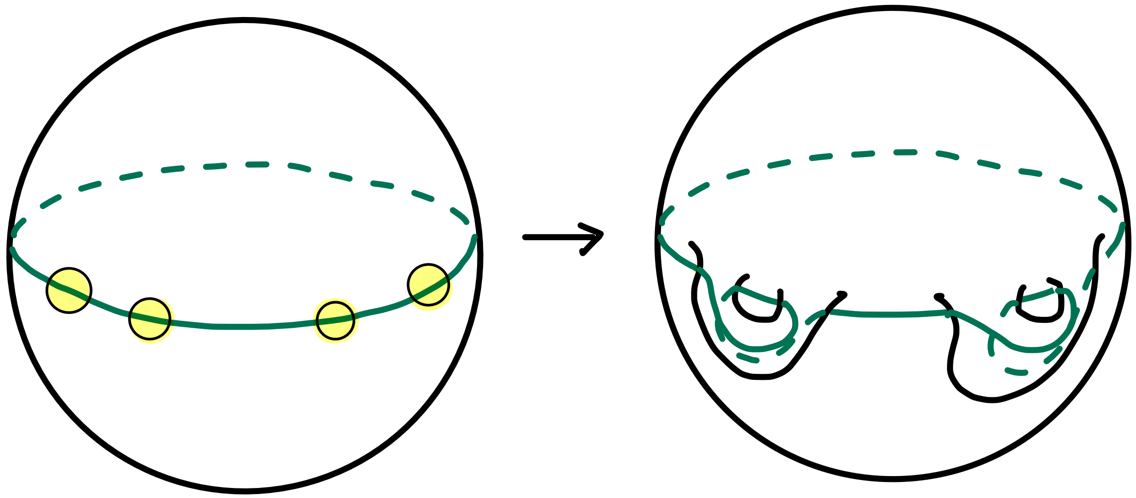
CONTACT HANDLEDECOMPOSITION

THM (GIROUX) ANY CONTACT 3-MANIFOLD ADMITS A CONTACT HANDLEDECOMPOSITION

CONTACT HANDLEBODY

0-HANDLE ∪ SOME 1-HANDLES

E.G.

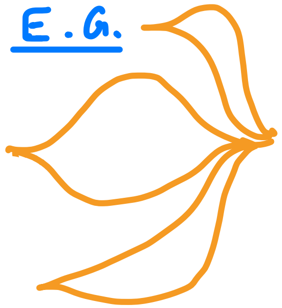


$(U, \partial U, \Gamma)$

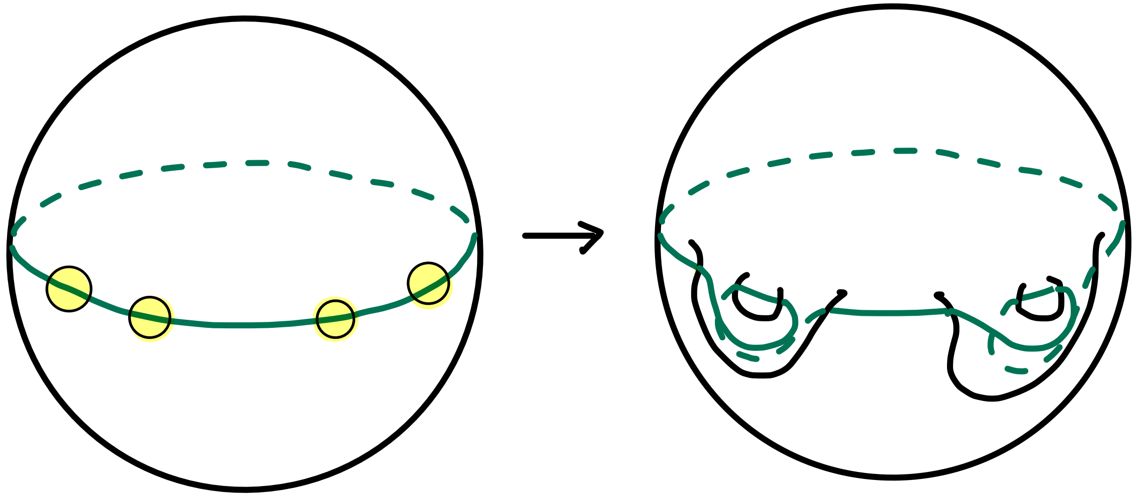
Contact Handlebody

0-HANDLE $\cup$ SOME 1-HANDLES

E.G.



NEIGHBOURHOOD OF A LEGENDRIAN GRAPH:



$(U, \partial U, \Gamma)$

Contact Handlebody

0-HANDLE $\cup$ SOME 1-HANDLES

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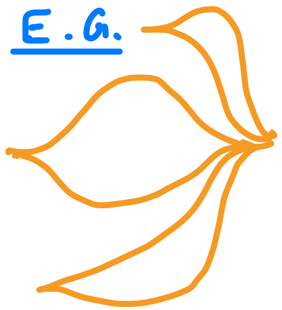

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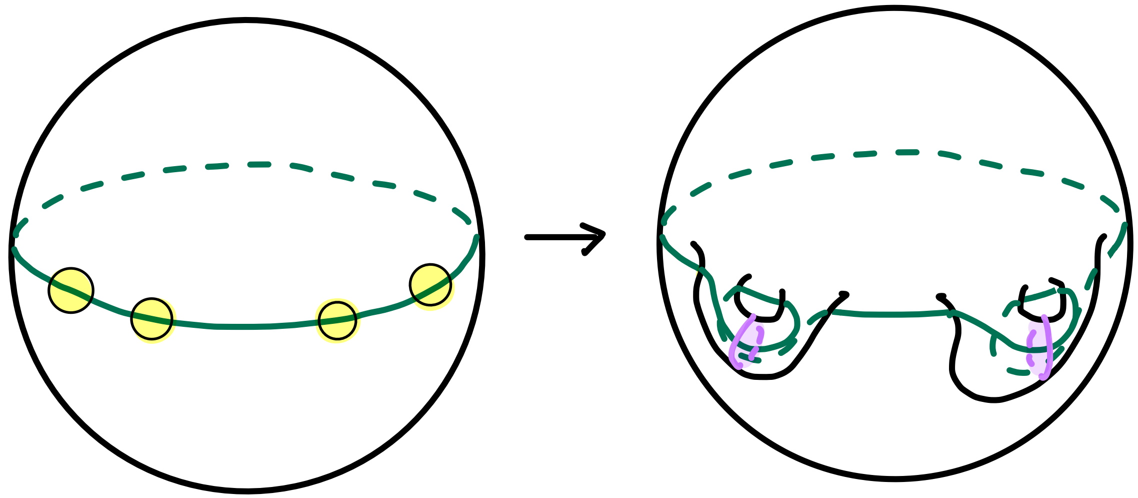

$(U, \partial U, \Gamma)$

NOTE: CONTACT HANDLEBODIES ARE PRODUCT DISC DECOMPOSABLE

RECALL: A PRODUCT DISC DECOMPOSABLE HANDLEBODY U ADMITS A UNIQUE TIGHT CONTACT STRUCTURE WITH DIVIDING CURVE Γ ON ∂U.

THUS

> **THM:** A PRODUCT DISC DECOMPOSABLE HANDLEBODY U WITH TIGHT CONTACT STRUCTURE ξ IS A CONTACT HANDLEBODY

Rearranging Contact Handles

JUST AS IN THE SMOOTH CASE

IN A CONTACT HANDLE DECOMPOSITION ONE CAN ASSUME THAT:

- CONTACT 0-h's ARE ATTACHED FIRST
- CONTACT 3-h's ARE ATTACHED LAST

Rearranging Contact Handles

JUST AS IN THE SMOOTH CASE

IN A CONTACT HANDLE DECOMPOSITION ONE CAN ASSUME THAT:

- CONTACT 0-h's ARE ATTACHED FIRST
- CONTACT 3-h's ARE ATTACHED LAST

BUT! CONTACT 1-h's CANNOT ALWAYS BE ATTACHED BEFORE CONTACT 2-h's

INTERLUDE – CONTACT MORSE FUNCTIONS

ONLY FOR FUN

(M^3, ξ) CONTACT MANIFOLD

DEF: $f: M \longrightarrow \mathbb{R}$ MORSE FUNCTION IS CONTACT IF $\exists$ X CONTACT VECTORFIELD THAT IS AN ALMOST GRADIENT FOR f

$\pitchfork$ TO LEVEL SETS & "LIKE" ∇f NEAR CRITICAL POINTS

RMK: NONCRITICAL LEVELSETS ARE CONVEX

DEF: CRITICAL SUBMANIFOLD: $C^2 = \{ X \in \xi \}$

($\approx$ UNION OF DIVIDING CURVES OF LEVELSETS)

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($\approx$ UNION OF DIVIDING CURVES OF LEVELSETS)

THM (GIROUX): $f|_C : C \longrightarrow \mathbb{R}$ IS ALSO MORSE WITH THE SAME CRITICAL POINTS AS f WITH INDICES

f	$f\|_C$
0	0
1	1
2	1
3	2

INTERLUDE - CONTACT MORSE FUNCTIONS

ONLY FOR FUN

(M^3, ξ) CONTACT MANIFOLD

DEF: $f: M \longrightarrow \mathbb{R}$ MORSE FUNCTION IS CONTACT IF $\exists$ X CONTACT VECTORFIELD THAT IS AN ALMOST GRADIENT FOR f

$\boxed{\pitchfork \text{ TO LEVEL SETS \& ``LIKE'' } \nabla f \text{ NEAR CRITICAL POINTS}}$

RMK: NONCRITICAL LEVELSETS ARE CONVEX

DEF: CRITICAL SUBMANIFOLD: $C^2 = \{ x \in \xi \}$

($\approx$ UNION OF DIVIDING CURVES OF LEVELSETS)

