SRNI - LECTURE 1

INTRODUCTION

SUBMANIFOLDS OF CONTACT STRUCTURES

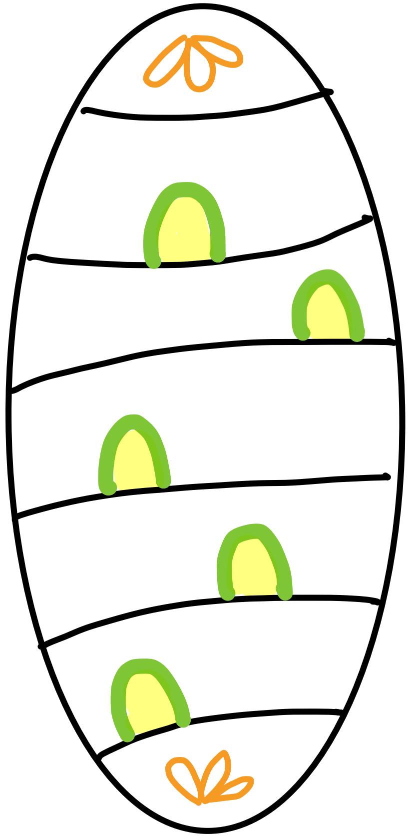
- CONTACT STRUCTURES
- (1D) LEGENDRIAN & TRANSVERSE KNOTS, LEGENDRIAN GRAPHS
- (2D) CONVEX SURFACES
- NEIGHBOURHOOD THEOREMS
- TIGHT & OVERTWISTED CONTACT STRUCTURES

THE GIROUX CORRESPONDENCE VIA CONVEX SURFACES

VERA VÉRTESI

JOINT WORK WITH JOAN LICATA

UNIVERSITY OF VIENNA



INTRODUCTION

1973 VINKELNKEMPER: FIRST USED THE WORD "OPEN BOOK DECOMPOSITION"

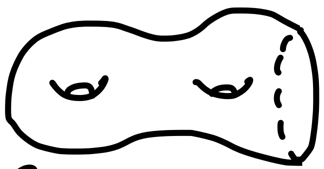
BUT IT WAS ALREADY KNOWN & STUDIED UNDER DIFFERENT NAMES:

- GLOBAL POINCARÉ-BIRKHOFF SECTION
- RELATIVE MAPPING TORUS
- LEFSHETZ / MILNOR FIBRATION
- FIBERED LINKS
- SPINNABLE STRUCTURES

DEF: OPEN BOOK: (S, φ)

$\varphi: S \circlearrowleft$ HOMEOMORPHISM : MONODROMY

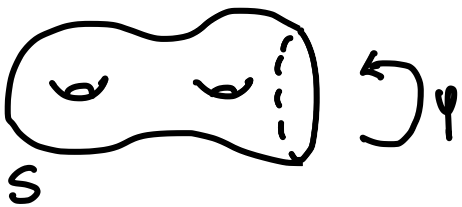
SURFACE WITH BOUNDARY



S

INTRODUCTION

$$\text{OPEN BOOKS} \xrightarrow[\text{VINKENKEMPLER}]{\text{THURSTON}} \text{CONTACT STRUCTURES}$$



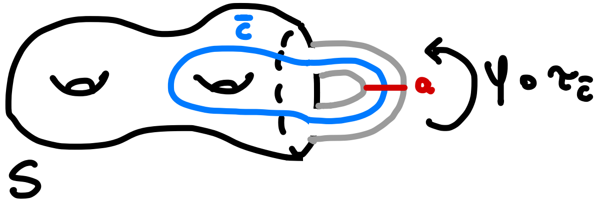
$$(M^3, \xi)$$

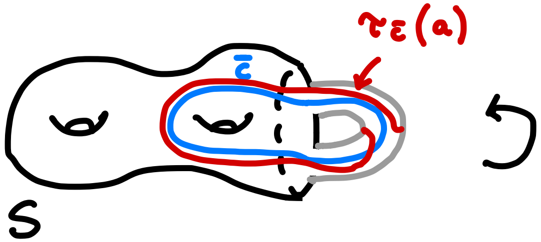
INTRODUCTION

OPEN BOOKS $\xrightarrow[\text{VINKENKEMPLER}]{\text{THURSTON}}$ CONTACT STRUCTURES

(M^3, ξ)

$\Big\downarrow$ POSITIVE STABILISATION



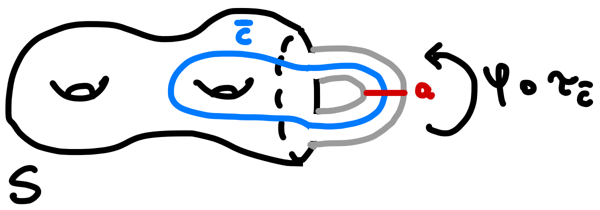


Introduction

OPEN BOOKS $\qquad\qquad\qquad\qquad$ **CONTACT STRUCTURES**

$$\xrightarrow[\text{WINKELNKEMPER}]{\text{THURSTON}} \quad (M^3, \xi)$$

$\big\downarrow$ POSITIVE STABILISATION



$$(S, h) \xleftarrow{\text{GIROUX}} (M^3, \xi)$$

GIROUX CORRESPONDENCE

$$\text{OPEN BOOKS} \big/ \text{POSITIVE STABILISATION} \longleftrightarrow \text{CONTACT STRUCTURES} \big/ \text{CONTACTOMORPHISM}$$

INTRODUCTION

GIROUX CORRESPONDENCE

OPEN BOOKS / POSITIVE STABILISATION $\longleftrightarrow$ CONTACT STRUCTURES / CONTACTOMORPHISM

GC HAS BEEN EXTENSIVELY USED TO PROVE THNS ABOUT CTCT 3-MFDS

2000 THE ORIGINAL PROOF OF GIROUX WAS INCOMPLETE

(MASSOT WROTE DOWN A COMPLETE PROOF BUT DIDN'T PUBLISH IT)

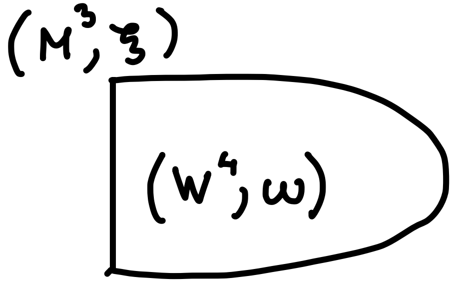
2023 BREEN-HONDA-HUANG: PROOF OF THE GIROUX CORRESPONDENCE FOR CONTACT STRUCTURES IN ANY ODD DIMENSIONS

2023 LICATA-V.: PROOF OF THE GIROUX CORRESPONDENCE FOR TIGHT CONTACT 3-MANIFOLDS (INDEPENDENT)

2024 LICATA-V.: EXTENDED OUR PROOF TO WORK FOR ANY CONTACT 3-MANIFOLD

APPLICATIONS

IN CONTACT TOPOLOGY



→ FILLABILITY

GIROUX: TOPOLOGICAL DESCRIPTION OF STEIN-FILLABLE CONTACT 3-MANIFOLDS

ELIASHBERG, ETNYRE: ANY WEAK SYMPLECTIC FILLING OF A CONTACT 3-MANIFOLD CAN BE EMBEDDED INTO A CLOSED SYMPLECTIC MANIFOLD

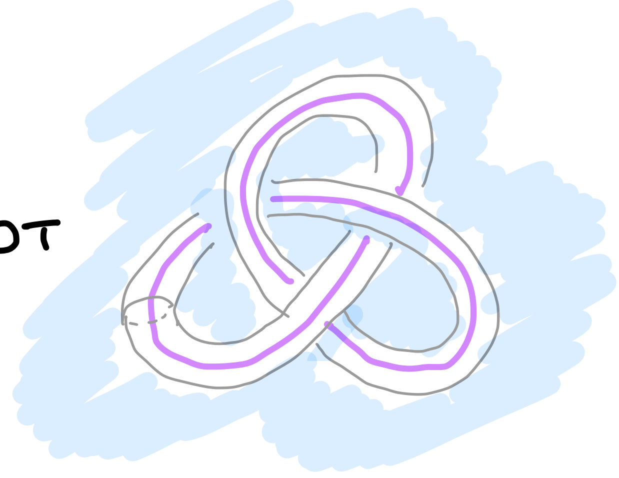
→ CONTACT SURGERY

WAND: CONTACT SURGERY PRESERVES TIGHTNESS

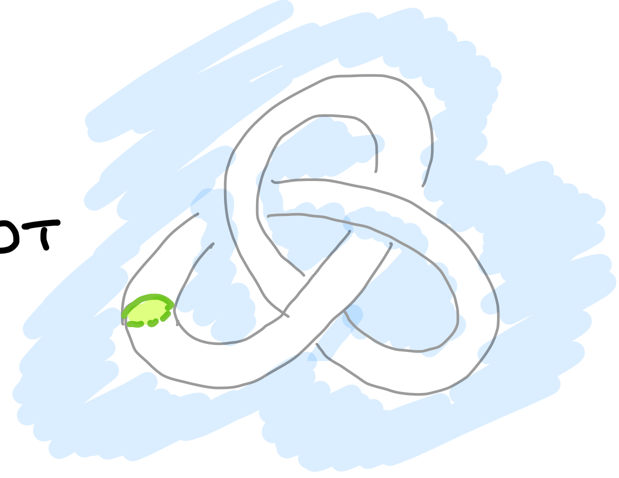
KEGEL-STENHENDE-V-ZUDDAS: CLASSIFICATION OF LEGENDRIAN SURGERY DIAGRAMS DESCRIBING THE SAME CONTACT MANIFOLD

APPLICATIONS

SURGERY: REMOVE NEIGHBOURHOOD OF A KNOT

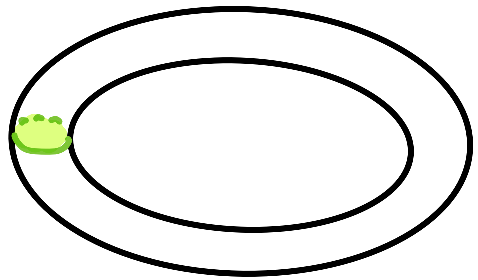


APPLICATIONS



SURGERY: REMOVE NEIGHBOURHOOD OF A KNOT

& GLUE BACK A $D^2 \times S^1$ DIFFERENTLY



APPLICATIONS

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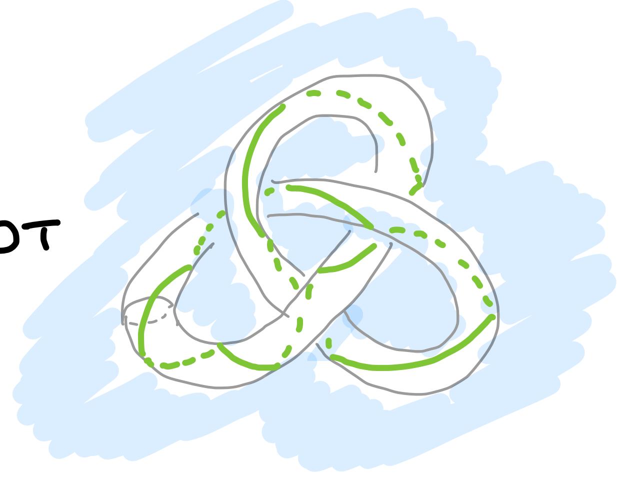

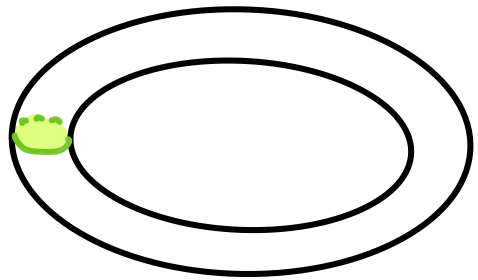

TOPOLOGY

KRONHEIMER - MROWKA: EVERY NONTRIVIAL KNOT HAS PROPERTY P

OZSVÁTH - SZABÓ: THE UNKNOT, TREFOIL & FIGURE-EIGHT KNOT ARE CHARACTERISED BY THEIR SURGERIES

OZSVÁTH - SZABÓ: THE THURSTON NORM IS DETERMINED BY HEEGAARD FLOER HOMOLOGY

GIROUX - GOODMAN: INDUCTIVE CONSTRUCTION OF FIBERED KNOTS IN S^3

PLAN OF TALKS

- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES
- LECTURE 2: DESCRIBING CONTACT STRUCTURES
- LECTURE 3: PROOF OF GIROUX CORRESPONDENCE

PLAN OF TALKS

- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES
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- LECTURE 2: DESCRIBING CONTACT STRUCTURES
- LECTURE 3: PROOF OF GIROUX CORRESPONDENCE

Plan of Talks

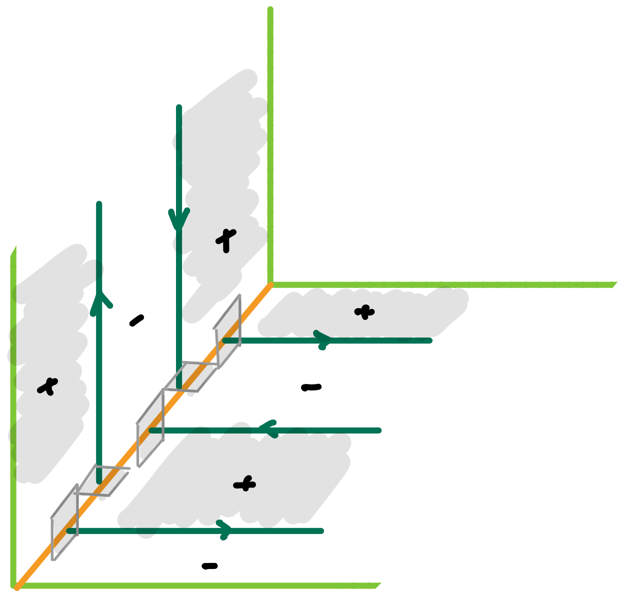
- Lecture 1: Submanifolds of contact structures
- Lecture 2: Describing contact structures
  - contact cell decompositions
  - convex surface theory - bypasses
  - contact Heegaard splittings (proof of existence)
  - open book deconpositions
  - open book deconpositions & contact Heegaard splittings
- Lecture 3: Proof of Giroux correspondence

PLAN OF TALKS

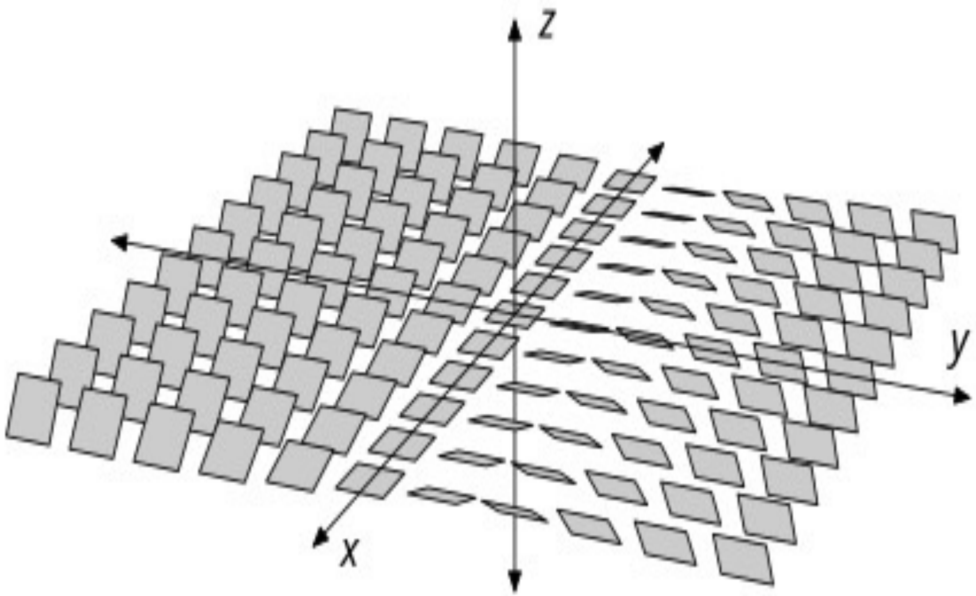
- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES
- LECTURE 2: DESCRIBING CONTACT STRUCTURES
- LECTURE 3: PROOF OF GIROUX CORRESPONDENCE
  - STABILISATION
  - STATEMENT OF GIROUX CORRESPONDENCE
  - IDEA OF PROOF
  - FURTHER DIRECTIONS

LECTURE 1

SUBMANIFOLDS IN CONTACT STRUCTURES



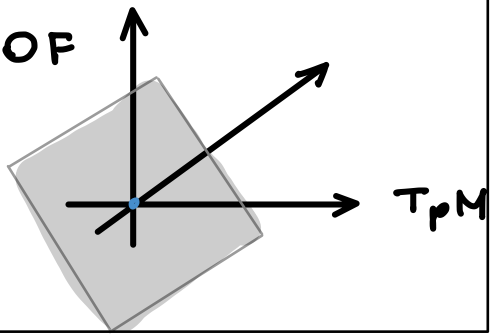
CONTACT STRUCTURES



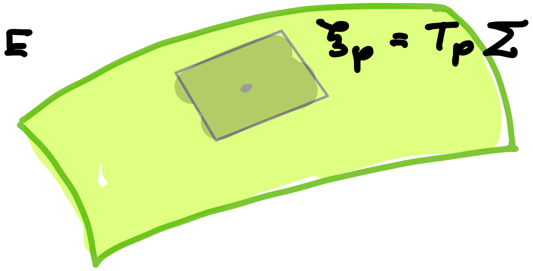
Contact Structures

DEF: A **CONTACT STRUCTURE** ON A CLOSED, ORIENTED SMOOTH 3-MANIFOLD M^3 IS A TOTALLY NONINTEGRABLE 2-PLANE-DISTRIBUTION $\xi \subset TM$

SMOOTH CHOICE OF $\xi_p^2 \subset T_pM \quad \forall p$



NOT TANGENT TO ANY SURFACE $\Sigma \subseteq M$



$\Updownarrow$ FROBENIUS

$\alpha \wedge d\alpha > 0$

LOCALLY: $\xi = \ker \alpha \qquad \alpha \in \Omega^1(M)$

COORIENTED CONTACT STRUCTURE: GLOBAL α

CONTACT STRUCTURES

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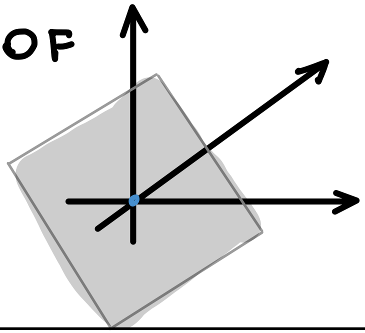

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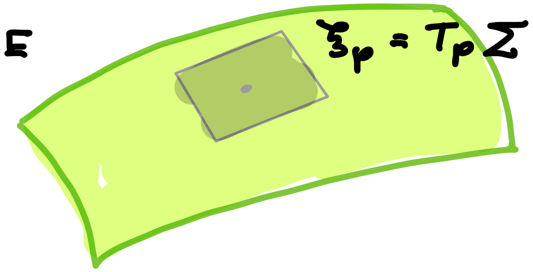


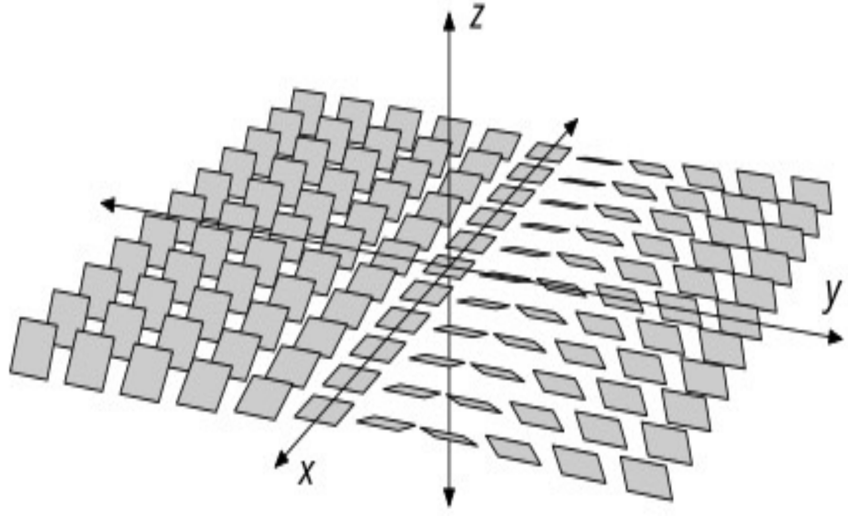
$\Updownarrow$ FROBENIUS

LOCALLY: $\xi = \ker \alpha \qquad \alpha \in \Omega^1(M) \qquad\qquad \alpha \wedge d\alpha > 0$

COORIENTED CONTACT STRUCTURE: GLOBAL α

DARBOUX THM: LOCALLY ANY CONTACT STRUCTURE IS **CONTACTOMORPHIC** TO $\mathbb{R}^3$, $\xi_{st} = \ker(dz - y\,dx)$

DIFFEOMORPHISM THAT CARRIES ξ TO ξ'



EQUIVALENCE OF CONTACT STRUCTURES

(M,ξ) & (M',ξ') CONTACT STRUCTURES

- **CONTACTOMORPHISM:** $(M,\xi)\cong(M',\xi')$ IF $\exists$ DIFFEOMORPHISM $\phi: M\to M'$ THAT CARRIES ξ TO ξ' : $\phi_*\xi=\xi'$

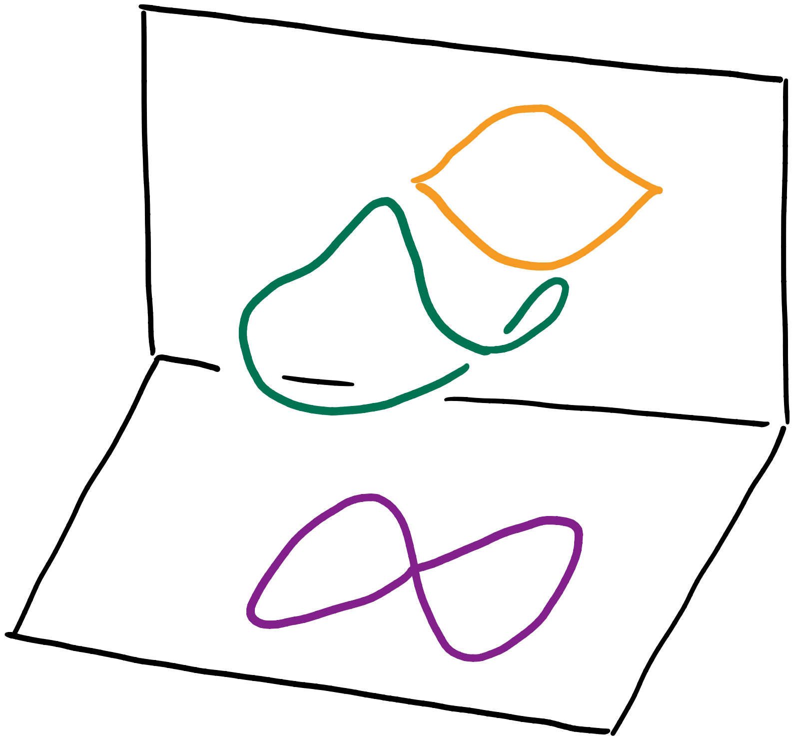
WHEN $\boxed{M=M'}$

- **HOMOTOPY:** $\xi\simeq\xi'$ IF $\exists$ 1-PARAMETER FAMILY OF CONTACT STRUCTURES $(\xi_t)_{t\in[0,1]}$ ON M WITH $\xi=\xi_0$ & $\xi'=\xi_1$

- **ISOTOPY:** $\xi\approx\xi'$ IF $\exists$ 1-PARAMETER FAMILY OF SELF-DIFFEOMORPHISM $(\phi_t)_{t\in[0,1]}$ OF M WITH
  - $\phi_0=\mathrm{Id}$ &
  - $\xi'=(\phi_1)_*\xi$

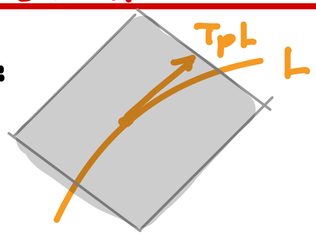
THM (GRAY STABILITY): "HOMOTOPY = ISOTOPY"
ANY HOMOTOPY $(\xi_t)_{t\in[0,1]}$ OF CONTACT STRUCTURES IS INDUCED BY AN ISOTOPY $(\phi_t)_{t\in[0,1]}$:
- $\phi_0=\mathrm{Id}$ &
- $\xi_t=(\phi_t)_*\xi_0$

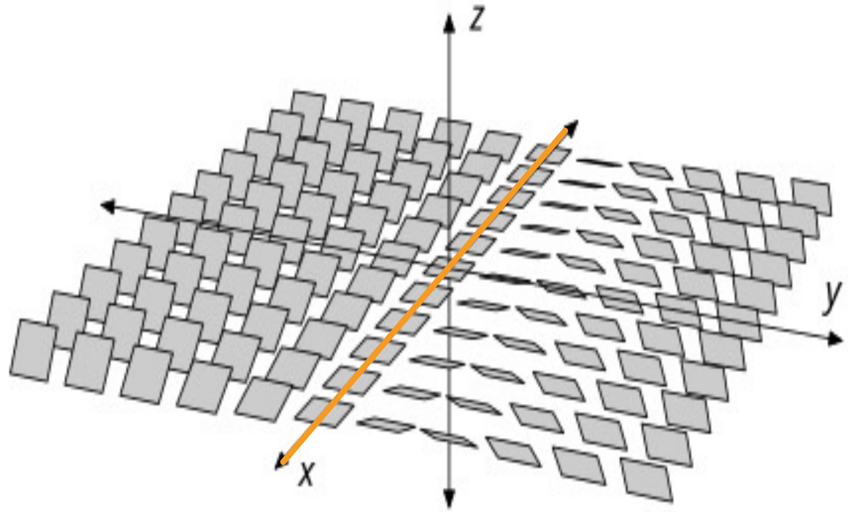
1-DIM : KNOTS



Knots in Contact Structures

DEF: $L^1 \subset M$ is a **Legendrian knot** if $T_pL \subset \xi_p \ \forall p$:

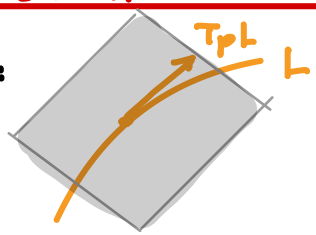


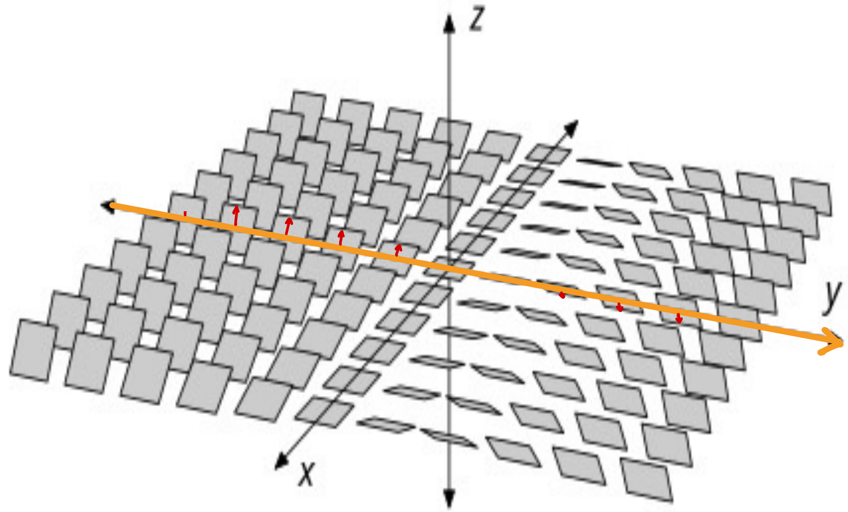


$L = \{y = z = 0\}$

KNOTS IN CONTACT STRUCTURES

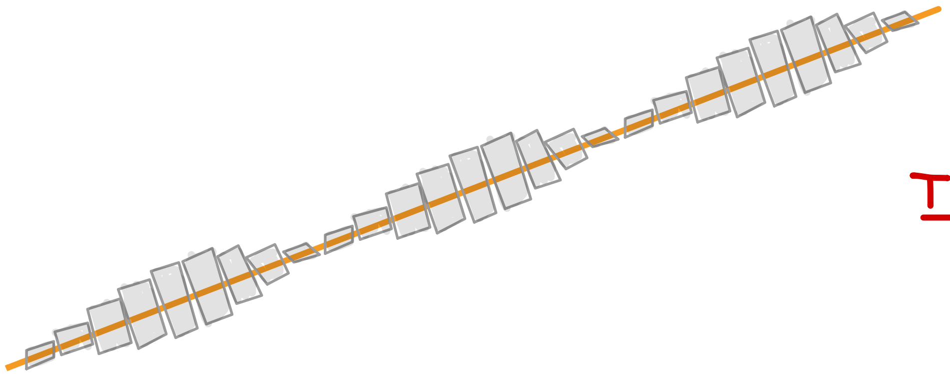
DEF: $L^1 \subseteq M$ IS A LEGENDRIAN KNOT IF $T_pL \subset \xi_p \ \forall p$:





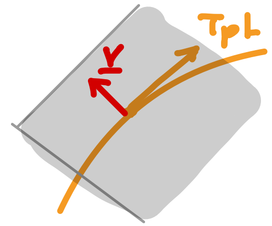
$L = \{x = z = 0\}$

MOTTO: THE CONTACT STRUCTURE ALWAYS "ROTATES" ALONG LEGENDRIANS



THURSTON-BENNEQUIN FRAMING:

PUSH L IN THE DIRECTION OF $\underline{v}$ WHERE $\underline{v}_p \perp T_pL$ & $\underline{v}_p \in \xi_p$



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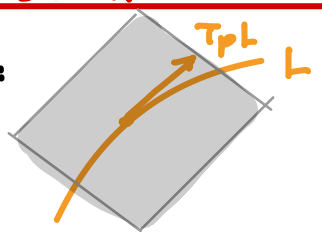

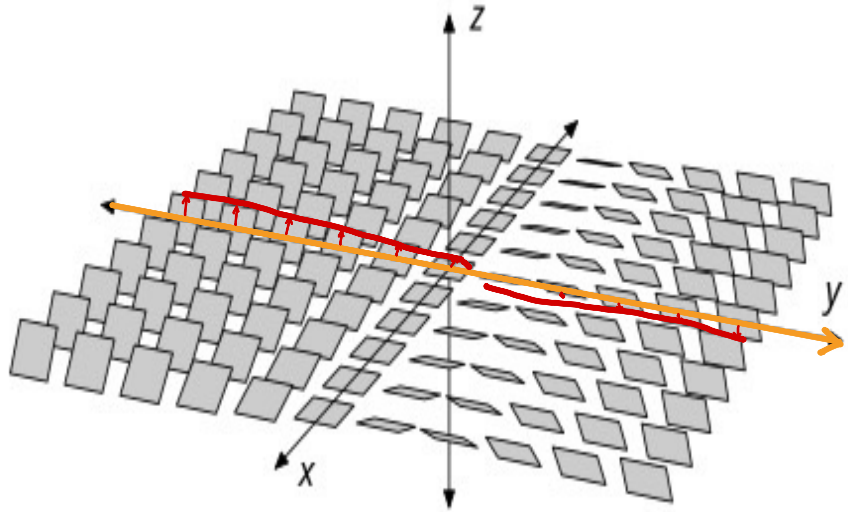

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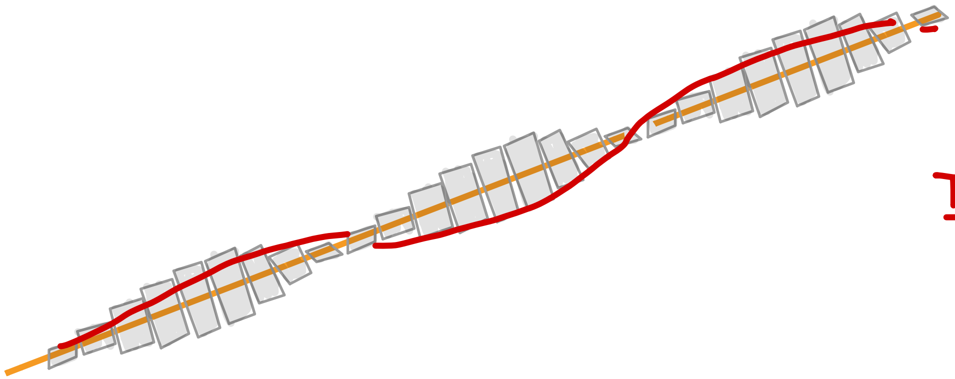

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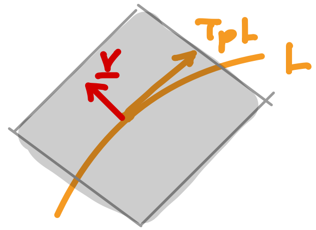
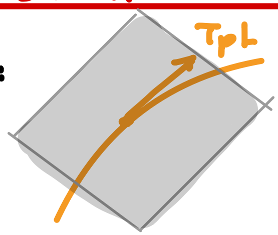
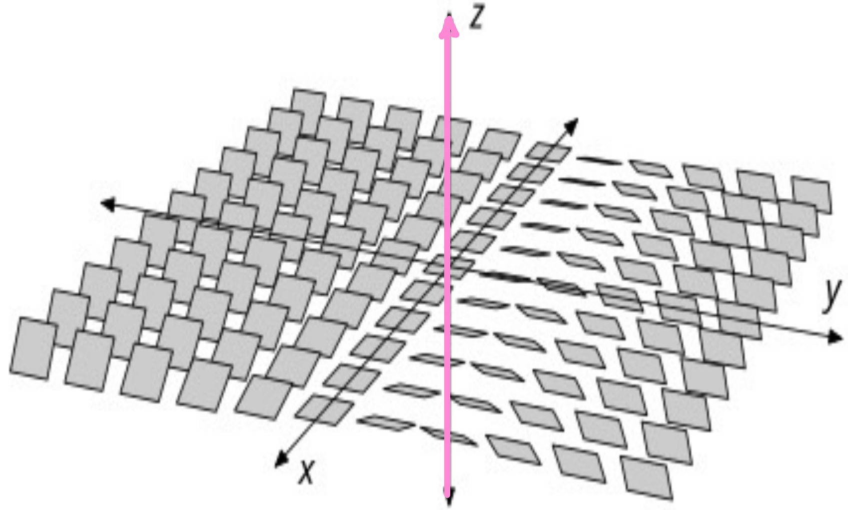