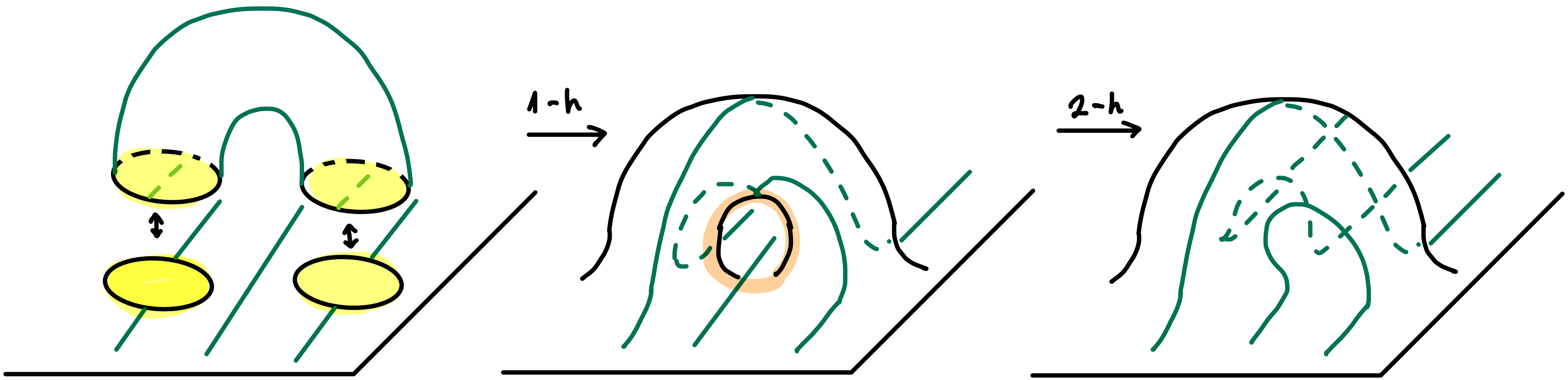
THM (GIROUX): $f|_C : C \longrightarrow \mathbb{R}$ IS ALSO MORSE WITH THE SAME CRITICAL POINTS AS f WITH INDICES

f	$f\|_C$
0	0
1	1
2	1
3	2

THIS EXPLAINS WHY WE CANNOT MOVE 1-HANDLES UNDER 2-HANDLES

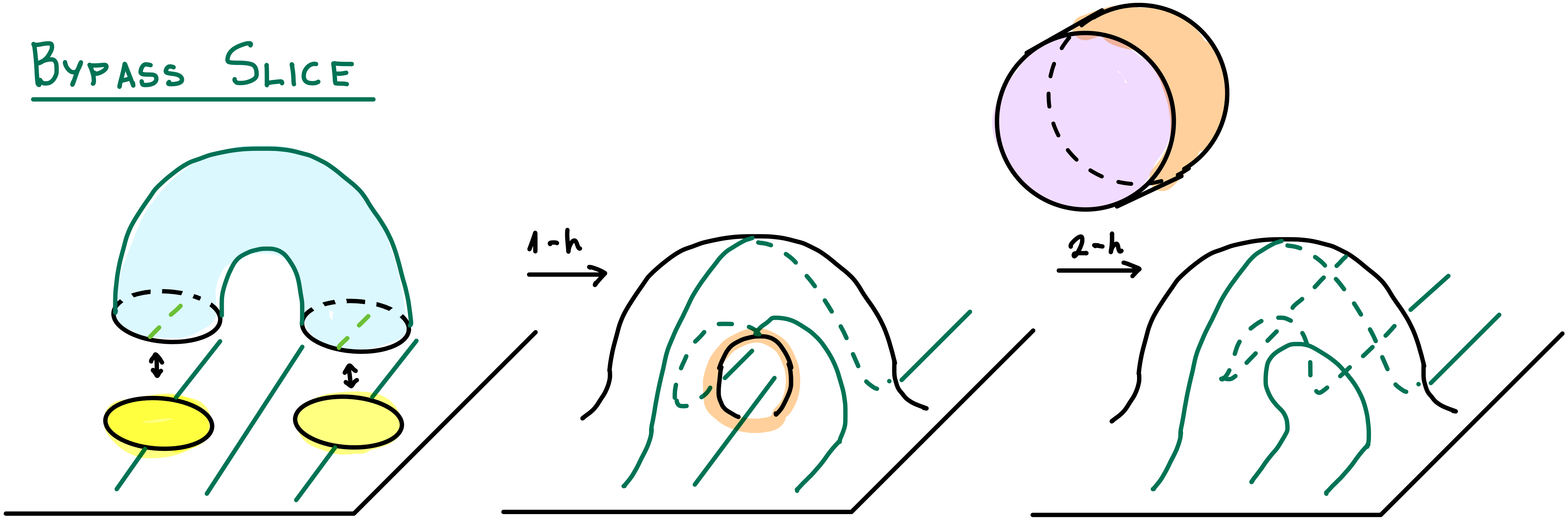
(THEY ARE BOTH 1-HANDLES FOR C)

Bypass Slice

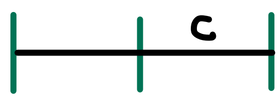


FROM THE TOP

BYPASS SLICE



FROM THE TOP

THE ABOVE PAIR OF CONTACT 1- & 2-h CAN BE ATTACHED TO ANY CONVEX SURFACE (Σ, Γ) ALONG ANY ARC c INTERSECTING Γ AS 

WILL SEE: BYPASSES ARE BASIC BUILDING BLOCKS OF CONTACT STRUCTURES ON $\Sigma \times I$

CONVEX SURFACE THEORY

0-PARAMETER

RECALL (GIROUX): CONVEX SURFACES ARE C^∞- GENERIC

1-PARAMETER

THM (GIROUX, COLIN, REPHRASED BY HONDA) ANY 1- PARAMETER FAMILY OF SURFACES $(\Sigma_t)_{t\in[0,1]}$ WITH Σ_0, Σ_1 CONVEX CAN BE ISOTOPED TO $(\Sigma'_t)_{t\in[0,1]}$ SO THAT

- $\Sigma_t = \Sigma'_t$ NEAR $t=0$ & 1

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- $\Sigma_t = \Sigma_t'$ NEAR $t=0$ & 1
- Σ_t' IS CONVEX EXCEPT AT DISCRETE TIMES $\{t_1, .., t_k\} \subseteq [0,1]$

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- $\Sigma'_{t_i - \varepsilon}$ & $\Sigma'_{t_i+\varepsilon}$ COBOUND A BYPASS SLICE $(i = 1 \dots k)$

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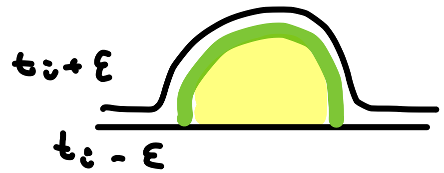

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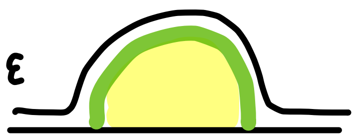
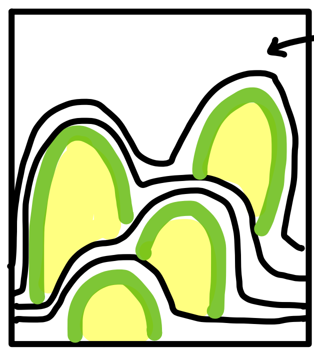


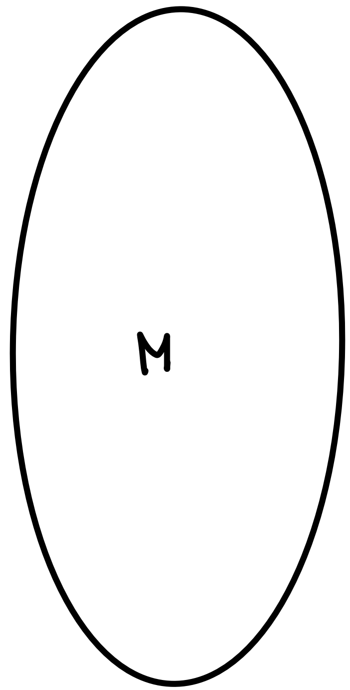
ANY CONTACT STRUCTURE ON $\Sigma\times I$ IS CONTACTOMORPHIC TO A STACK OF BYPASS SLICES



Existence of Contact Heegaard Decompositions

THM (Giroux): Any contact 3-manifold (M,ξ) admits a contact Heegaard decomposition

PROOF (Licata - V)

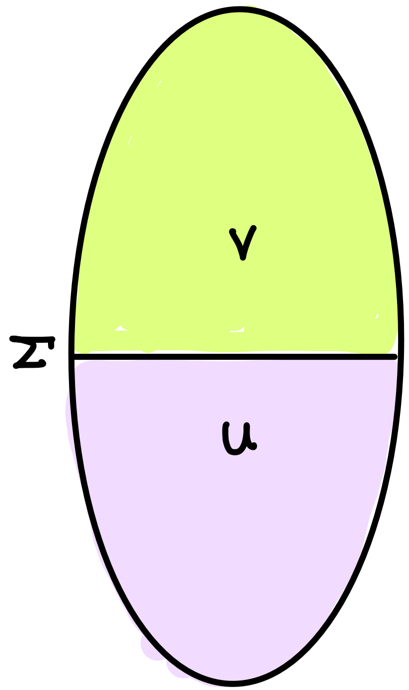


EXISTENCE OF CONTACT HEEGAARD DECOMPOSITIONS

THM (GIROUX): ANY CONTACT 3-MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

PROOF (LICATA - V)