KNOTS IN CONTACT STRUCTURES

DEF: $L^1 \subset M$ IS A **LEGENDRIAN KNOT**

IF $T_pL < \xi_p$ $\forall p$:





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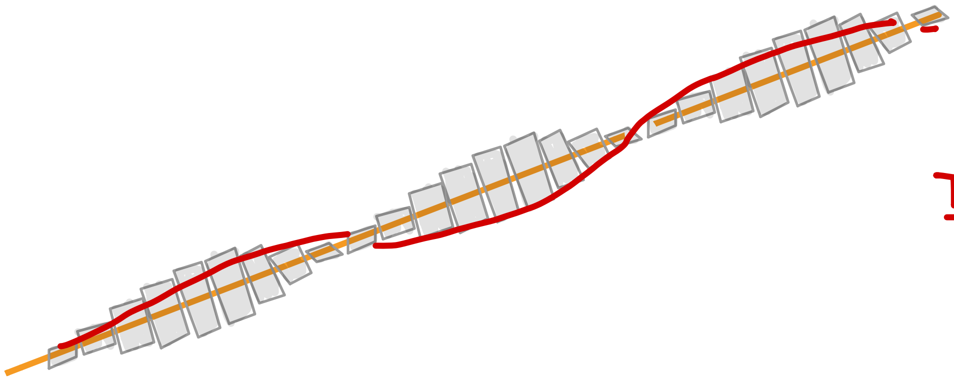

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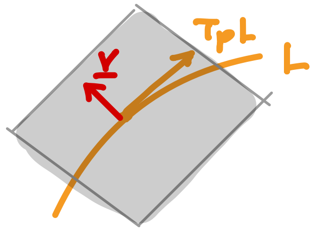
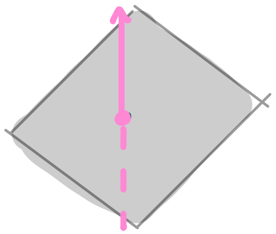


DEF: $T^1 \subset M$ IS A **TRANSVERSE KNOT** IF $T_pT \pitchfork \xi_p$ $\forall p$:

$\alpha(T_pT) > 0$

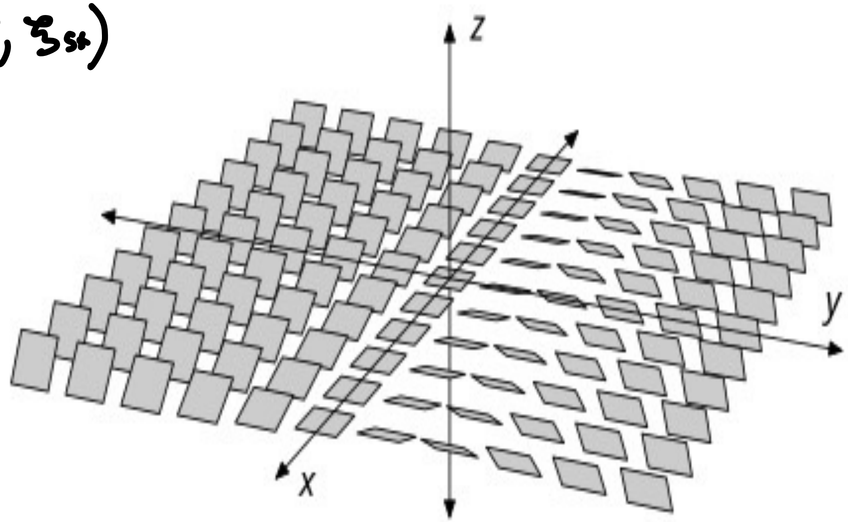


LEGENDRIAN APPROXIMATION

THM: ANY KNOT $K \hookrightarrow (M, \xi)$ CAN BE C^0-APPROXIMATED BY A LEGENDRIAN KNOT

IDEA OF PROOF: ENOUGH TO APPROXIMATE LOCALLY & BY DARBOUX THM WE CAN WORK IN $(\mathbb{R}^3, \xi_{st})$

$$\xi_{st} = \ker(dz - y\,dx)$$



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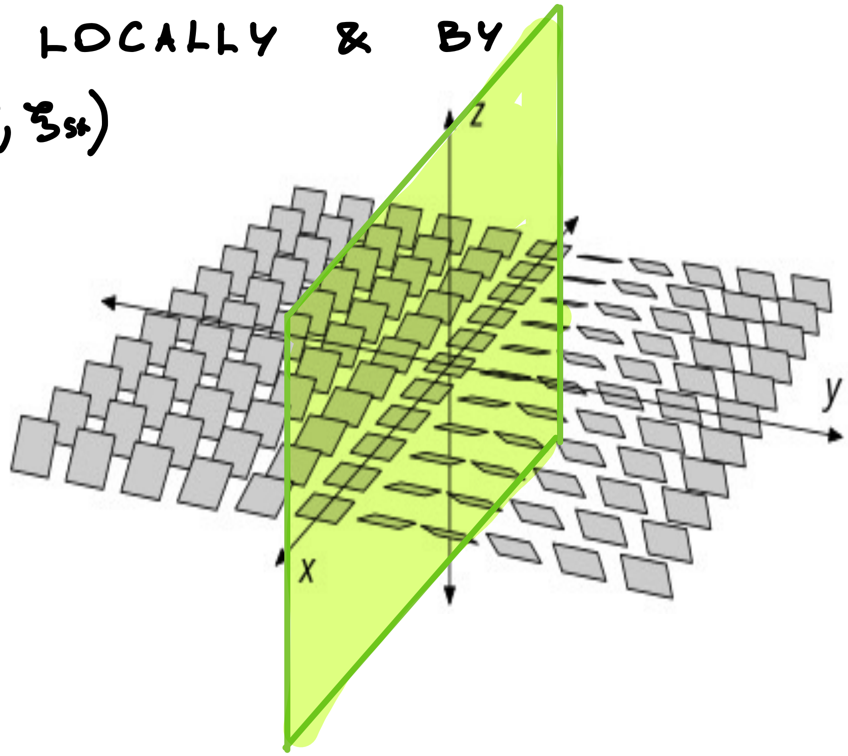
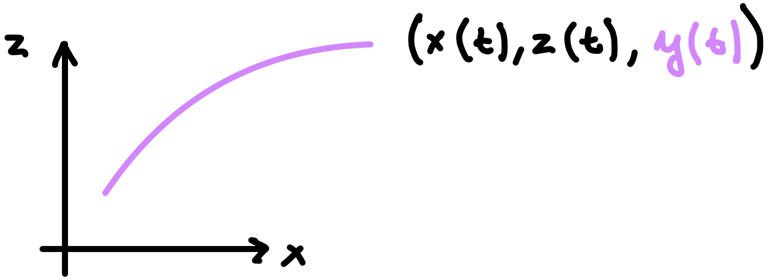


$$\xi_{st} = \ker(dz - y\,dx) \longleftrightarrow \text{"}\, y = \frac{dz}{dx} \,\text{"}$$

WE CAN READ OFF y-COORDINATE FROM THE PROJECTION TO (x,z)-PLANE

- PROJECT K TO THE (x,z)-PLANE



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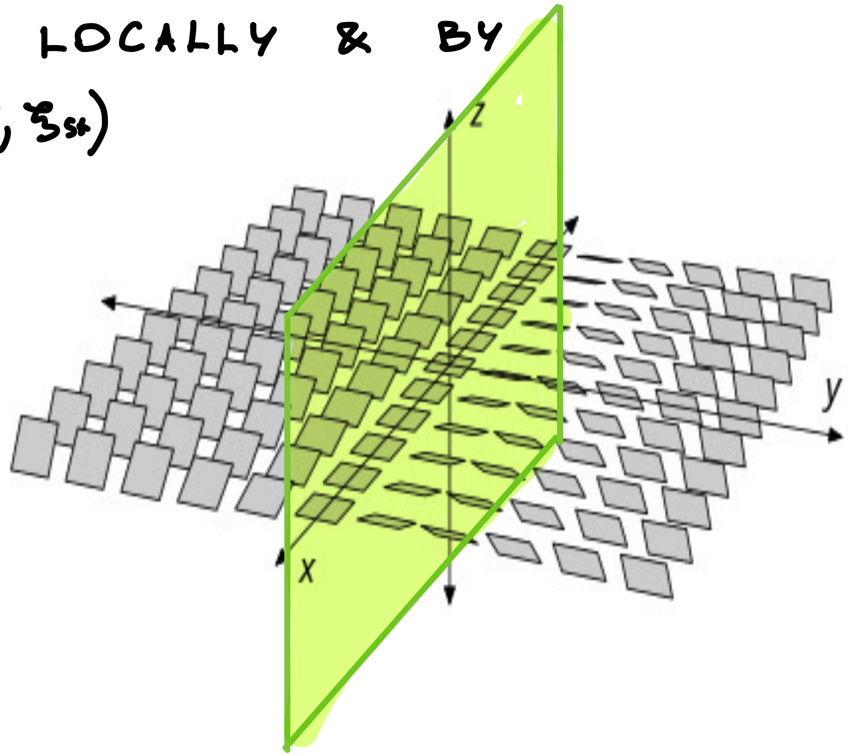

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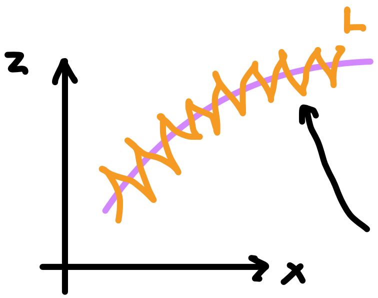
$\boxed{\text{WE CAN READ OFF } y\text{-COORDINATE FROM THE PROJECTION TO } (x,z)\text{-PLANE}}$

- PROJECT K TO THE (x,z)-PLANE



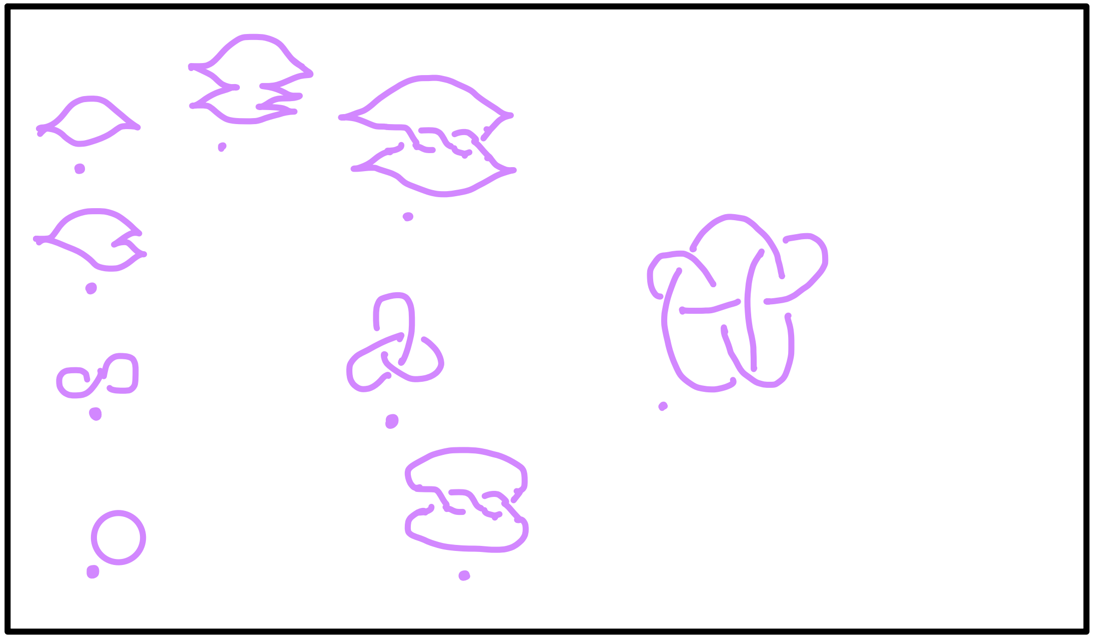
$(x(t), z(t), y(t))$

SLOPE $\frac{dz}{dx}$ "CLOSE" TO $y(t)$ $\Rightarrow$ L IS C^0-CLOSE TO K $\blacksquare$

COR: ANY SMOOTH KNOT CAN BE REPRESENTED BY A LEGENDRIAN KNOT

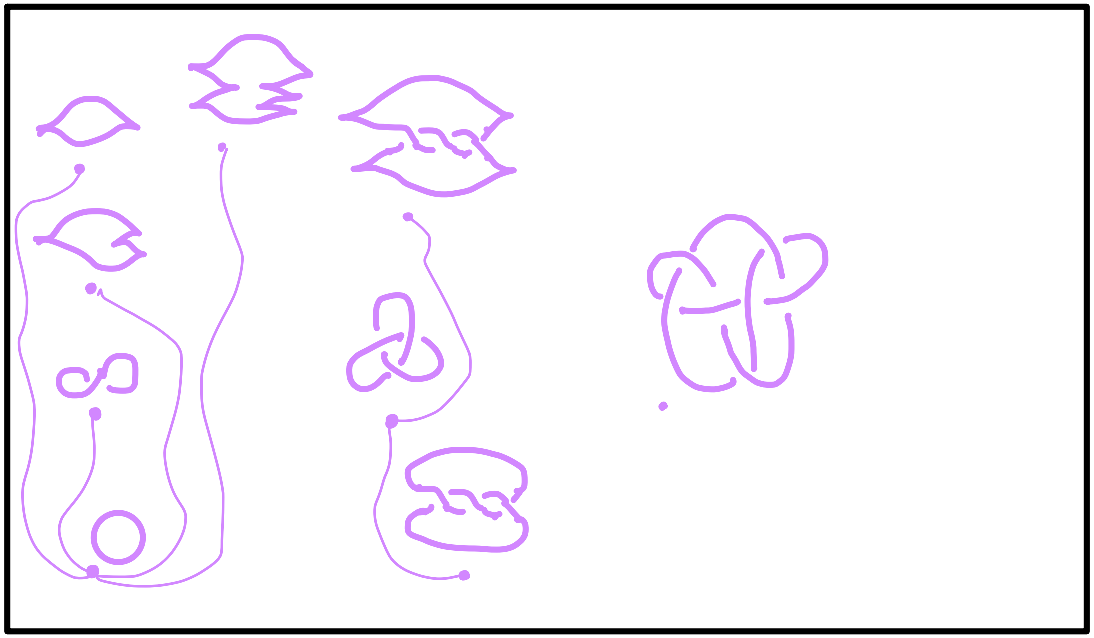
Isotopies of Knots

Smooth Knots



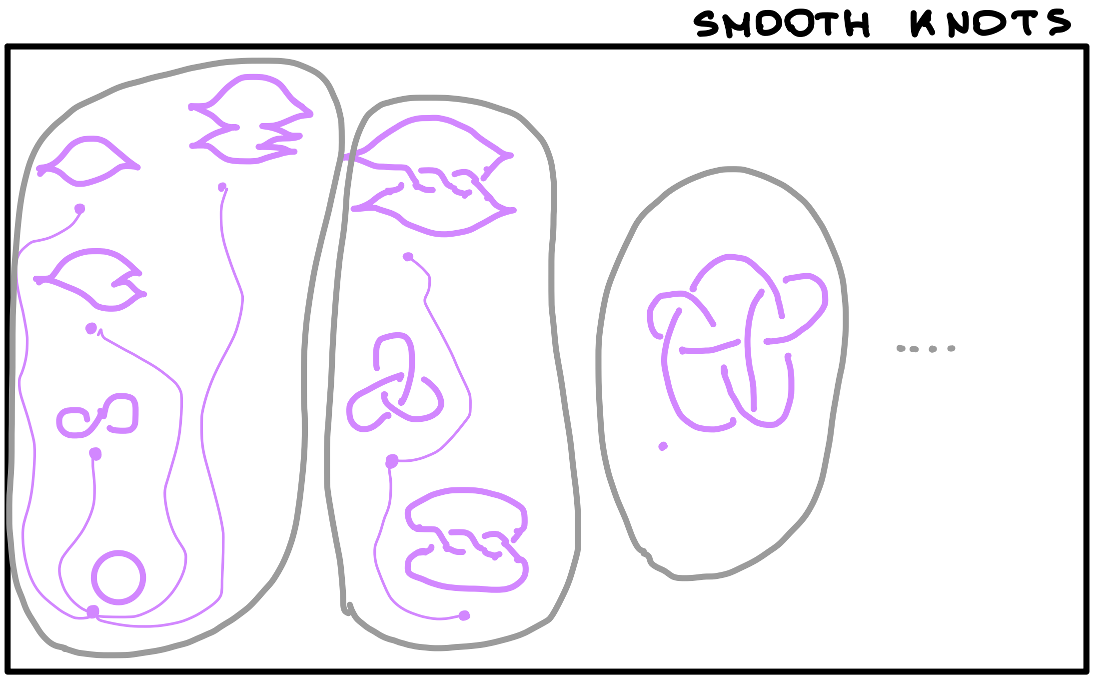
ISOTOPIES OF KNOTS

SMOOTH KNOTS



ISOTOPY: PATH IN THE SPACE OF KNOTS

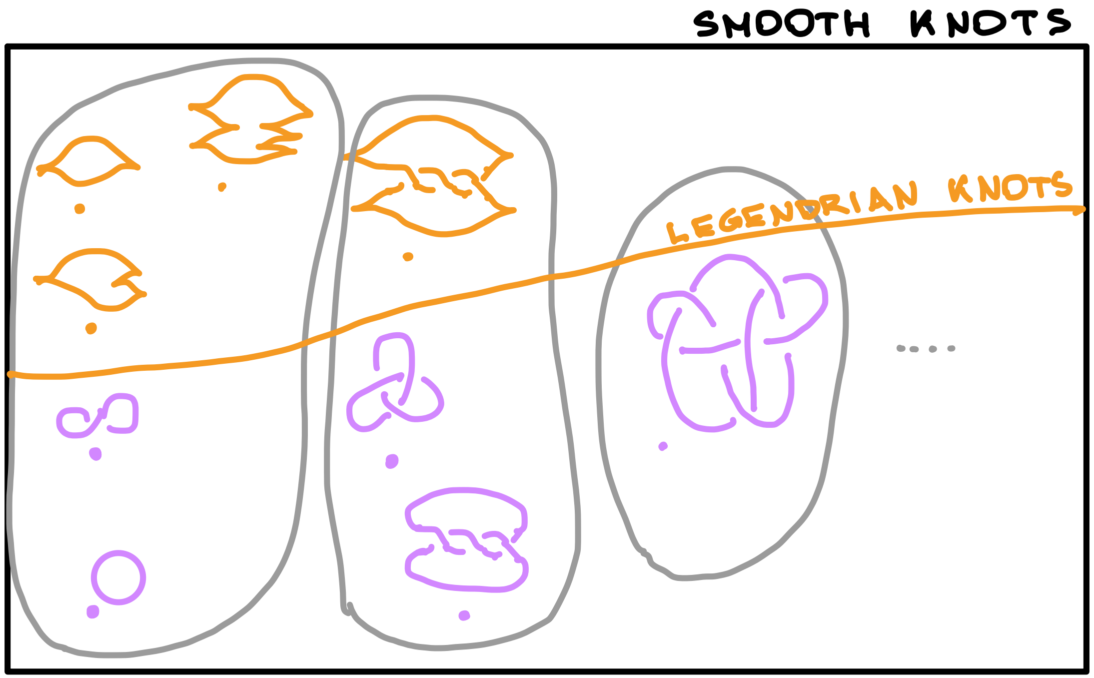
Isotopies of Knots



ISOTOPY: PATH IN THE SPACE OF KNOTS

ISOTOPY CLASS: CONNECTED COMPONENT

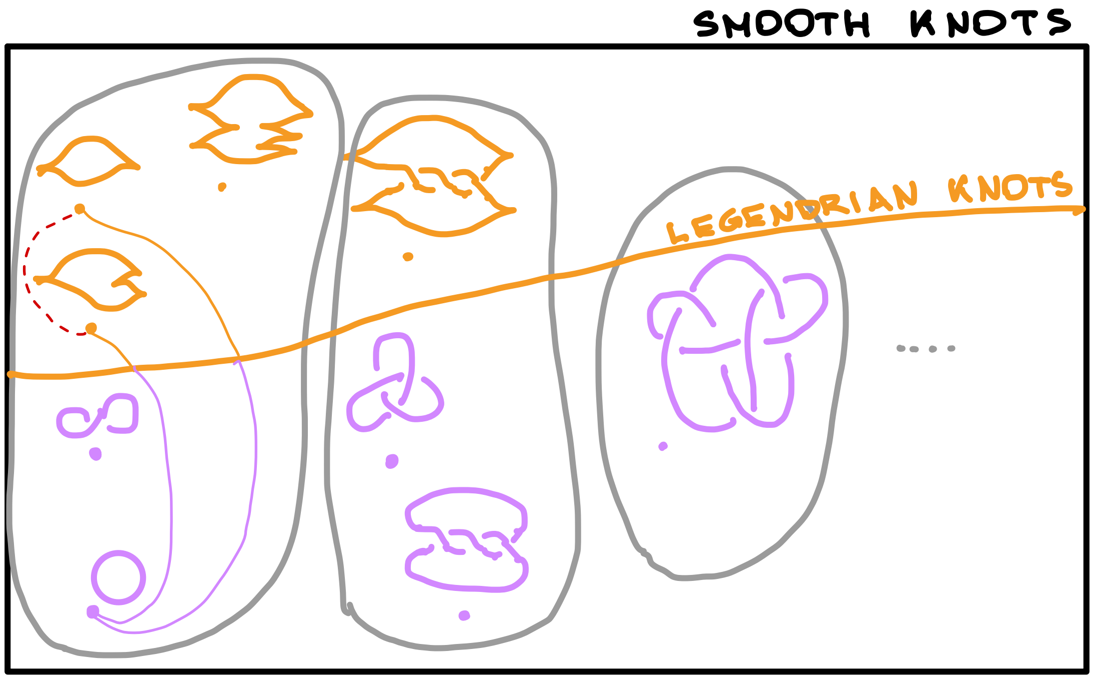
Isotopies of Knots



Isotopy: path in the space of knots

Isotopy class: connected component

ISOTOPIES OF KNOTS



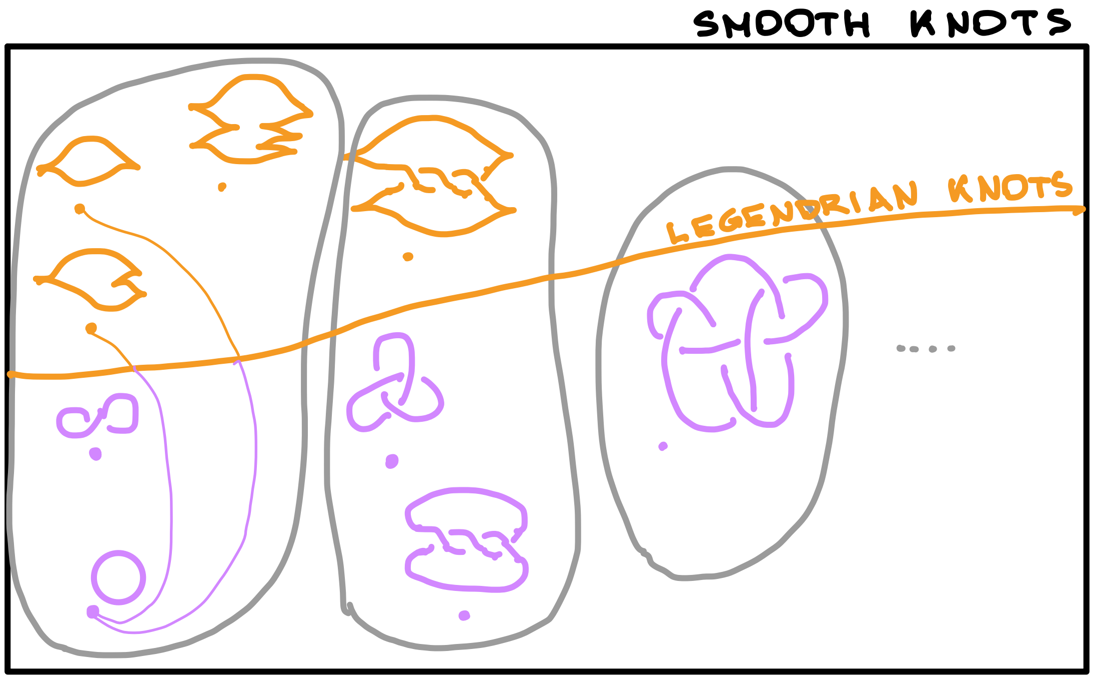
ISOTOPY: PATH IN THE SPACE OF KNOTS

ISOTOPY CLASS: CONNECTED COMPONENT

LEGENDRIAN ISOTOPY: PATH IN THE SPACE OF LEGENDRIAN KNOTS

IF L ISOTOPIC TO L' IMPLIES L LEGEDRIAN ISOTOPIC TO L'?

ISOTOPIES OF KNOTS



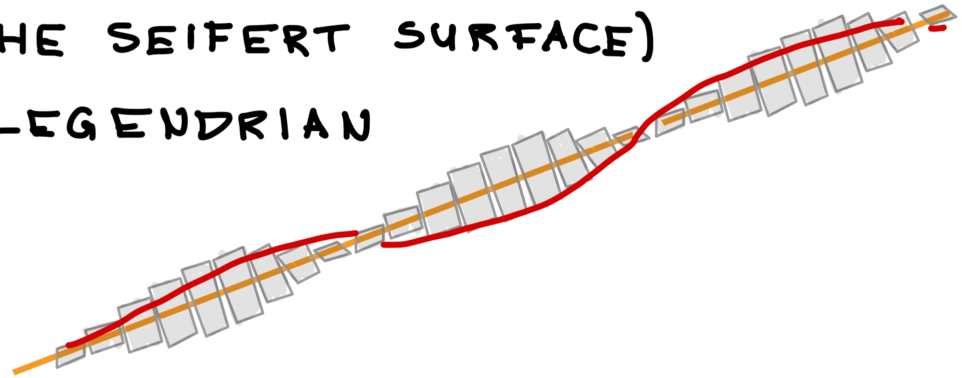
ISOTOPY: PATH IN THE SPACE OF KNOTS

ISOTOPY CLASS: CONNECTED COMPONENT

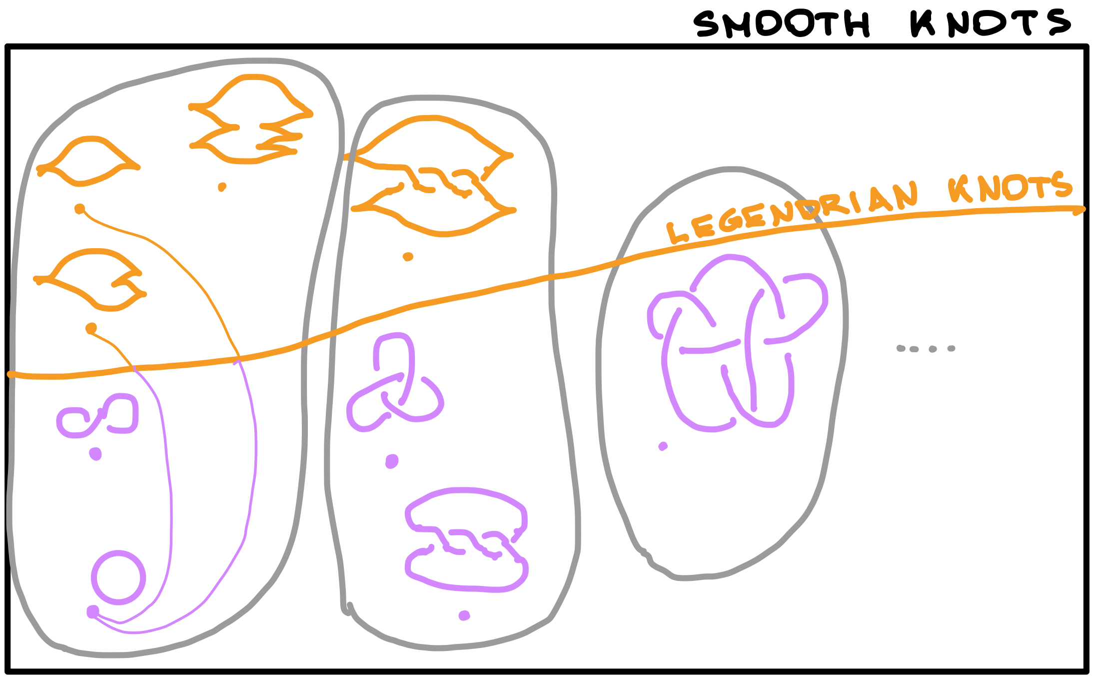
LEGENDRIAN ISOTOPY: PATH IN THE SPACE OF LEGENDRIAN KNOTS

IF L ISOTOPIC TO L' IMPLIES L LEGEDRIAN ISOTOPIC TO L'?

NO! - THE TWISTING OF ξ (W.R.T THE SEIFERT SURFACE) DOESN'T CHANGE DURING LEGENDRIAN ISOTOPY



ISOTOPIES OF KNOTS



ISOTOPY: PATH IN THE SPACE OF KNOTS

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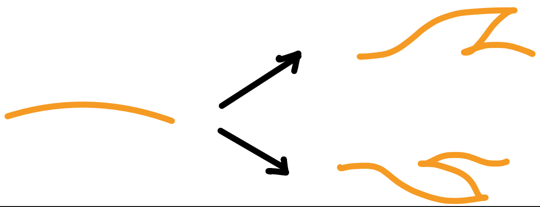
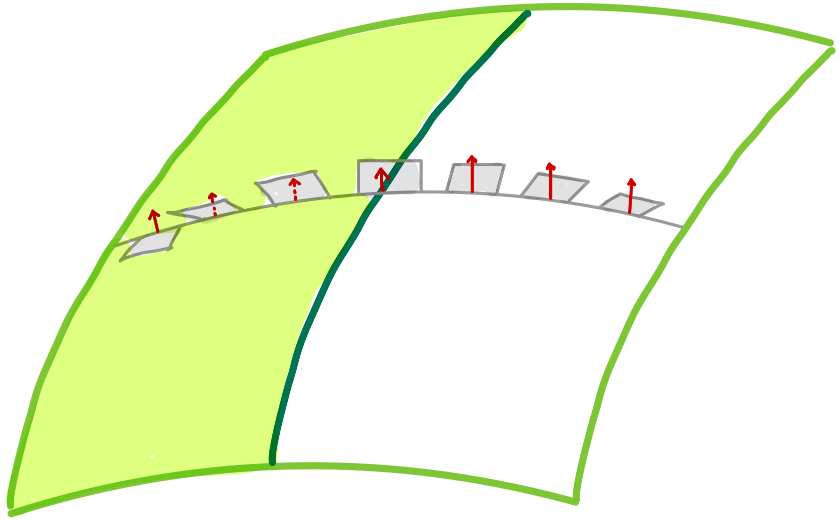
STABILISATION:  CHANGES TWISTING

THM (FUCHS-TABACHNIKOV): L IS ISOTOPIC TO L' $\Longleftrightarrow$ AFTER SOME STABILISATIONS St(L) IS LEGEDRIAN ISOTOPIC TO St(L')

2-DIM: SURFACES

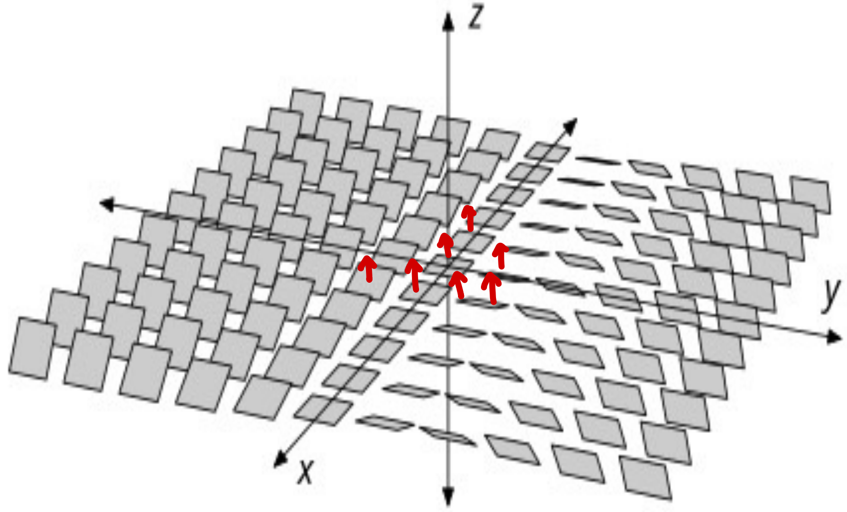


Surfaces in Contact Structures

DEF: A <u>contact vectorfield</u> $X \in \mathfrak{X}(M)$ is a vectorfield whose flow preservs ξ

$$\Updownarrow$$

$$\mathcal{L}_X \alpha = g\alpha \quad \text{for some } g : M \to \mathbb{R}$$



$X = \frac{\partial}{\partial z}$

SURFACES IN CONTACT STRUCTURES

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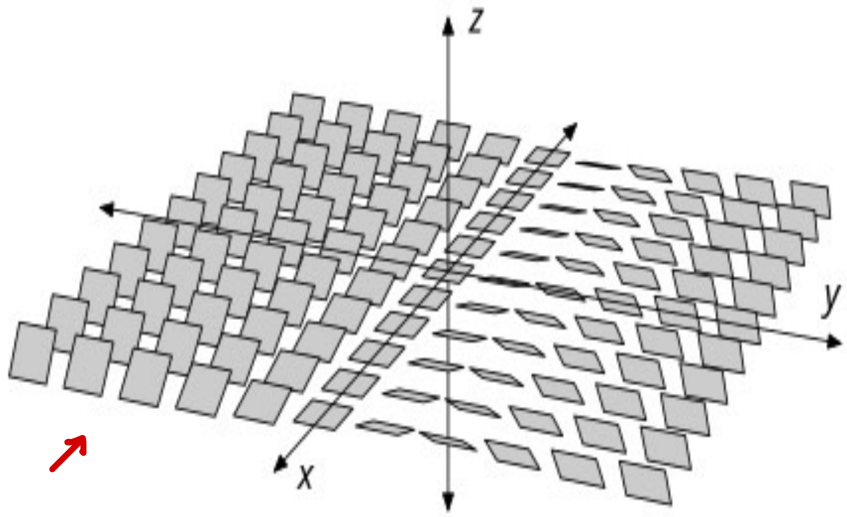