STEP 1: TAKE ANY SMOOTH HEEGAARD DECOMPOSITION OF M: $M = U \cup_\Sigma V$



Existence of Contact Heegaard Decompositions

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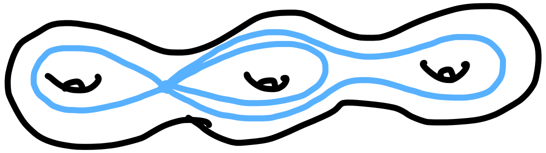
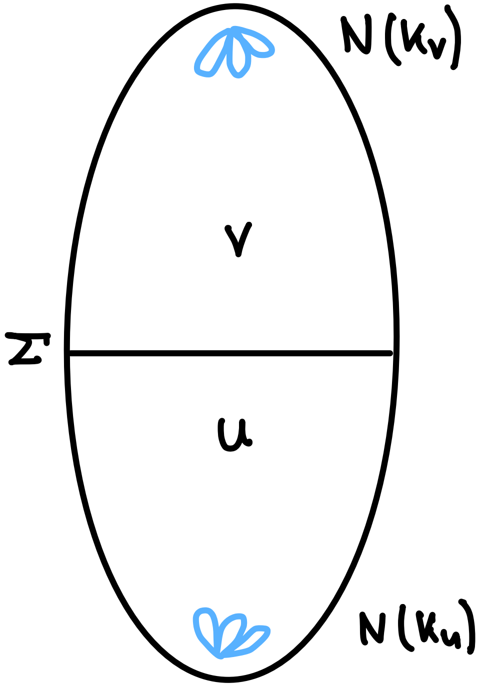
STEP 1: Take any smooth Heegaard decomposition of M: $M = U \cup_\Sigma V$

STEP 2: Take spines $K_U \subset U$ & $K_V \subset V$





Existence of Contact Heegaard Decompositions

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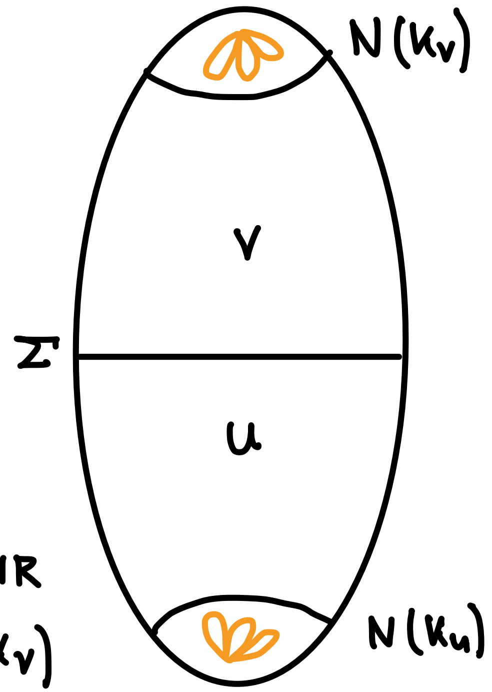

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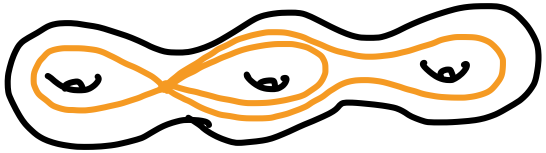

STEP 3: LEGENDRIAN REALISE K_U & K_V & TAKE THEIR STANDARD CONTACT NEIGHBOURHOODS $N(K_U)$ & $N(K_V)$

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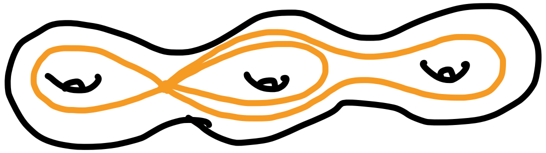

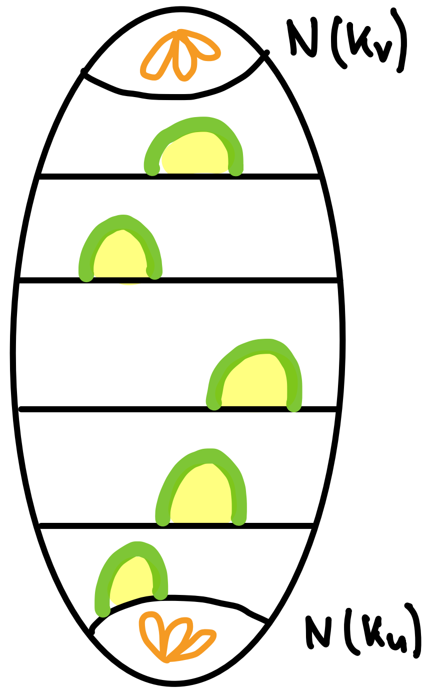

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(AWAY FROM THE HANDLES ξ IS I-INVARIANT)

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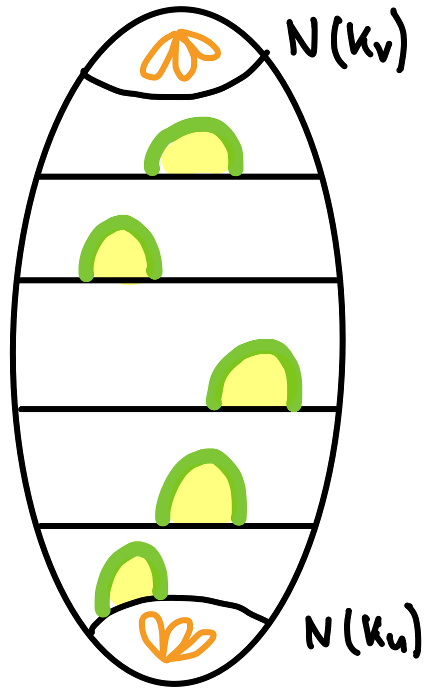

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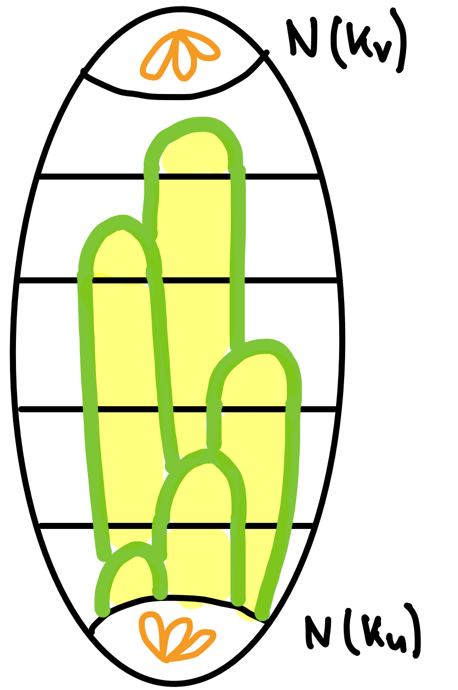

STEP 5: Use this flow to extend the handles

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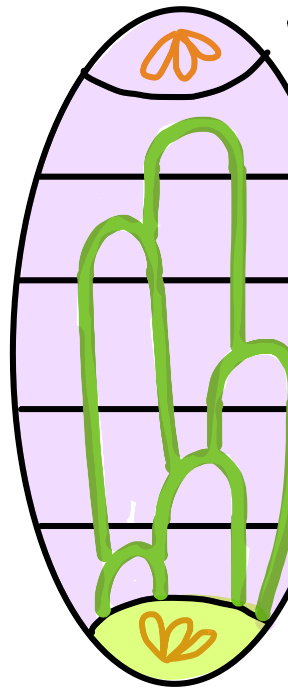

STEP 5: Use this flow to extend the handles

CONSIDER: $\hat{U} = N(K_u) \cup (\cup h_i^1)$

Contact handlebody; contact 1-handles

$\Longrightarrow \hat{U}$ is a contact handlebody

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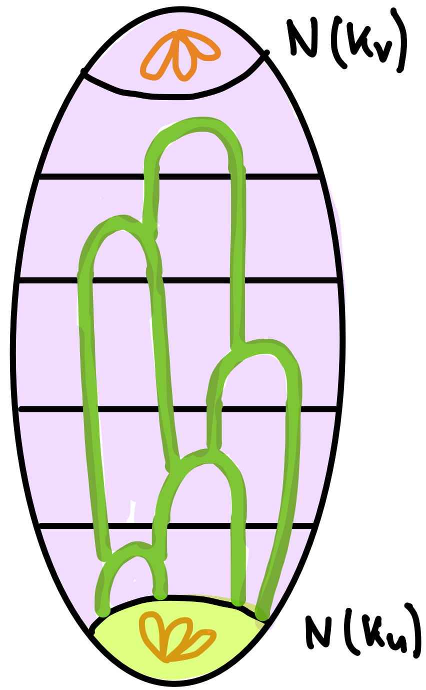

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CAN BE WRITTEN AS A STACK

OF BYPASS-SLICES $\cong h_i^1 \cup h^2$

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STEP 5: USE THIS FLOW TO EXTEND THE HANDLES

CONSIDER: $\hat{U} = N(K_u) \cup (\cup h_i^1)$

CONTACT HANDLEBODY

CONTACT 1-HANDLES

$\Longrightarrow$ $\hat{U}$ IS A CONTACT HANDLEBODY

UPSIDE DOWN: $\hat{V} = M \setminus \hat{U} \cong N(K_v) \cup (\cup h_i^2)$ IS A CONTACT HANDLEBODY

$\Longrightarrow$ $M = \hat{U} \cup \hat{V}$ IS A CONTACT HEEGAARD DECOMPOSITION ∎

Bridging

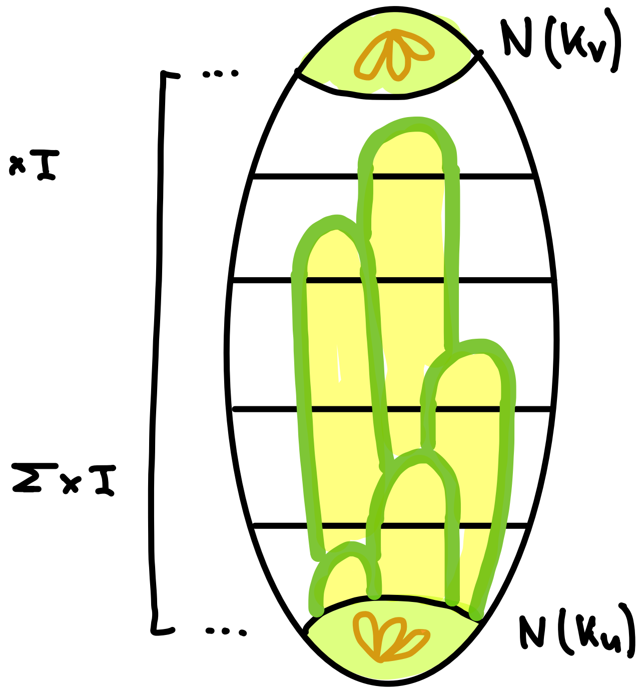
$$\underset{\mathcal{H}}{M = U \cup V} \quad \overset{\text{BRIDGING}}{\rightsquigarrow} \quad \underset{\hat{\mathcal{H}}}{M = \hat{U} \cup \hat{V}}$$

BRIDGE

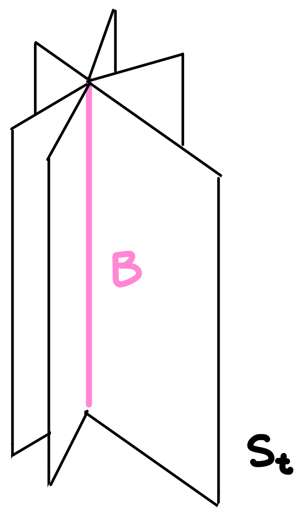
WHERE
- $\hat{U} = N(K_u) \cup (\cup h_i^1)$
- $\hat{V} = \overline{M \setminus \hat{U}}$

NOTE: THE BRIDGE DEPENDS ON
- $\mathcal{H}$
- K_u & K_v
- THE BYPASSES $\mathcal{B}$ BUILDING UP $\xi|_{\Sigma \times I}$

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}(K_u, K_v, \mathcal{B})$$



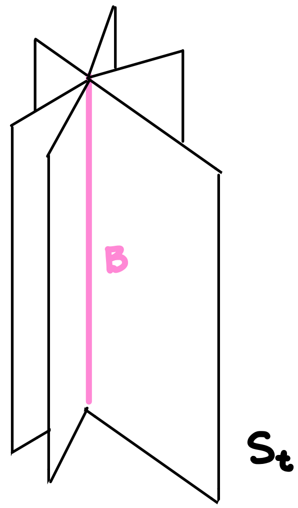
OPEN BOOK DECOMPOSITIONS



Open Book Decompositions

DEF: PAIR (B, π), WHERE

- $B^1 \hookrightarrow M$ EMBEDDED 1-MANIFOLD: **BINDING**
- $\pi : M - B \longrightarrow S^1$ FIBRATION SUCH THAT
  - $\rightarrow \forall\, t \in S^1 \quad S_t := \overline{\pi^{-1}(t)}$ IS A SEIFERT SURFACE FOR B
  - $\rightarrow$ & ON $N(B) \cong B \times D^2 \quad \pi = $ ANGLE



DEF: $S_t := \pi^{-1}(t)$ ARE THE **PAGES** OF (B, π)

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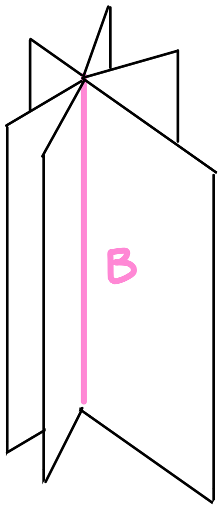

S_t

E.G.: $M = S^3 = \{|z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$

$B = \{|z| = 0\} \cong S^1$, $\quad \pi : S^3 \setminus B \longrightarrow S^1$

$$(z,w) \longmapsto \frac{z}{|z|}$$

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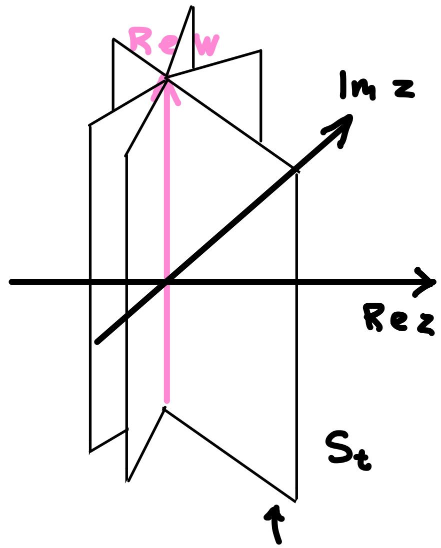

$\{\arg(z) = \frac{z}{|z|} = t\}$

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ON $\mathbb{R}^3$ WITH COORDINATES: $\frac{1}{1 - \operatorname{Im} w}(z, \operatorname{Re} w)$

HERE $S_t \cong D^2 \quad \forall\, t$

Open Book Decompositions - Another Example

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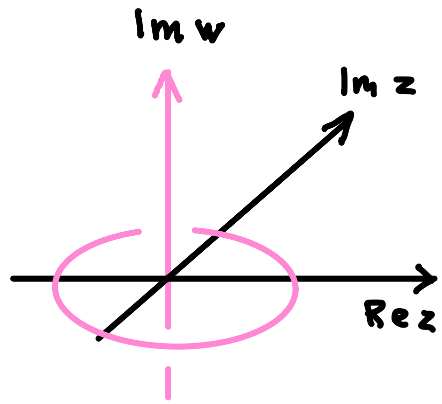
$B = \{|z| = 0\} \amalg \{|w| = 0\}$

$\pi : S^3 \setminus B \longrightarrow S^1$

$(z,w) \longmapsto \frac{zw}{|zw|}$



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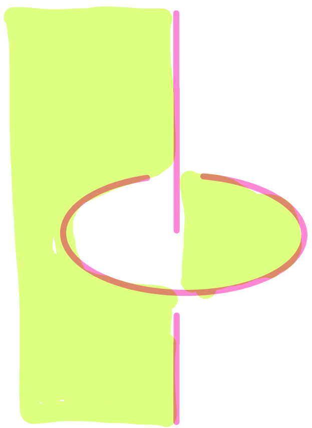

Video

THM (ALEXANDER): ANY 3-MANIFOLD ADMITS AN OPEN BOOK DECOMPOSITION

PROOF LATER

Open Book Decompositions - Another Example

E.G.: $M = S^3 = \{|z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$

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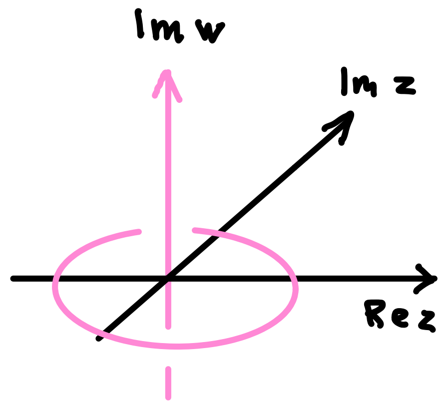

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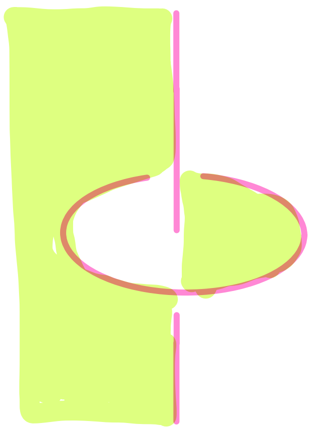


Video

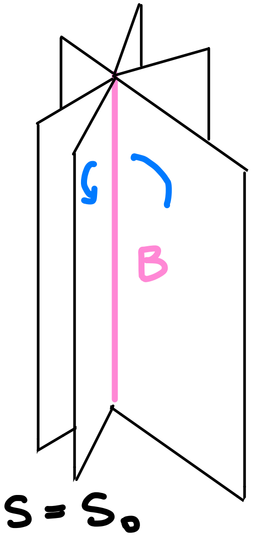
THM (ALEXANDER): ANY 3-MANIFOLD ADMITS AN OPEN BOOK DECOMPOSITION

PROOF LATER

Abstract Open Books

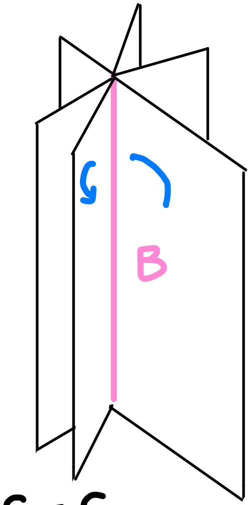
FIX $S := S_0$ & LOOK AT THE FIRST RETURN-MAP OF $\pi : M - B \longrightarrow S^1$ $\rightsquigarrow$ GET (S, φ) WHERE

- S IS AN ORIENTED SURFACE WITH BOUNDARY
- & $\varphi : S \circlearrowleft S$ HOMEOMORPHISM THAT FIXES $N(\partial S)$



DEF: THE PAIR (S, φ) IS AN **ABSTRACT OPEN BOOK**

ABSTRACT OPEN BOOKS



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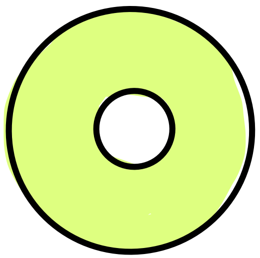
E.G.: THE PREVIOUS EXAMPLE GIVES

$S =$ 

ABSTRACT OPEN BOOKS

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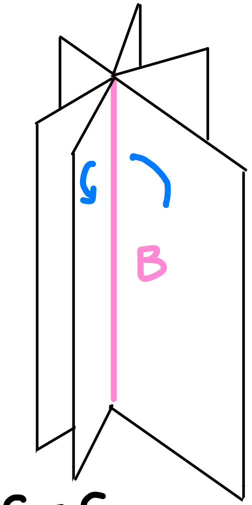