$X = \frac{\partial}{\partial x}$

Surfaces in Contact Structures

DEF: A CONTACT VECTORFIELD $X \in \mathfrak{X}(M)$ IS A VECTORFIELD WHOSE FLOW PRESERVS ξ

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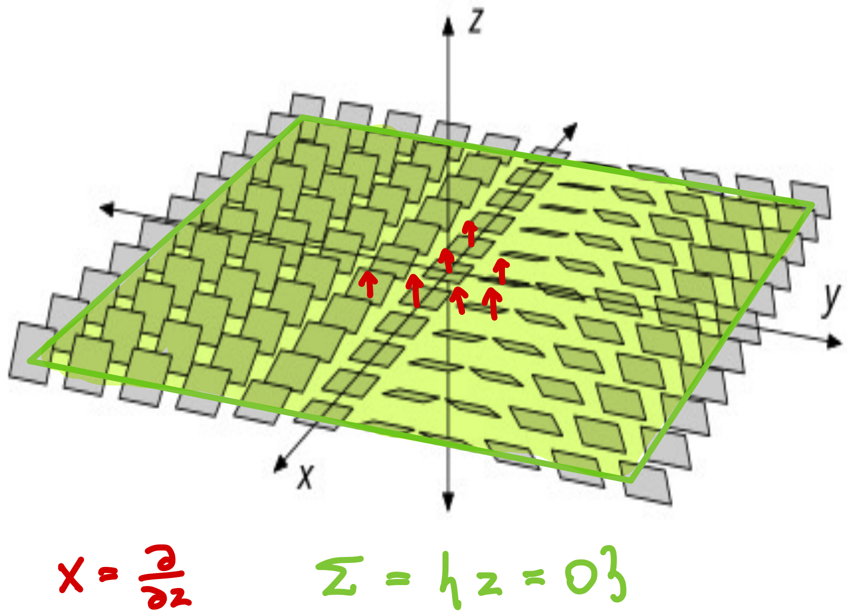
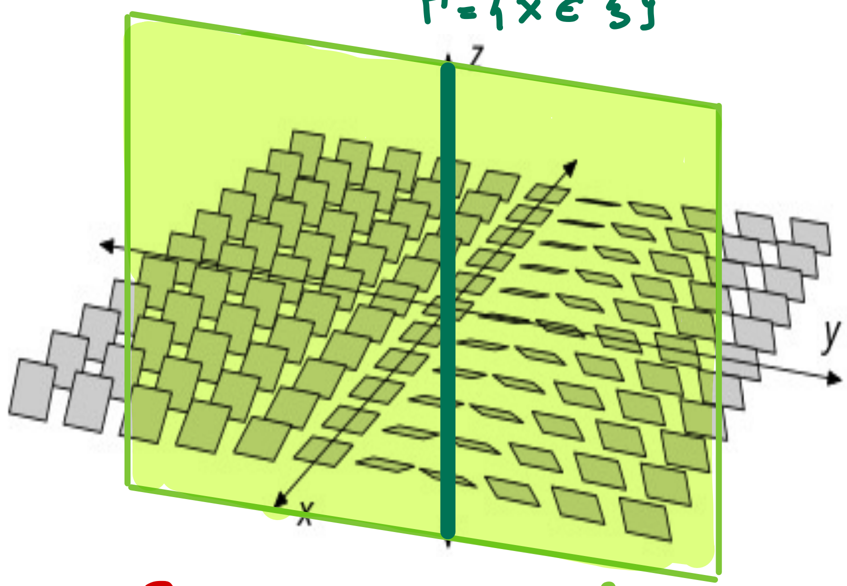


DEF: $\Sigma \hookrightarrow M$ IS CONVEX IF $\exists$ X CONTACT VECTORFIELD $X \pitchfork \Sigma$

SURFACES IN CONTACT STRUCTURES



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DEF: $\Sigma \hookrightarrow M$ IS CONVEX IF $\exists$ X CONTACT VECTORFIELD $X \pitchfork \Sigma$

EQUIVALENTLY: Σ HAS A NEIGHBOURHOOD $N(\Sigma) \cong \Sigma \times \underset{t}{I}$ WITH I-INVARIANT CONTACT STRUCTURE ξ

$$\Updownarrow$$

$\alpha = \beta + g\,dt$ WHERE $\beta \in \Omega^1(\Sigma)$ & $g : M \to \mathbb{R}$

FACT (GIROUX): TO UNDERSTAND ξ ON $N(\Sigma)$ ONE ONLY NEEDS TO KNOW $\Gamma = \{g = 0\} = \{\frac{\partial}{\partial t} \in \xi\} = \{X \in \xi\}$

CONVEX SURFACES (GIROUX)

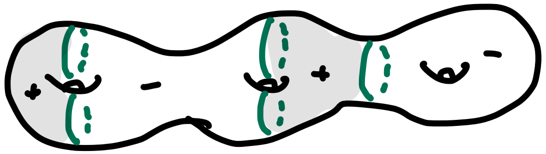
DEF: $\Sigma \hookrightarrow M$ IS **CONVEX** IF $\exists$ CONTACT VECTORFIELD X : $\Sigma \pitchfork X$

DEF: $\Gamma = \{x \in \xi\} = \{\alpha(X) = 0\} \subset \Sigma$ IS THE **DIVIDING CURVE**

PROP: - THE ISOTOPY CLASS OF Γ IS INDEPENDENT OF THE CHOICE OF X

- Γ DIVIDES Σ INTO TWO PIECES: $\Sigma_+ = \{\alpha(X) > 0\}$

$\Sigma_- = \{\alpha(X) < 0\}$



THM (THE DIVIDING CURVE DETERMINES ξ NEAR Σ)

Σ, Σ' CONVEX SURFACES W/ ISOTOPIC DIVIDING CURVES
$\implies \exists$ $N(\Sigma), N(\Sigma')$ NEIGHBOURHOODS THAT ARE CONTACTOMORPHIC

THM (CONVEX SURFACES ARE C^∞-GENERIC): ANY SURFACE Σ CAN BE C^∞-SMALL ISOTOPED TO BE CONVEX

Contact Manifolds with Boundary

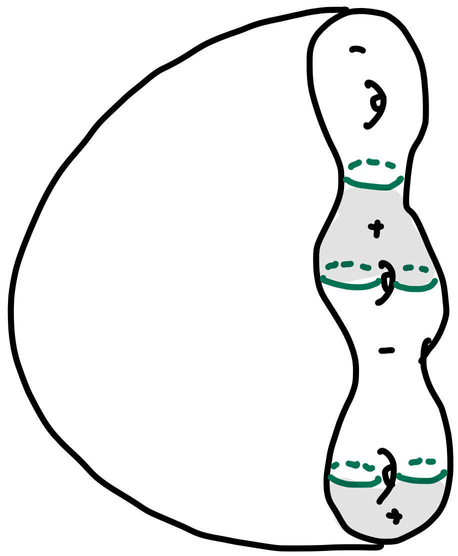
DEF: $(\Sigma_t)_{t\in[0,1]}$ IS A **CONVEX ISOTOPY** IF Σ_t IS CONVEX $(\forall t \in [0,1])$

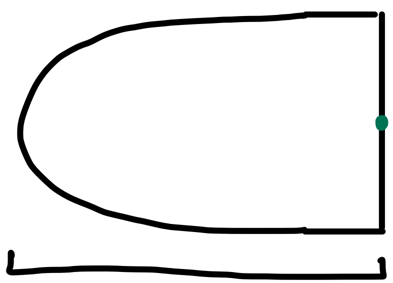
WE WILL WORK WITH

M^3 3-MANIFOLD WITH BOUNDARY,

ξ CONTACT STRUCTURE ON M, S.T

∂M IS CONVEX





CONTACT MANIFOLDS WITH BOUNDARY

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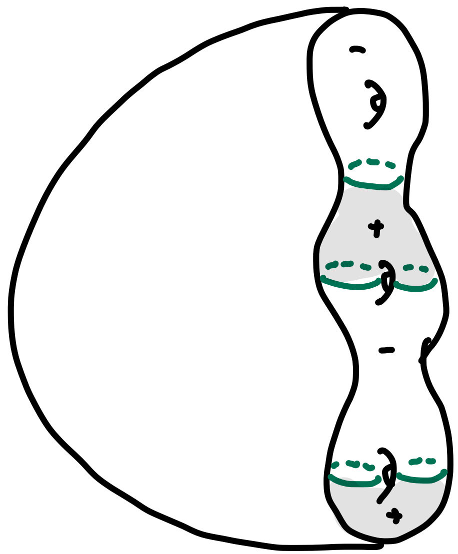


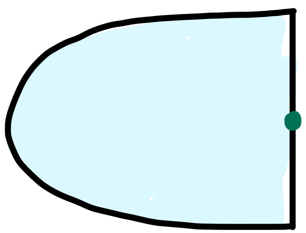
WE WILL WORK WITH

M^3 3-MANIFOLD WITH BOUNDARY,

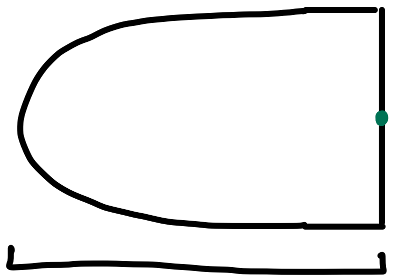
ξ CONTACT STRUCTURE ON M, S.T

∂M IS CONVEX

- SAME FOR (M', ξ')



(M', ξ')



(M, ξ)

Contact Manifolds with Boundary

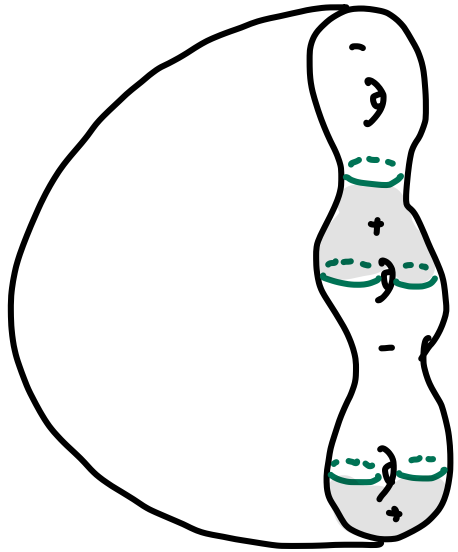
DEF: $(\Sigma_t)_{t\in[0,1]}$ is a **convex isotopy** if Σ_t is convex $(\forall t \in [0,1])$

We will work with

M^3 3-manifold with boundary,

ξ contact structure on M, s.t

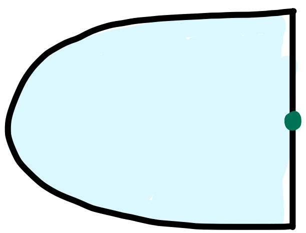
∂M is convex



- same for (M',ξ')

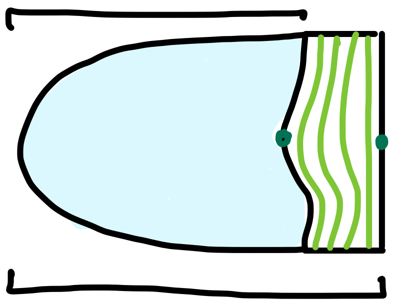
DEF: **Weakly contactomorphic**: $\exists$ embedding $(M',\xi') \xhookrightarrow{\iota} (M,\xi)$

such that $\iota(\partial M')$ is convex isotopic to ∂M



(M',ξ')

$(\iota(M'), \iota_* \xi')$

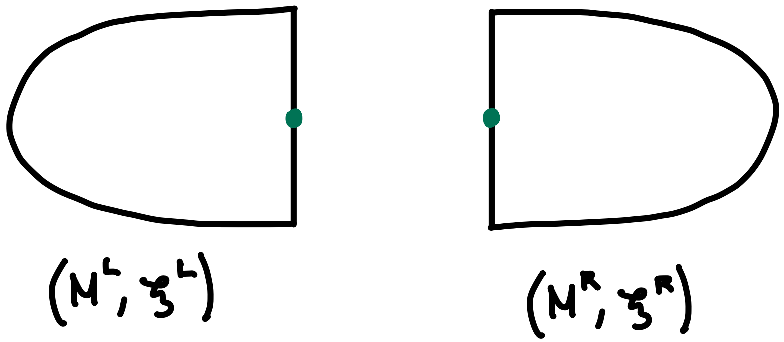


(M,ξ)

GLUING CONTACT STRUCTURES

WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH MATCHING DIVIDING CURVES

IDEA



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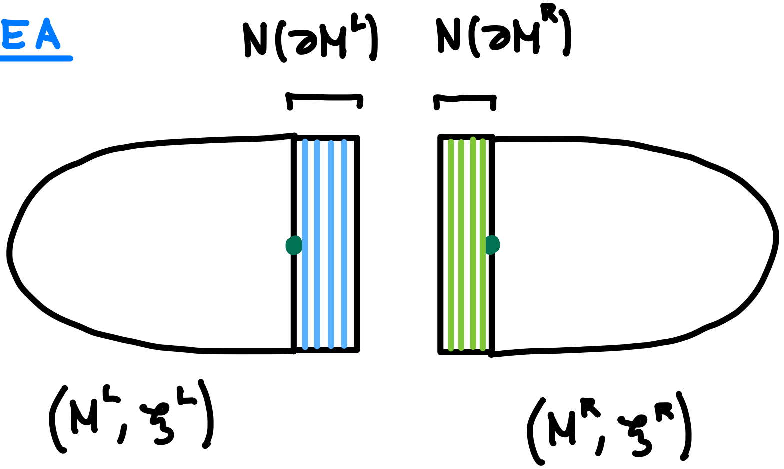

STEP 1: ADD I-INVARIANT PART TO EACH

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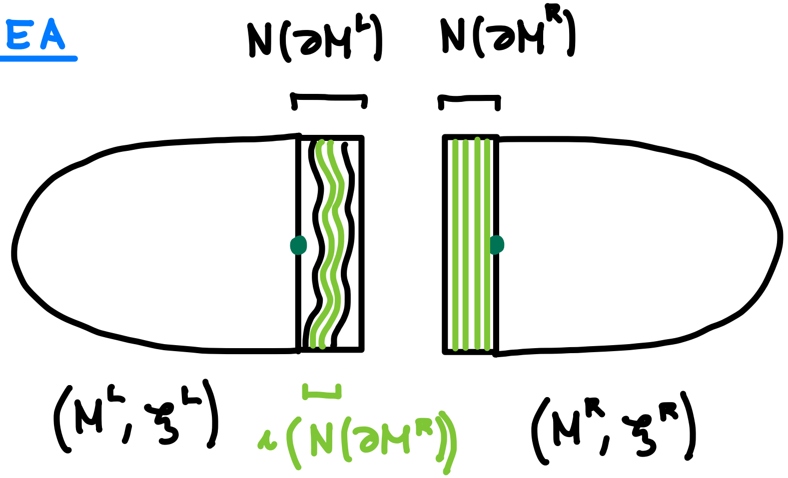
WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH MATCHING DIVIDING CURVES