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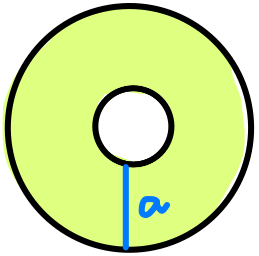
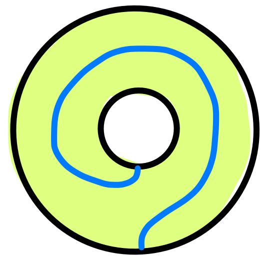


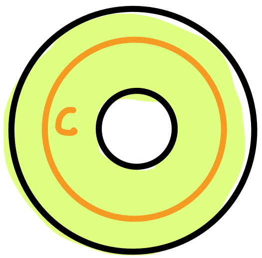
$S^1 \times I$

& φ MAPPING a TO $\varphi(a)$



(THIS DETERMINES φ UP TO ISOTOPY)

DEF: THE ABOVE MAP IS A RIGHT HANDED DEHN-TWIST ALONG c

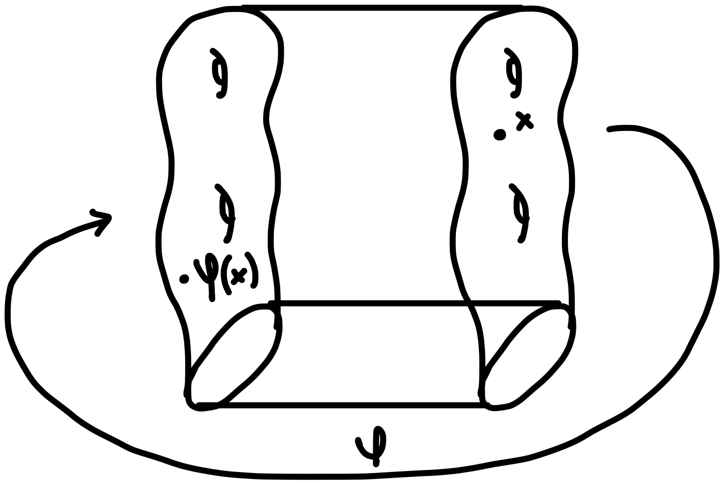


ABSTRACT OPEN BOOKS

CONVERSALY: AN ABSTRACT OB (S,φ) DETERMINES A 3-MANIFOLD M TOGETHER WITH AN OPEN BOOK DECOMPOSITION

PROOF: • TAKE THE MAPPING TORUS OF φ:

$$M_\varphi = \frac{S \times I}{(x,1) \sim (\varphi(x),0)}$$



• (AS φ FIXES ∂S) $\partial M_\varphi = \partial S \times S^1$

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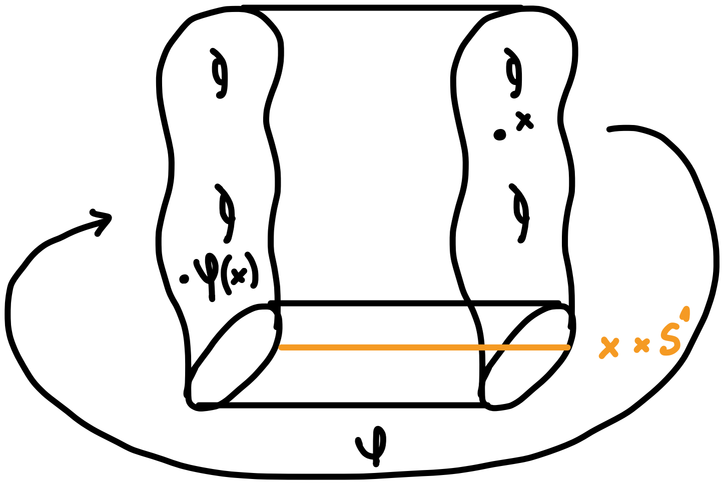
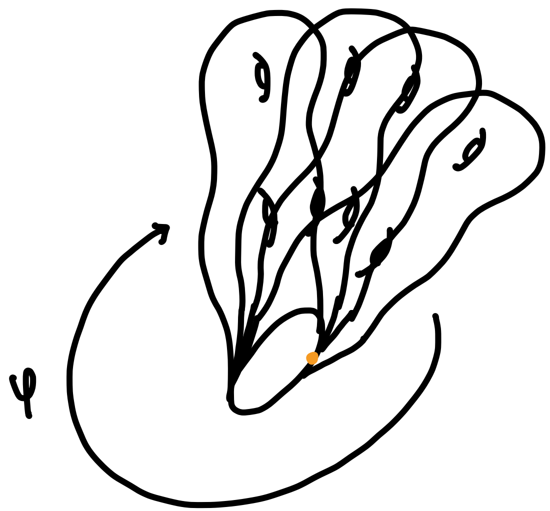


• (AS φ FIXES ∂S) $\partial M_\varphi = \partial S \times S^1$

• & WE COLLAPSE EACH CIRCLE $x \times S^1$ $(x \in \partial S)$

$$M := M_\varphi \big/ (x,t) \sim (x,t') \quad x \in \partial S$$



THEN: $B = \partial S/\sim$

& $\pi : M \setminus B \longrightarrow S^1$

$(x,t) \longrightarrow t$

GIVES AN OBD ■

INTERLUDE - DESCRIBING MONODROMIES

FOR FUN ONLY

POWER OF OPEN BOOKS: TO DESCRIBE CTCT STRUCTURES

WE NEED TO UNDERSTAND $\{\varphi : S \to S : \varphi|_{\partial S} = id\} /_{\text{ISOTOPY}} =: MCG(S)$

(MAPPING CLASS GROUP)

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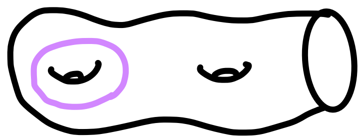
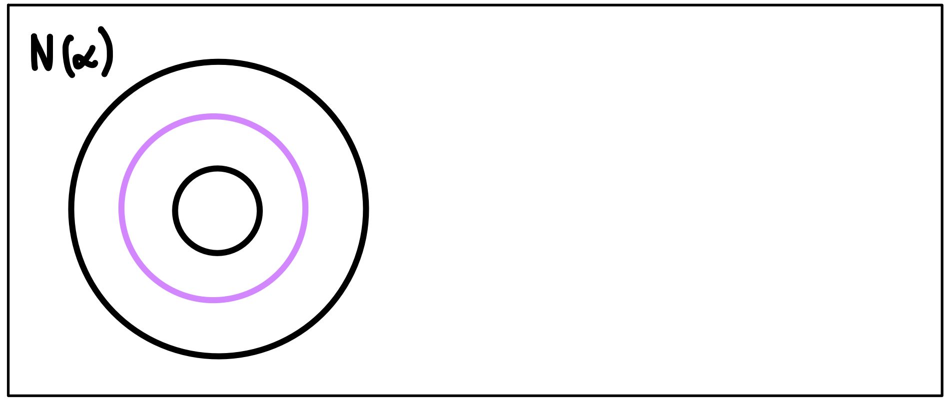
THM (DEHN): $MCG(S)$ IS GENERATED BY DEHN TWISTS ALONG SIMPLE CLOSED CURVES α





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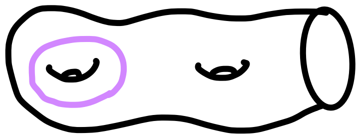

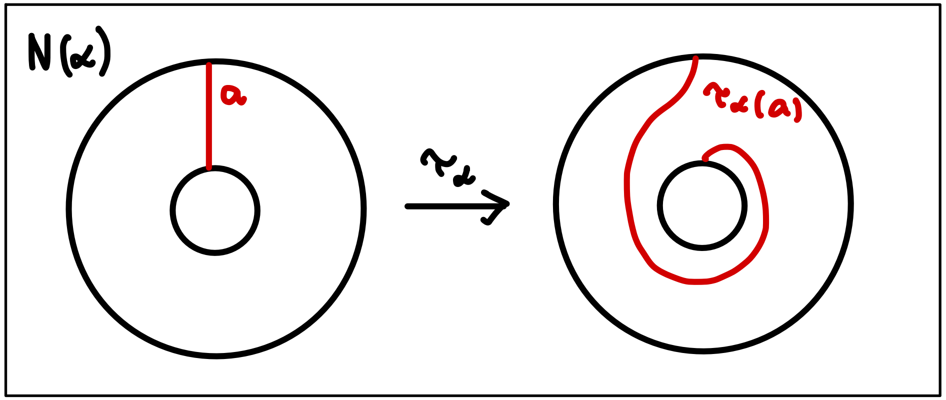

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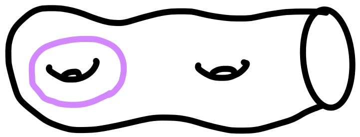
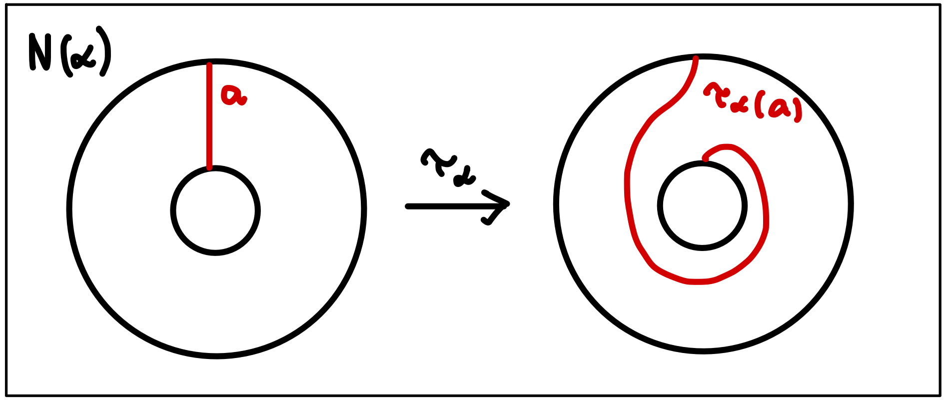
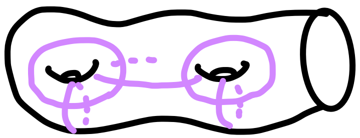




THM (LICKORISH) MCG(S) IS GENERATED BY DEHN TWISTS ALONG



$\rightsquigarrow$ WE CAN REPHRASE/SOLVE PROBLEMS ABOUT CTC STRUCTURES COMBINATORIALLY (CURVES ON A SURFACE)

INTERLUDE – DESCRIBING MONODROMIES

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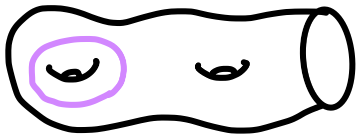

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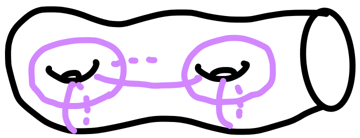
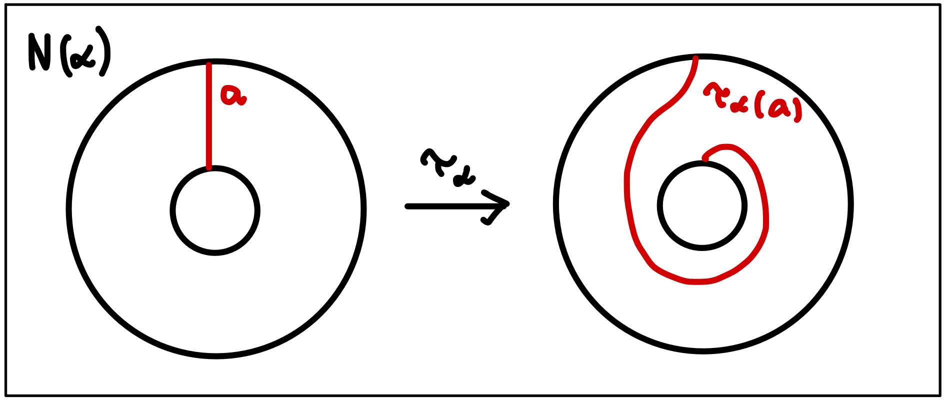


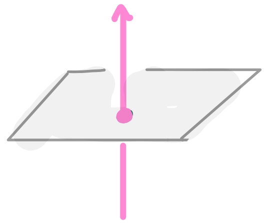


⟶ WE CAN REPHRASE/SOLVE PROBLEMS ABOUT CTC STRUCTURES COMBINATORIALLY (CURVES ON A SURFACE)

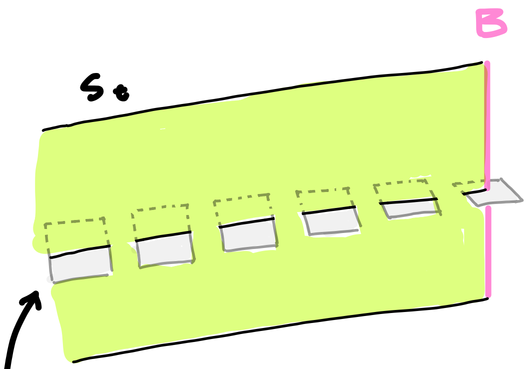
GIROUX CORRESPONDANCE ALLOWS US TO USE OPEN BOOKS TO DEFINE INVARIANTS OF CTC STRUCTURES

Open Books & Contact Structures

DEF: An OBD (B, π) supports a contact structure if

- B is <u>transverse</u> ($TB \pitchfork \xi$ ): $\boxed{\alpha|_B > 0}$
- $\boxed{d\alpha|_{T\mathring{S}_t} > 0}$

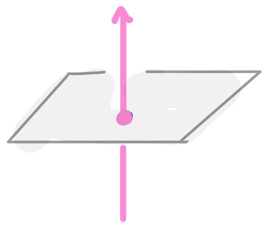
$\pi^{-1}(t)$ positive area form on S_t:



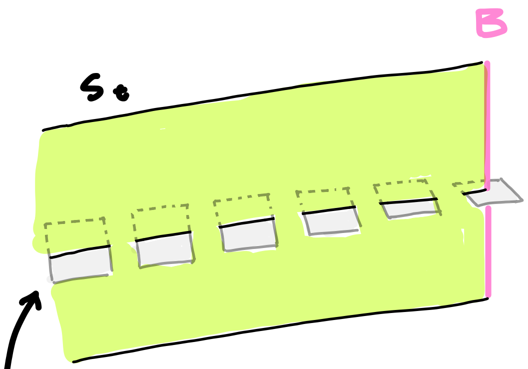
ξ never gets $= TS_t$

Open Books & Contact Structures

DEF: AN OBD (B,π) SUPPORTS A CONTACT STRUCTURE IF

- B IS TRANSVERSE ($TB \pitchfork \xi$ ): $\boxed{\alpha|_B > 0}$
- $\boxed{d\alpha|_{T\mathring{S}_t} > 0}$

$\pi^{-1}(t)$ POSITIVE AREA FORM ON S_t:



ξ NEVER GETS $= TS_t$

WE WILL HAVE A MORE TOPOLOGICAL DEF LATER

> **CONSTRUCTION (THURSTON-WINKENKEMPER):** ANY OBD SUPPORTS A CONTACT STRUCTURE THAT IS UNIQUE UP TO ISOTOPY

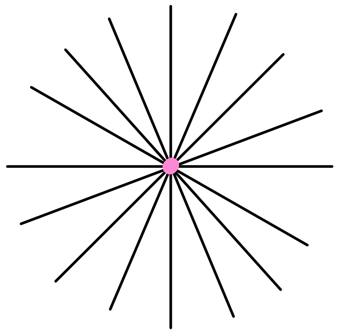
IDEA: • 1-PARAMETER FAMILY OF AREA FORM $\{\beta_t\}$

$\rightsquigarrow \alpha = \beta_t + K\,dt$ WHERE $K \gg 0$

• + LOCAL CONSTRUCTION NEAR BINDING ∎

Open Book- & Contact Heegaard Decompositions

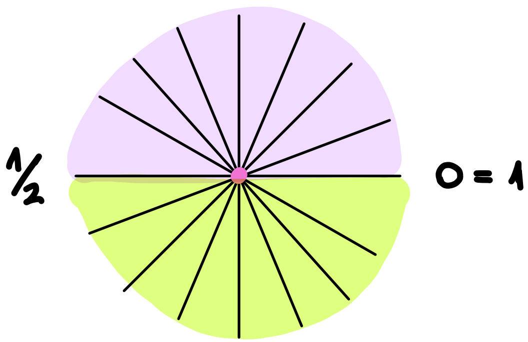
GIVEN AN OBD (B,π)



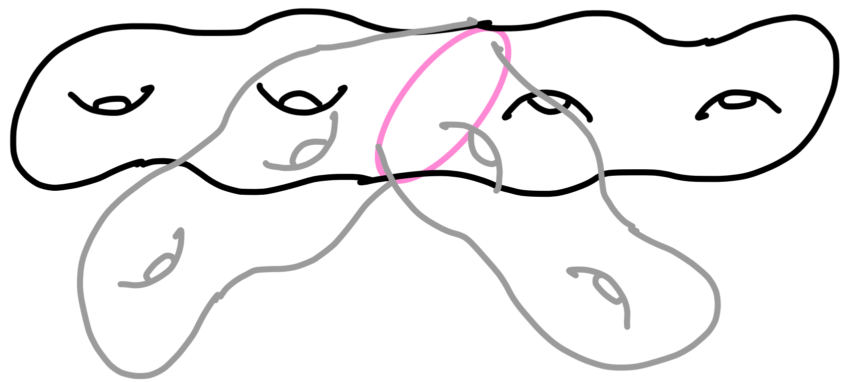
OPEN BOOK DECOMPOSITIONS $\rightsquigarrow$ HEEGAARD DECOMPOSITIONS

GIVEN AN OBD $(B,\pi) \rightsquigarrow$ CONSIDER $M = U \cup_\Sigma V$ WHERE

$U = \overline{\pi^{-1}([0,1/2])}$ $\quad$ $V = \overline{\pi^{-1}([1/2,1])}$



HERE $\Sigma = S_0 \cup S_1$



PROP: U & V ARE HANDLEBODIES

Open Book Decompositions $\rightsquigarrow$ Heegaard Decompositions

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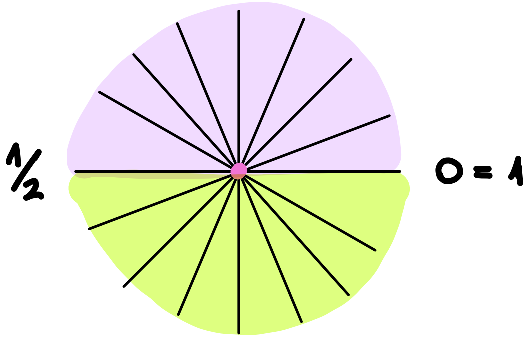

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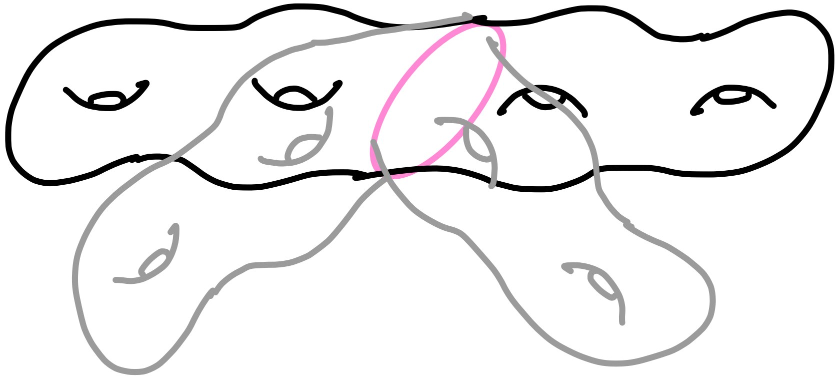
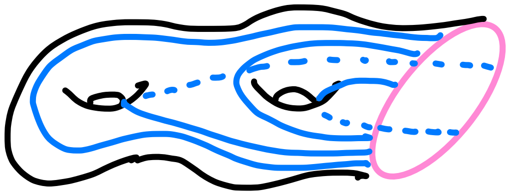