IDEA



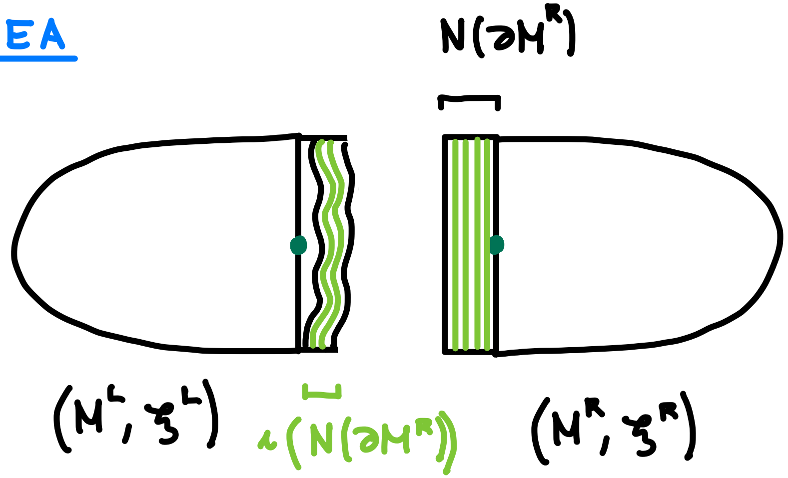
STEP 1: ADD I-INVARIANT PART TO EACH

STEP 2: FIND $N(\partial M^R)$ IN $N(\partial M^L)$

GLUING CONTACT STRUCTURES

WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH MATCHING DIVIDING CURVES

IDEA



STEP 1: ADD I-INVARIANT PART TO EACH

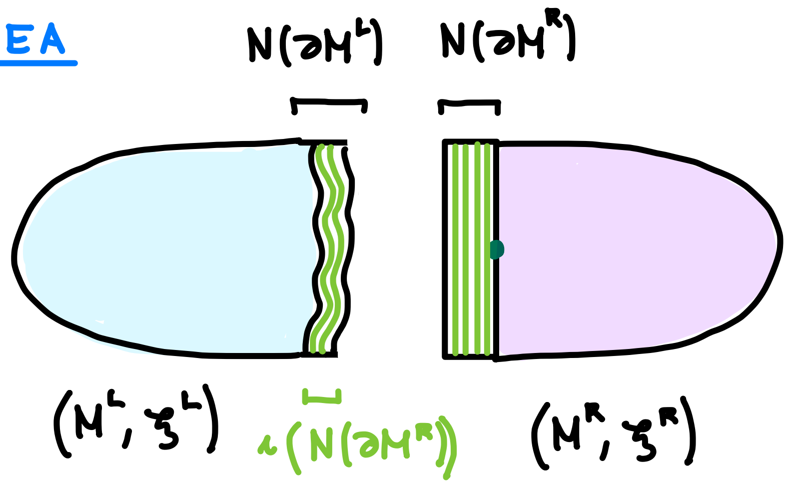
STEP 2: FIND $N(\partial M^R)$ IN $N(\partial M^L)$

STEP 3: TRUNCATE M^L AT $\iota(\partial M^R \times \{1\})$

GLUING CONTACT STRUCTURES

WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH MATCHING DIVIDING CURVES

IDEA

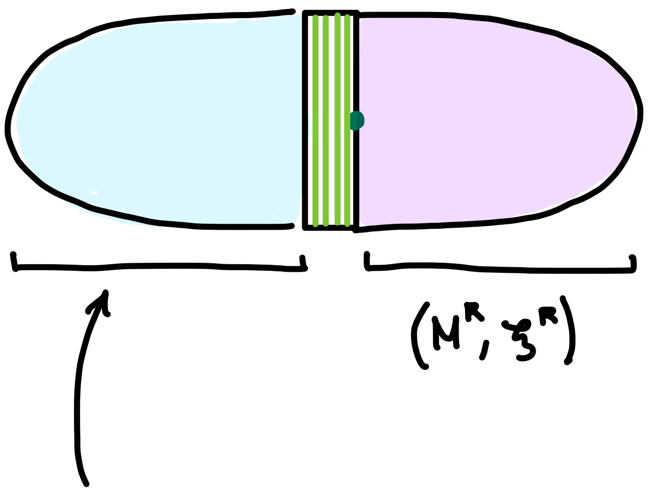


STEP 1: ADD I-INVARIANT PART TO EACH

STEP 2: FIND $N(\partial M^R)$ IN $N(\partial M^L)$

STEP 3: TRUNCATE M^L AT $\iota(\partial M^R \times \{1\})$

STEP 4: OVERLAP $\iota(N(\partial M^R))$ WITH $N(\partial M^R)$



THE OBTAINED CONTACT MANIFOLD IS $(M^L, \xi^L) \cup (M^R, \xi^R)$

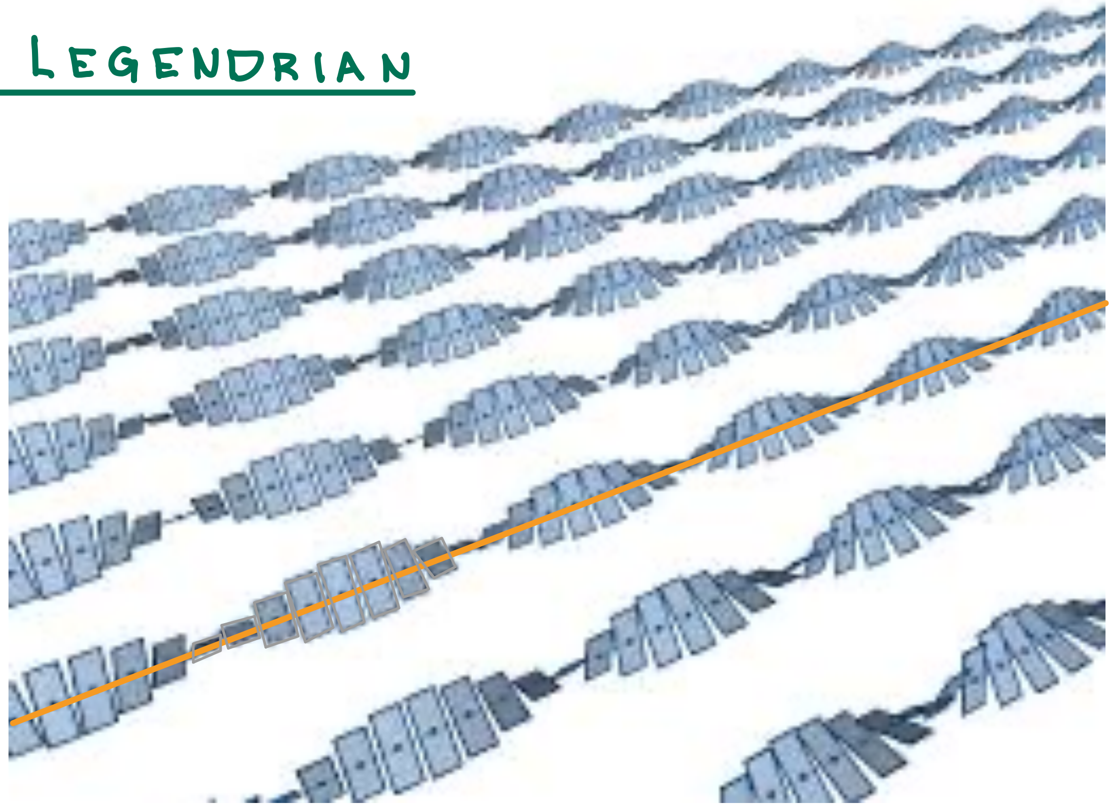
& IT IS WELL DEFINED UP TO CONTACTOMORPHISM

WEAKLY CONTACT ISOTOPIC TO (M^L, ξ^R)

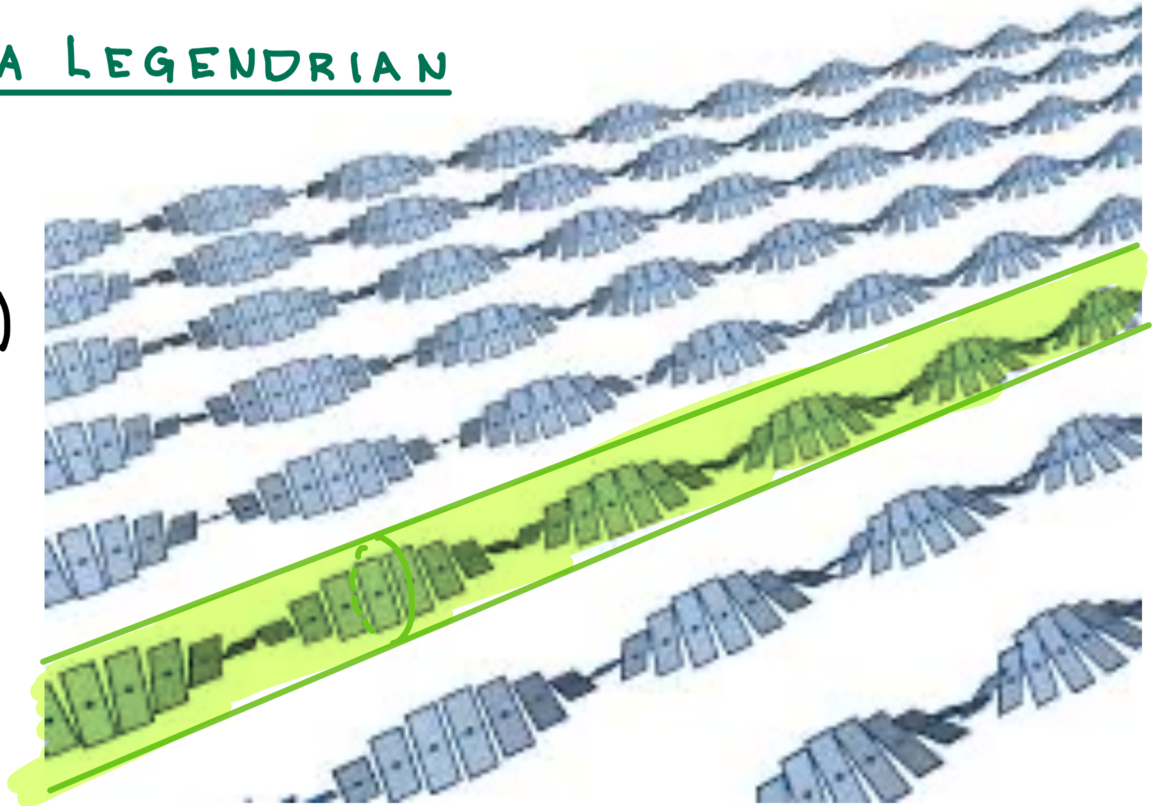
STANDARD NEIGHBOURHOOD OF A LEGENDRIAN

E.G: $\xi = \ker\left(\cos(z)\,dx - \sin(z)\,dy\right)$

(ISOTOPIC TO ξ_{st})



STANDARD NEIGHBOURHOOD OF A LEGENDRIAN



E.G: $\xi = \ker\left(\cos(z)\,dx - \sin(z)\,dy\right)$

(ISOTOPIC TO ξ_{st})

IDENTIFY $(x,y,z) \sim (x,y,z+2\pi n)$

$\rightsquigarrow$ CONTACT STRUCTURE

ON $\mathbb{R}^2 \times S^1$

WITH LEGENDRIAN KNOT

$L_0 = (0,0) \times S^1 \subset \mathbb{R}^2 \times S^1$

NEIGHBOURHOOD $N(L_0) = D^2 \times S^1$

STANDARD NEIGHBOURHOOD OF A LEGENDRIAN

E.G: $\xi = \ker\left(\cos(z)\,dx - \sin(z)\,dy\right)$

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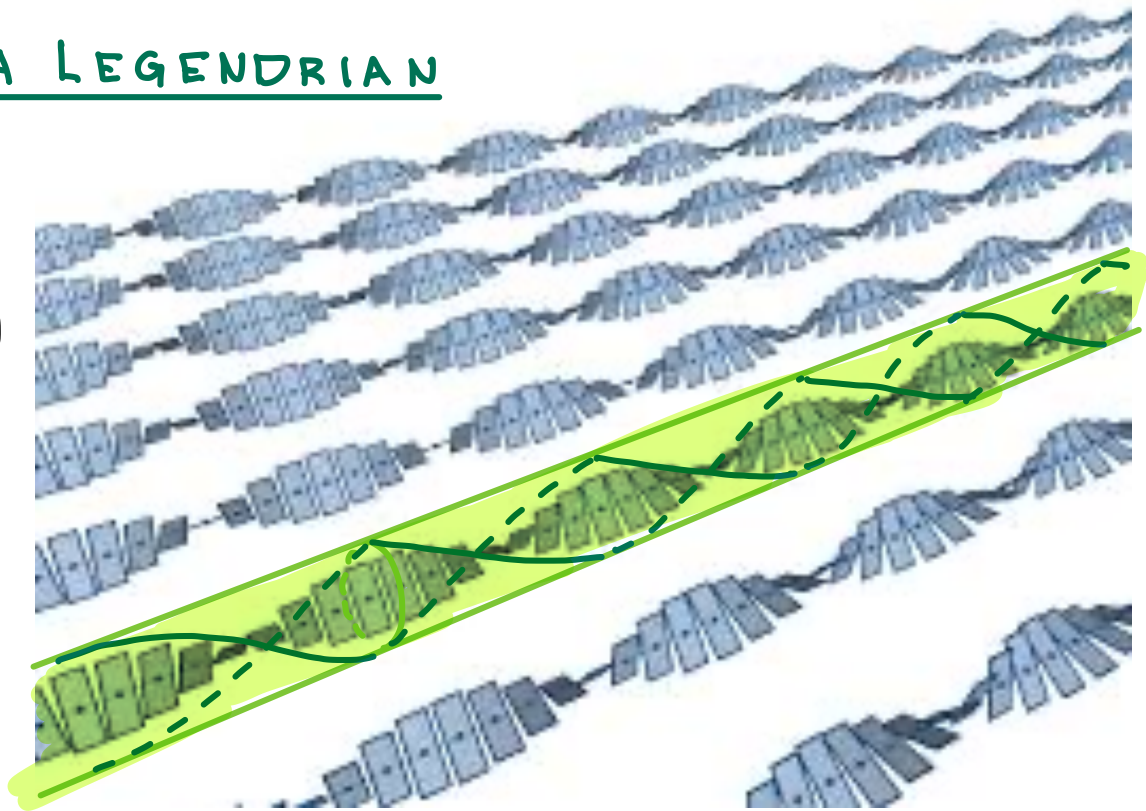

RMK: $\partial N(L_0)$ IS NOT CONVEX, BUT BY A C^∞-SMALL ISOTOPY IT CAN BE MADE CONVEX WITH A TWO COMPONENT DIVIDING CURVE PARALLEL TO $\left(\sin(z), \cos(z), z\right)/\sim$

THIS GIVES THE THURSTON-BENNEQUIN FRAMING

STANDARD NEIGHBOURHOOD OF A LEGENDRIAN

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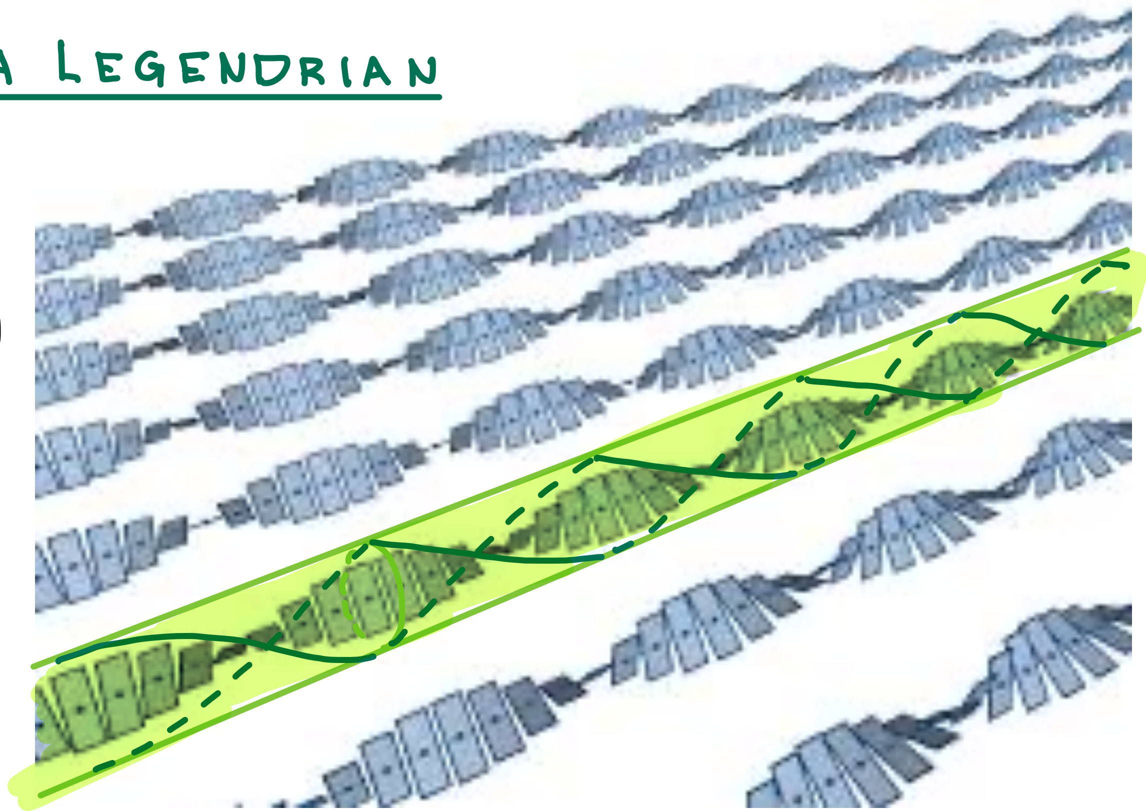