PROP: U & V ARE HANDLEBODIES

PROOF: FOR U:

- LET $a_1, \ldots, a_{2g+b}$ BE ARCS ON S SUCH THAT $S \setminus \cup a_i \cong D^2$



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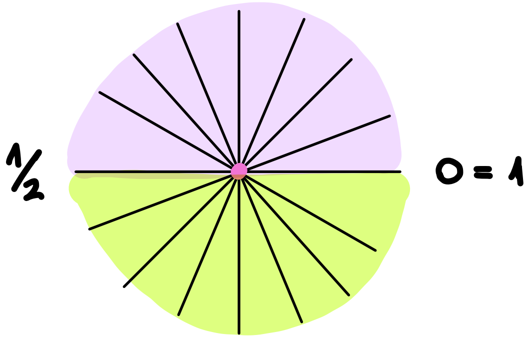

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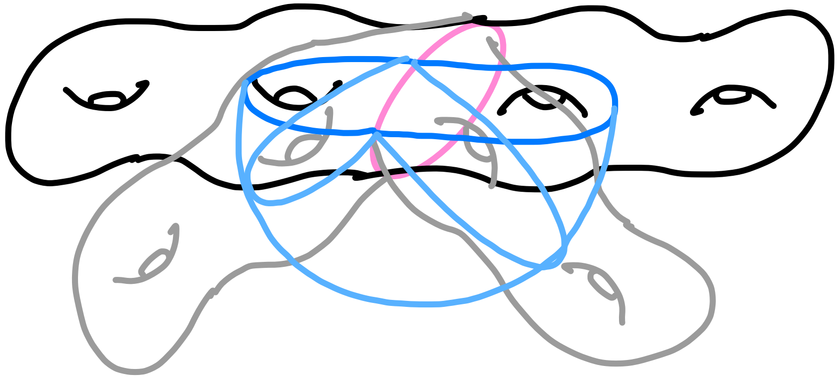

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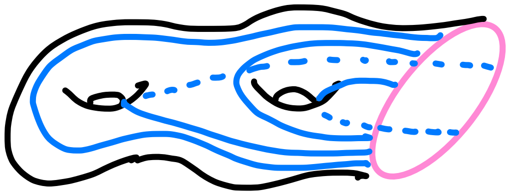
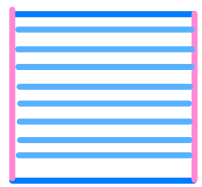
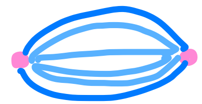


THEN $D_i := a_i \times [0,1/2] / \sim$ =



=



$\cong D^2$

Open Book Decompositions $\rightsquigarrow$ Heegaard Decompositions

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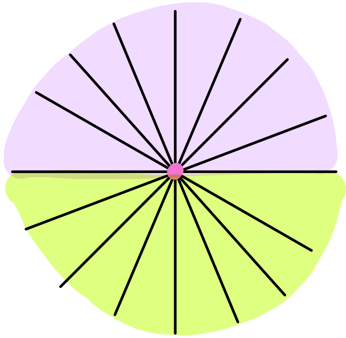
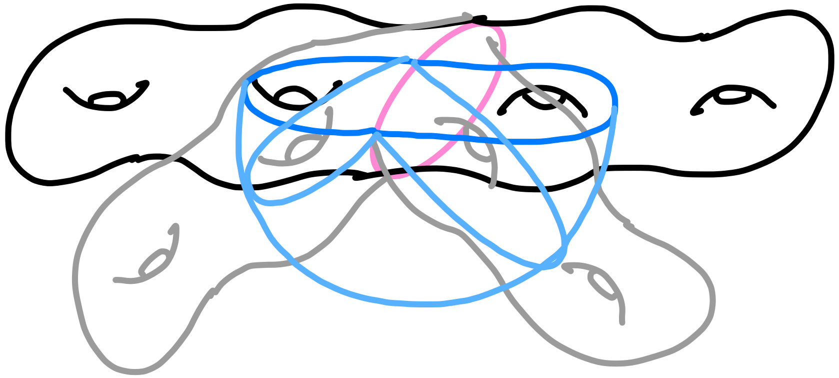
1/2  0 = 1

HERE $\Sigma = S_0 \cup S_1$



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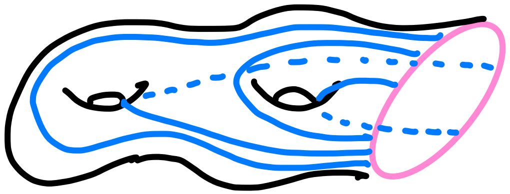
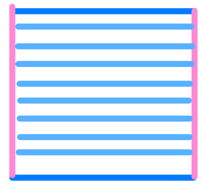
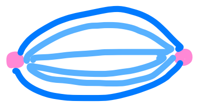


THEN $D_i := a_i \times [0,1/2]/\sim$ =  =  $\cong D^2$

- & $U \setminus \cup D_i = (S \setminus \cup a_i) \times [0,1/2]/\sim \;\cong D^2 \times [0,1/2]/\sim \;\cong D^3$

Open Book Decompositions $\rightsquigarrow$ Heegaard Decompositions

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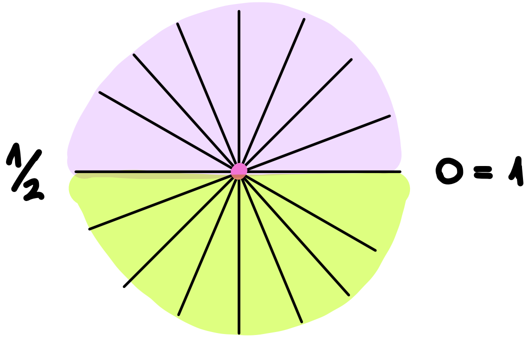

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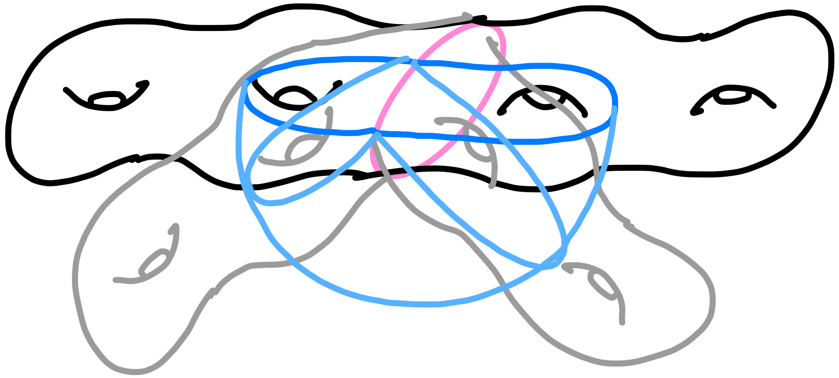

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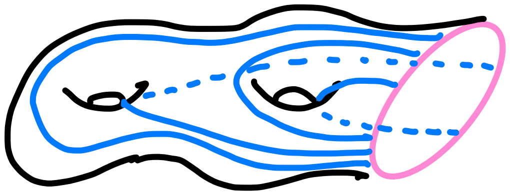
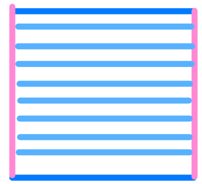
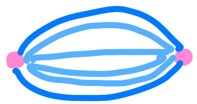


THEN $D_i := a_i \times [0,1/2]/\sim$ =  =  $\cong D^2$

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$\Longrightarrow$ U IS A HANDLEBODY ∎

OPEN BOOK DECOMPOSITIONS $\rightsquigarrow$ HEEGAARD DECOMPOSITIONS

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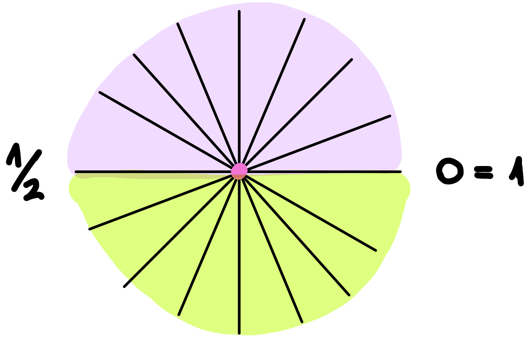

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HERE $\Sigma = S_0 \cup S_1$

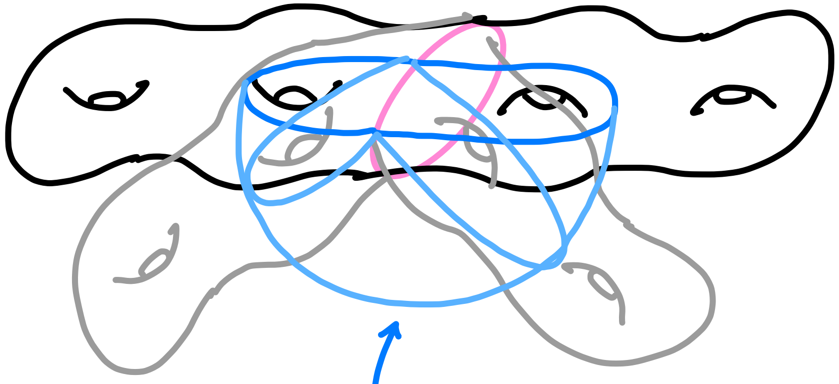
PROP: U & V ARE HANDLEBODIES

MOREOVER: $(U, \Sigma, \Gamma = B)$ IS

PRODUCT DISC DECOMPOSABLE

$\exists$ DISCS D_i SUCH THAT

- $U - \cup D_i \cong D^3$
- & $|\partial D_i \cap \Gamma| = 2 \quad \forall i$



$D_i = a_i \times [0, 1/2] / \sim$

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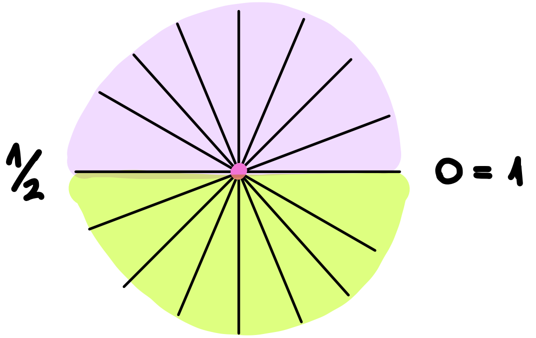