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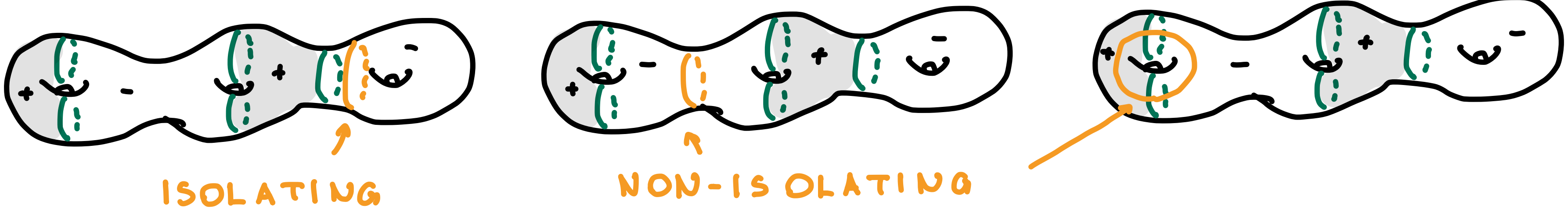
THIS GIVES THE THURSTON-BENNEQUIN FRAMING

THM: ANY LEGENDRIAN KNOT $L \subset (M,\xi)$ HAS A NEIGHBOURHOOD $N(L)$ CONTACTOMORPHIC TO $N(L_0)$

LEGENDRIANS ON CONVEX SURFACES

DEF: $C \subset (\Sigma, \Gamma)$ IS AN **ISOLATING** CURVE, IF SOME COMPONENT OF $\Sigma \setminus C$ IS DISJOINT FROM Γ:



THM (LEGENDRIAN REALISATION PRINCIPLE)

(Σ, Γ) CONVEX SURFACE, $C \subset \Sigma$ NON-ISOLATING CURVE

$\Longrightarrow$ Σ CAN BE ISOTOPED THROUGH CONVEX SURFACES $\psi_t(\Sigma)$ S.T. AFTER THE ISOTOPY $\psi_1(C) \subset \psi_1(\Sigma)$ IS LEGENDRIAN

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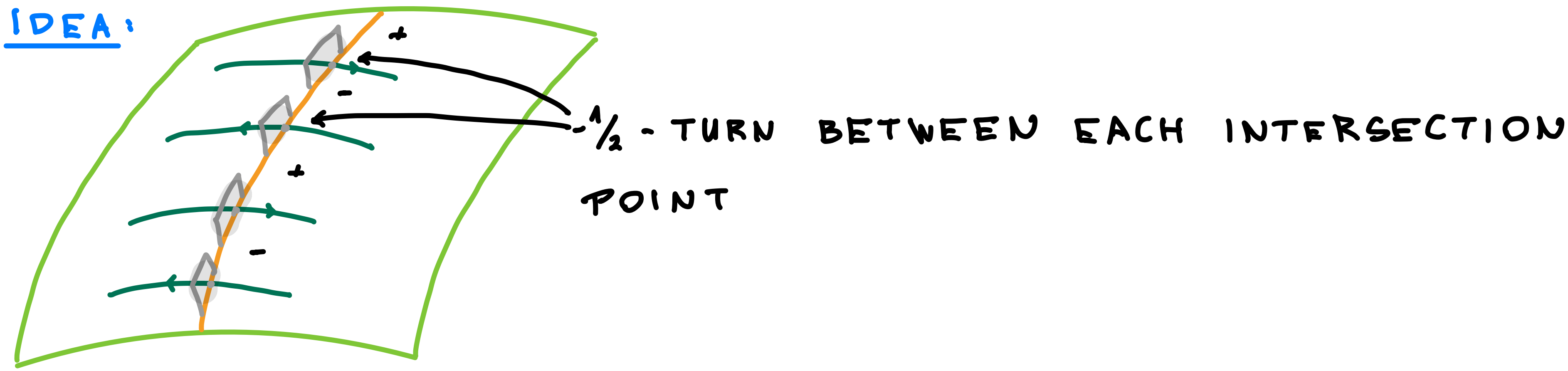
RMK: THE TWISTING OF ξ W.R.T. $T\Sigma$ ALONG C $= -\frac{1}{2}|C \cap \Gamma|$

IDEA:



CONVEX SURFACES WITH LEGENDRIAN BOUNDARY

THM (KANDA): Σ SURFACE WITH LEGENDRIAN BOUNDARY L CAN BE ISOTOPED REL ∂ TO BE CONVEX

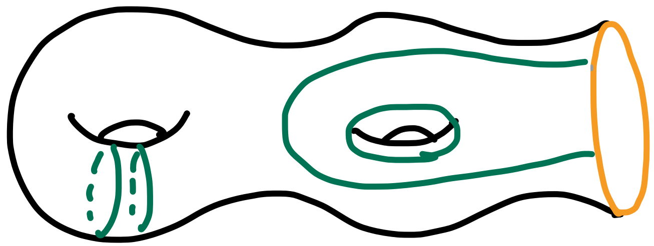
$\Updownarrow$

TWISTING OF ξ W.R.T. Σ ALONG L IS ≤ 0

- THE ISOTOPY CAN BE ASSUMED TO BE C^0-SMALL

  (C^∞-SMALL IF Σ IS ALREADY CONVEX NEAR $\partial\Sigma$)

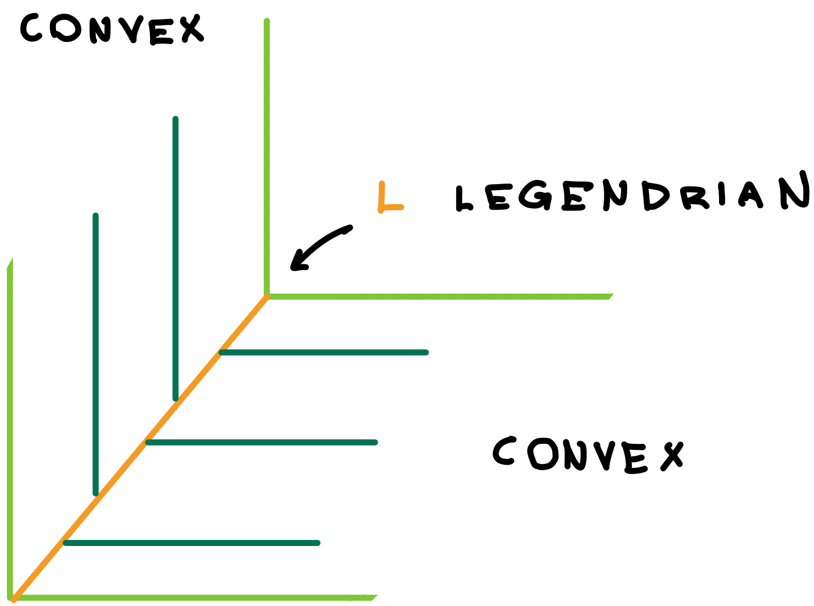
RMK: AFTER THE ISOTOPY



TWISTING OF ξ W.R.T. Σ ALONG $L = -\frac{1}{2}|\Gamma \cap L|$

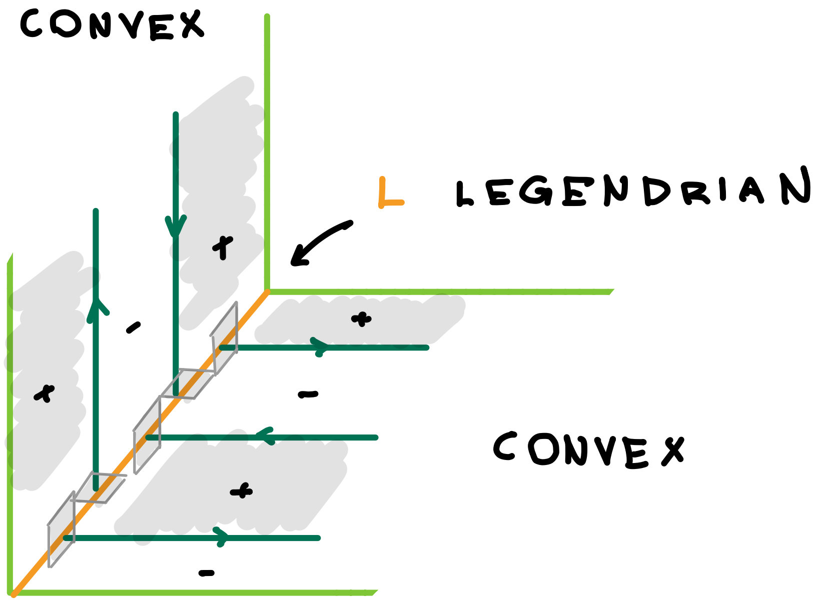
Rounding Edges

Σ_1 & Σ_2 CONVEX SURFACES WITH COMMON LEGENDRIAN BOUNDARY L



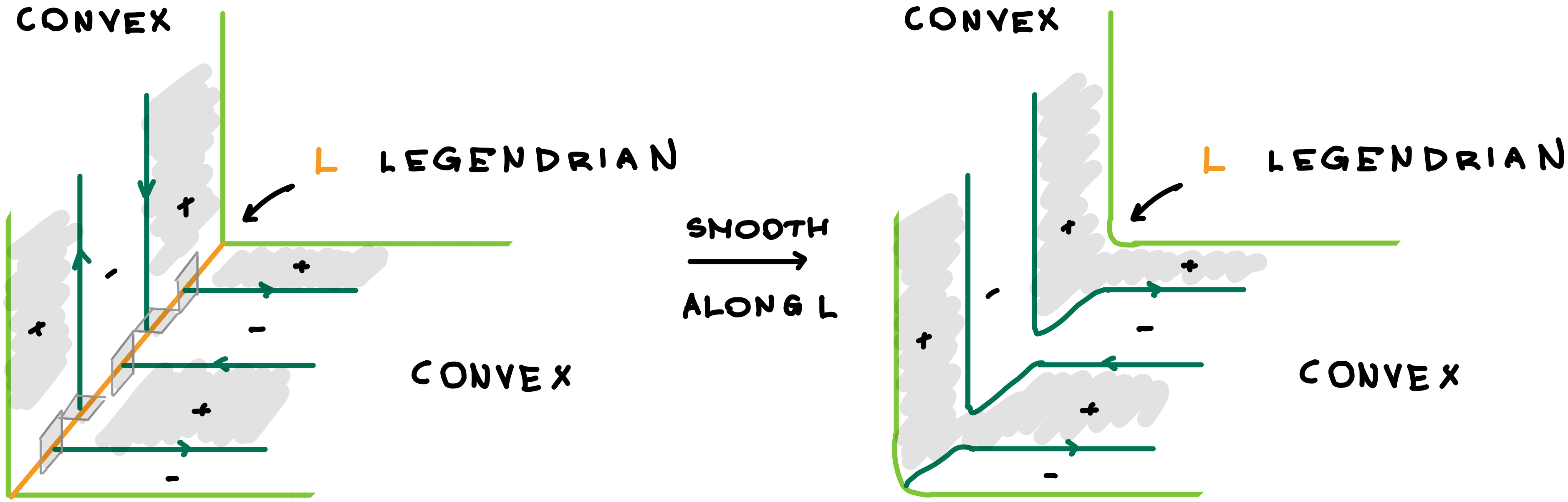
Rounding Edges

Σ_1 & Σ_2 CONVEX SURFACES WITH COMMON LEGENDRIAN BOUNDARY L



Rounding Edges

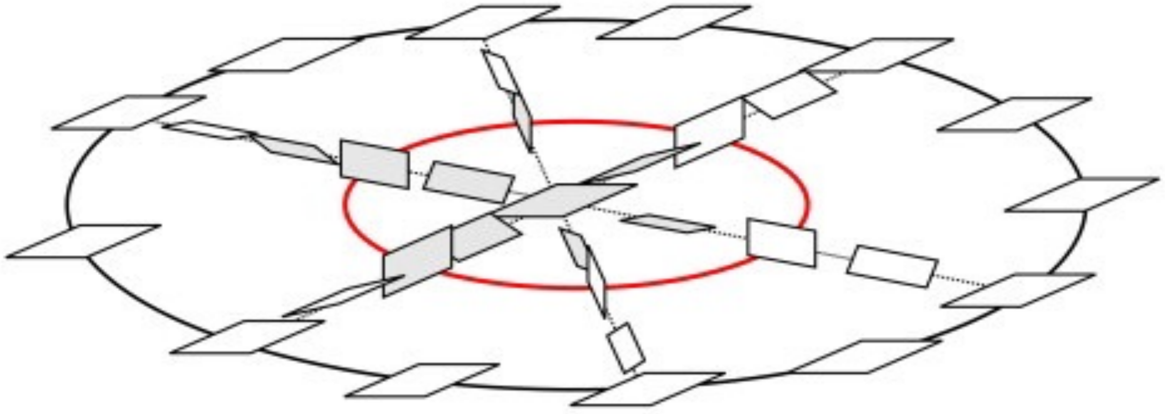
Σ_1 & Σ_2 CONVEX SURFACES WITH COMMON LEGENDRIAN BOUNDARY L THEN THE EDGE L CAN BE ROUNDED & WE GET A NEW SMOOTH CONVEX SURFACE Σ WITH DIVIDING CURVE AS BELOW



TIGHT

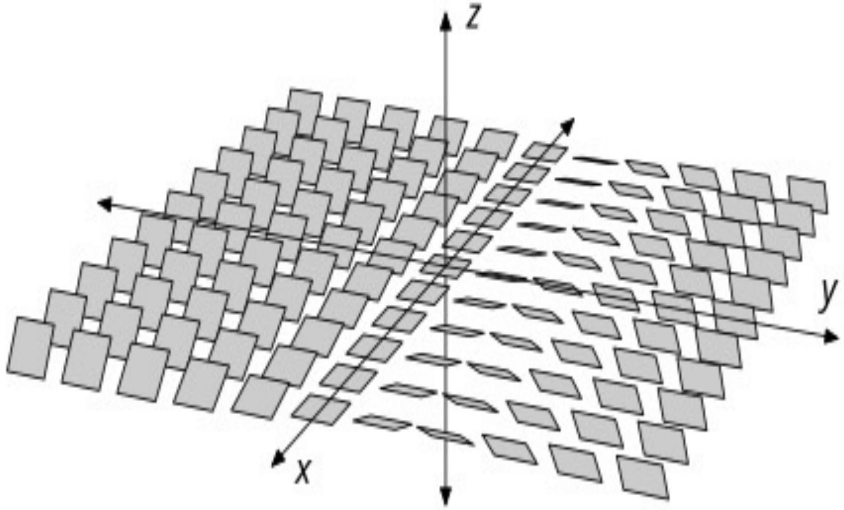
&

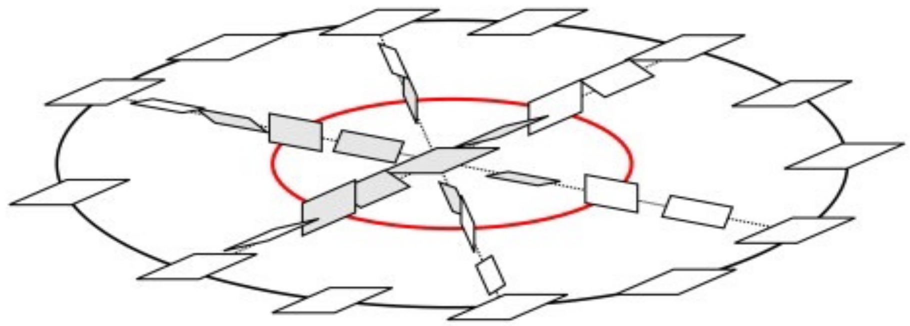
OVERTWISTED



Two Contact Structures

STANDARD CONTACT STRUCTURE OVERTWISTED CONTACT STRUCTURE





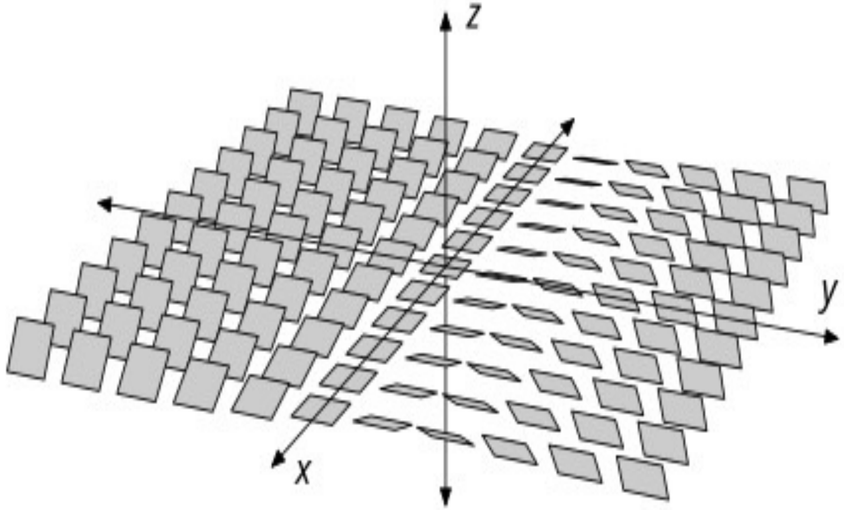
$\xi_{st} = \ker(dz - y\,dx)$

$\xi_{OT} = \ker(\cos(r)\,dz + r\sin(r)\,d\vartheta)$

ARE ξ_{st} & ξ_{OT} ISOTOPIC/CONTACTOMORPHIC ?