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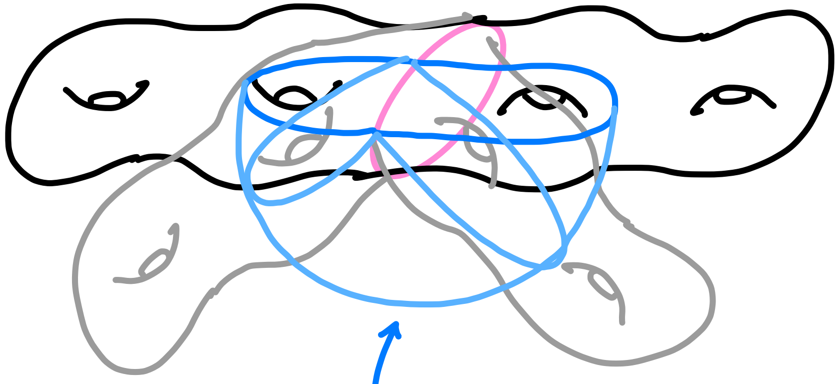

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- & $|\partial D_i \cap \Gamma| = 2 \quad \forall i$

WE HAVE SEEN: AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

WARNING: NEED Γ ON Σ

Open Book - & Contact Heegaard Decompositions

THM: AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

$$(B,\pi) \rightsquigarrow M = U \cup_{(\Sigma,\Gamma)} V$$

LET'S LOOK AT THE CONTACT STRUCTURE SUPPORTED BY (B,π)

THM (TORISU): THE SURFACE Σ IS CONVEX WITH DIVIDING CURVE Γ. THE CONTACT STRUCTURES $\xi|_U$ & $\xi|_V$ ARE TIGHT.

Open Book - & Contact Heegaard Decompositions

THM: AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

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SO: (U,ξ_U) & (V,ξ_V) ARE CONTACT HANDLEBODIES

THUS $\boxed{M = U \cup_{(\Sigma,\Gamma)} V \text{ IS A CONTACT HEEGAARD DECOMPOSITION}}$

Open Book - & Contact Heegaard Decompositions

THM: AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

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THM (TORISU): THE SURFACE Σ IS CONVEX WITH DIVIDING CURVE Γ. THE CONTACT STRUCTURES $\xi|_U$ & $\xi|_V$ ARE TIGHT.

SO: (U,ξ_U) & (V,ξ_V) ARE CONTACT HANDLEBODIES

THUS $\boxed{M = U \cup_{(\Sigma,\Gamma)} V \text{ IS A CONTACT HEEGAARD DECOMPOSITION}}$

THIS GIVES RISE TO AN EQUIVALENT DEFINITION:

DEF: ξ IS SUPPORTED BY THE OPEN BOOK (B,π) IF THE HEEGARD DECOMPOSITION DEFINED BY (B,π) IS A CONTACT HEEGAARD DECOMPOSITION

HEEGAARD DECOMPOSITIONS $\rightsquigarrow$ OPEN BOOK DECOMPOSITIONS

PROP: (U,Γ) PRODUCT DISC DECOMPOSABLE HANDLEBODY

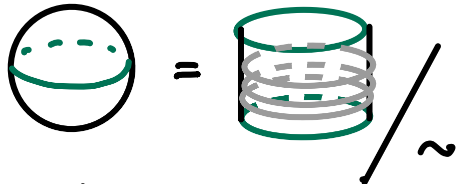
$$\Longrightarrow \quad U = S\times I/_{(x,t)\sim(x,t')} \quad x\in\partial S,\ t,t'\in I$$

SUCH THAT $\quad \partial U = S\times 0 \cup_{\Gamma} -S\times 1$

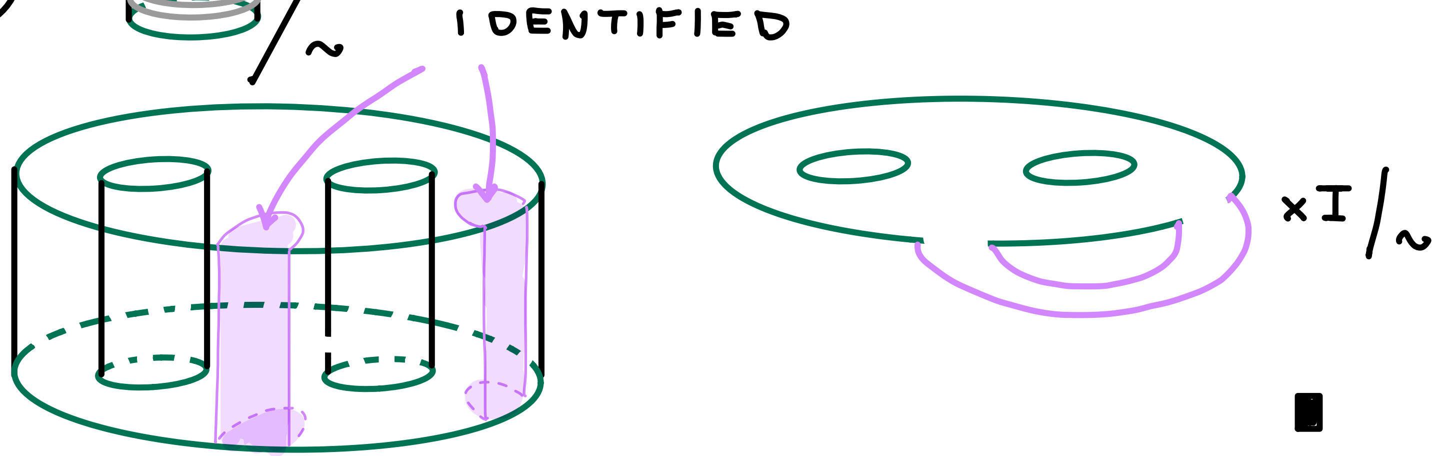
& $\Gamma = \partial S/_{\sim}$

IDEA: INDUCTION ON THE # OF PRODUCT DISCS

- $n=0$



- $n \rightsquigarrow n+1$



HEEGAARD DECOMPOSITIONS $\rightsquigarrow$ OPEN BOOK DECOMPOSITIONS

PROP: (U,Γ) PRODUCT DISC DECOMPOSABLE HANDLEBODY

$$\Longrightarrow U = S \times I \big/ (x,t) \sim (x,t') \quad x \in \partial S,\ t,t' \in I$$

SUCH THAT $\partial U = S \times 0 \cup_{\Gamma} -S \times 1$

& $\Gamma = \partial S / \sim$

GIVEN A CONTACT HEEGAARD DECOMPOSITION $M = U \cup_{(\Sigma,\Gamma)} V$

$U \qquad \partial U = \Sigma = R_+ \cup_{\Gamma} -R_-$

$\qquad\qquad\qquad\qquad\updownarrow \qquad\quad \updownarrow$

$V \qquad \partial V = -\Sigma = -R_+ \cup_{-\Gamma} R_-$

Heegaard Decompositions $\rightsquigarrow$ Open Book Decompositions

PROP: (U,Γ) PRODUCT DISC DECOMPOSABLE HANDLEBODY

$$\Longrightarrow \quad U = S \times I /_{(x,t)\sim(x,t')} \qquad x \in \partial S,\ t,t' \in I$$

SUCH THAT $\partial U = S\times 0 \cup_{\Gamma} -S \times 1$

& $\Gamma = \partial S /_{\sim}$

GIVEN A CONTACT HEEGAARD DECOMPOSITION $M = U \cup_{(\Sigma,\Gamma)} V$

BY PROP: $U = S\times[0,1/2]$

$$\partial U = \Sigma = \overset{\overset{S\times 0}{\parallel}}{R_+} \cup_{\Gamma} \overset{\overset{-S\times 1/2}{\parallel}}{-R_-}$$

$$\updownarrow \qquad \updownarrow$$

$V = S\times[1/2, 1]$

$$\partial V = -\Sigma = \underset{\underset{-S\times 1}{\parallel}}{-R_+} \cup_{-\Gamma} \underset{\underset{S\times 1/2}{\parallel}}{R_-}$$

$$\Longrightarrow \text{GLUES TO A FULL FIBRATION } (\overset{\overset{\Gamma}{\parallel}}{B},\pi)$$

GIVEN BY PROJECTION ONTO $[0,1/2] \cup [1/2, 1]$

Open Book – & Contact Heegaard Decompositions

SO WE GET A ONE-TO-ONE CORRESPONDANCE

$$\text{OPEN BOOK DECOMPOSITIONS OF } (M,\xi)\,/\,\text{ISOT} \longleftrightarrow \text{CONTACT HEEGAARD DECOMPOSITIONS OF } (M,\xi)\,/\,\text{ISOT}$$

SO WE CAN WORK WITH WHICHEVER IS MORE CONVENIENT

RECALL:

THM: EVERY CONTACT MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

COR: EVERY CONTACT MANIFOLD (M,ξ) ADMITS AN OPEN BOOK DECOMPOSITION

Coming up:

- Stabilisation
- Statement of Giroux correspondence
- Idea of proof