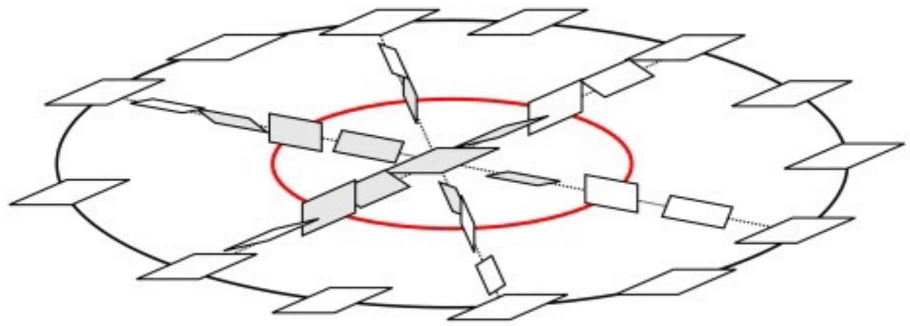
Two Contact Structures

STANDARD CONTACT STRUCTURE



$$\xi_{st} = \ker(dz - y\,dx)$$

OVERTWISTED CONTACT STRUCTURE



$$\xi_{OT} = \ker(\cos(r)\,dz + r\sin(r)\,d\vartheta)$$

ARE ξ_{st} & ξ_{OT} ISOTOPIC/CONTACTOMORPHIC ?

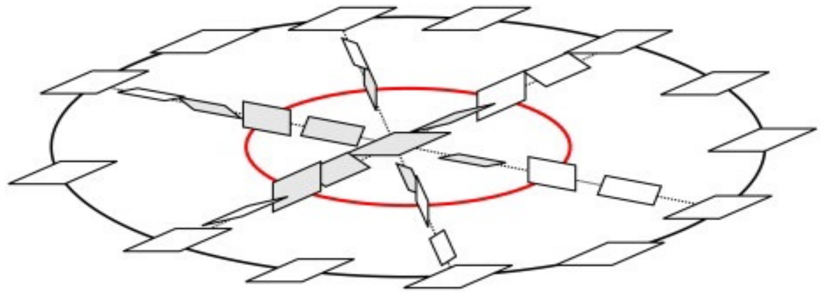
BENNEQUIN (1982): NO! CONSIDER $D = \{r \leq 2\pi\} \times \{0\}$

DEF: $D \hookrightarrow (M,\xi)$ IS AN OVERTWISTED DISK IF $TD|_{\partial D} = \xi|_{\partial D}$

RMK: ENOUGH TWISTING OF ξ ALONG ∂D W.R.T D IS 0

DEF:
- ξ IS OVERTWISTED IF ξ CONTAINS AN OVERTWISTED DISK
- ξ IS TIGHT IF IT IS NOT OVERTWISTED

OVERTWISTED CONTACT STRUCTURES



THM (ELIASHBERG) ξ, ξ' OVERTWISTED CONTACT STRUCTURES &
$\xi \simeq \xi'$ AS PLANEFIELDS
$\Longrightarrow \xi \simeq \xi'$

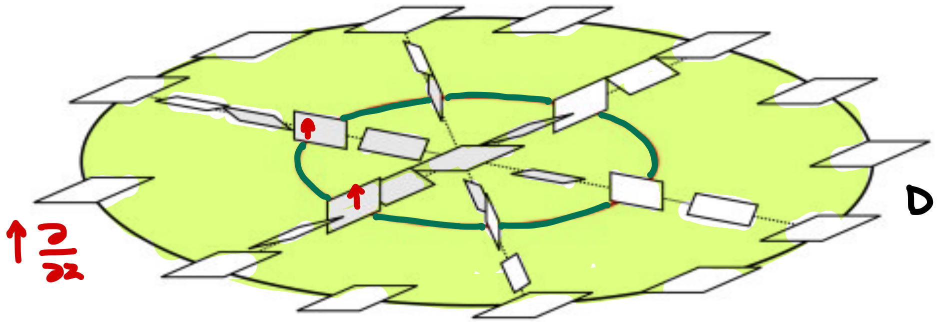
THM (LUTZ-MARTINEZ) ANY HOMOTOPY CLASS OF PLANEFIELDS IS REPRESENTED BY A(N OVERTWISTED) CONTACT STRUCTURE

OVERTWISTED CONTACT STRUCTURES CAN BE UNDERSTOOD THROUGH ALGEBRAIC TOPOLOGICAL INVARIANTS : d_2, d_3

RMK: TIGH CONTACT STRUCTURES ARE HARDER TO CLASSIFY

- ELIASHBERG: S^3 ADMITS A UNIQUE TIGHT CONTACT STRUCTURE
- GIROUX: $\exists$ ∞-LY MANY 3-MANIFOLDS WITH ∞-LY MANY TIGHT CONTACT STRUCTURES
- ETNYRE: $-\Sigma(2,3,5)$ ADMITS NO TIGHT CONTACT STRUCTURES
- ...

RECOGNISING OVERTWISTED CONTACT STRUCTURES



D IS CONVEX : $X = \frac{\partial}{\partial z}$ IS A CONTACT VECTORFIELD, $X \pitchfork D$

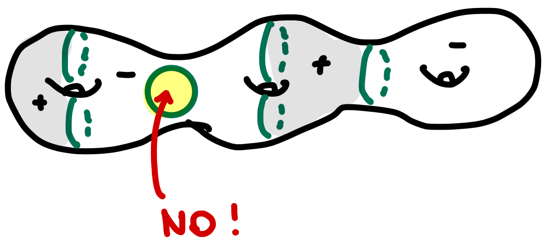
$\Gamma = \{\frac{\partial}{\partial z} \in \xi\} = \{r = \pi\} \times \{0\}$

$\xi_{OT} = \ker(\cos(r)\,dz + r\sin(r)\,d\vartheta)$

THM (GIROUX'S CRITERION): $\Sigma \hookrightarrow (M, \xi)$ CONVEX SURFACE ADMITS A TIGHT NEIGHBOURHOOD $(N(\Sigma), \xi|_{N(\Sigma)})$ IFF

- $\Sigma \cong S^2$ & $|\Gamma| = 1$
- $\Sigma \neq S^2$ & NO COMPONENT OF Γ BOUNDS A DISC

OR

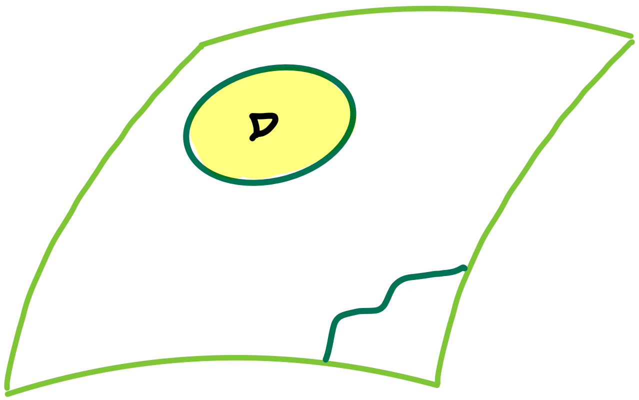


Idea of Proof

Thm (Giroux's criterion): $\Sigma \hookrightarrow (M,\xi)$ convex surface admits a tight neighbourhood $(N(\Sigma), \xi|_{N(\Sigma)})$ iff

- $\Sigma \cong S^2$ & $|\Gamma| = 1$
- $\Sigma \neq S^2$ & no component of Γ bounds a disc

$\boxed{\Rightarrow}$ **Will show:** Γ has a component that bounds a disc & this is not the only component $\Longrightarrow$ $\xi|_{N(\Sigma)}$ is tight



IDEA OF PROOF

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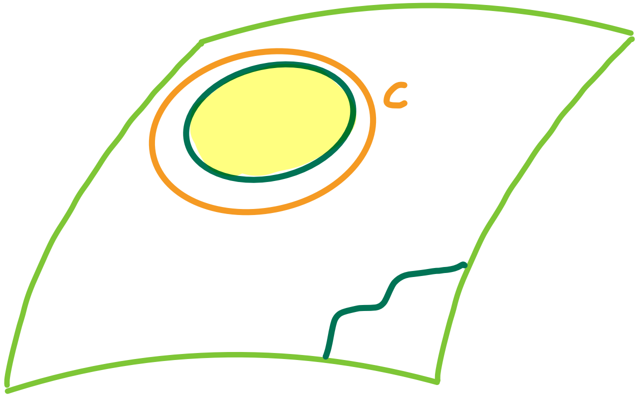
$\boxed{\Rightarrow}$ **WILL SHOW:** Γ HAS A COMPONENT THAT BOUNDS A DISC D & THIS IS NOT THE ONLY COMPONENT



$\Longrightarrow \xi|_{N(\Sigma)}$ IS TIGHT

CONSIDER: C ENCAPSULATING D

C IS NON-ISOLATING

$\Downarrow$ LEGENDRIAN REALISATION PRINCIPLE

WE CAN ISOTOPE Σ INSIDE $N(\Sigma)$ SUCH THAT C IS LEGENDRIAN

Idea of Proof

Thm (Giroux's Criterion): $\Sigma \hookrightarrow (M, \xi)$ convex surface admits a tight neighbourhood $(N(\Sigma), \xi|_{N(\Sigma)})$ iff
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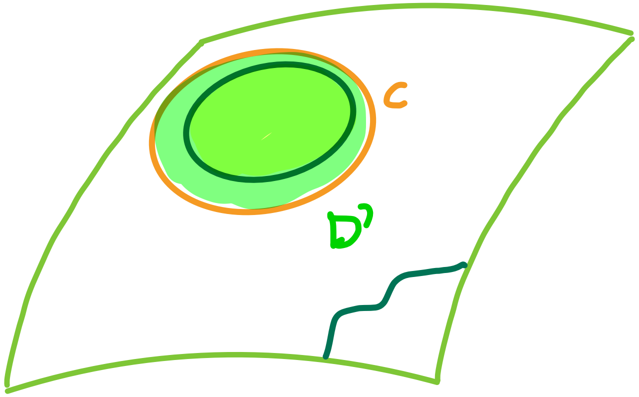


Consider: C encapsulating D

C is non-isolating

$\Downarrow$ Legendrian realisation principle

We can isotope Σ inside $N(\Sigma)$ such that C is Legendrian

Then twisting of ξ along C w.r.t. Σ $= \frac{1}{2}|\Gamma \cap C|$

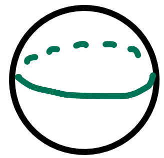
So D' is an overtwisted disc ✓

$\boxed{\Leftarrow}$ Universal cover of Σ ...

$\blacksquare$

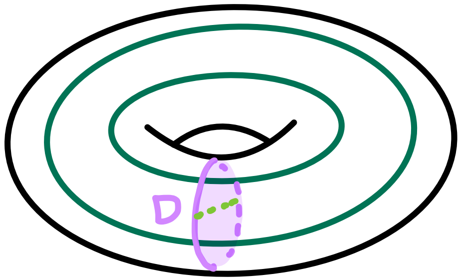
CLASSIFICATION OF TIGHT CONTACT STRUCTURES

THM (ELIASHBERG): D^3 ADMITS A UNIQUE TIGHT CONTACT STRUCTURE WITH CONNECTED DIVIDING CURVE ON S^2:



THIS RESULT ALLOWS US TO PROVE OTHER UNIQNESS RESULTS

E.G.: $M = D^2 \times S^1$ GIVEN ANY CONTACT STRUCTURF ξ ON M



- **STEP 1:** BY THE LEGENDRIAN REALISATION PRINCIPLE WE CAN ASSUME ∂D IS LEGENDRIAN

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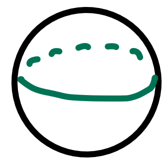

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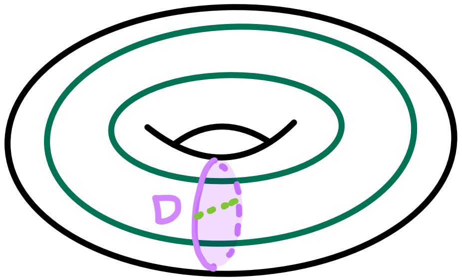

- **STEP 1:** BY THE LEGENDRIAN REALISATION PRINCIPLE WE CAN ASSUME ∂D IS LEGENDRIAN
- **STEP 2:** ISOTOPE D REL. ∂ TO BE CONVEX: THE DIVIDING CURVE ON D IS A SINGLE ARC

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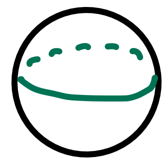

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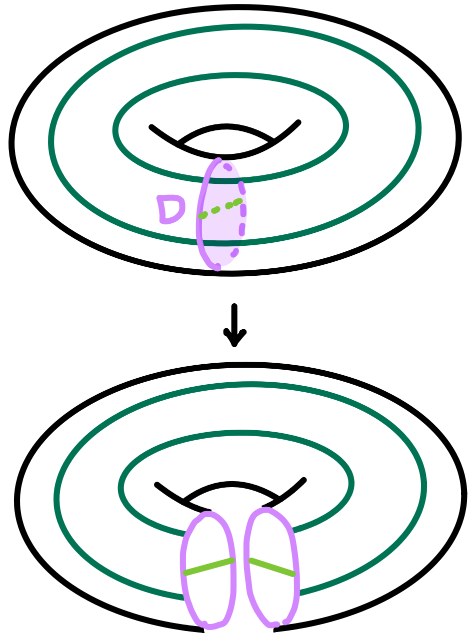

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- **STEP 2**: ISOTOPE D REL. ∂ TO BE CONVEX: THE DIVIDING CURVE ON D IS A SINGLE ARC
- **STEP 3**: CUT M ALONG D

CLASSIFICATION OF TIGHT CONTACT STRUCTURES

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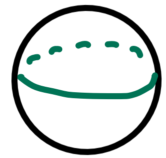

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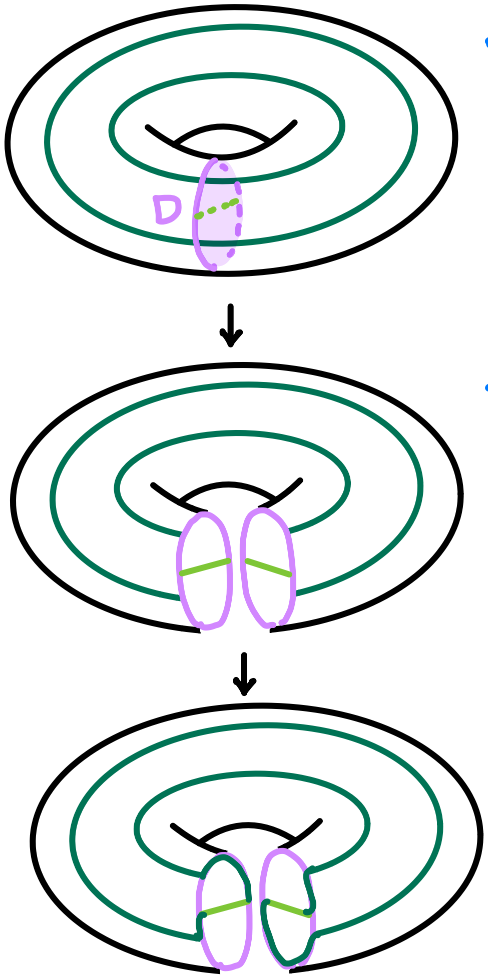
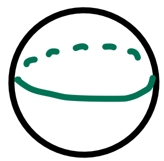


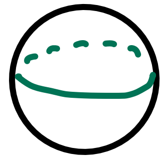
- **STEP 1:** BY THE LEGENDRIAN REALISATION PRINCIPLE WE CAN ASSUME ∂D IS LEGENDRIAN
- **STEP 2:** ISOTOPE D REL. ∂ TO BE CONVEX: THE DIVIDING CURVE ON D IS A SINGLE ARC
- **STEP 3:** CUT M ALONG D
- **STEP 4:** ROUND THE EDGES: WE GET A D^3 WHICH HAS A UNIQUE CONTACT STRUCTURE ξ_0

$\Longrightarrow$ ANY ξ CAN BE OBTAINED FROM ξ_0 BY GLUEING $\Longrightarrow$ ξ IS UNIQUE TOO ∎

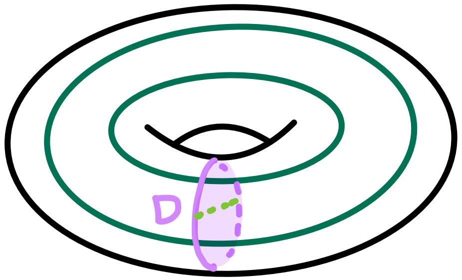
$\cong$ 

Classification of Tight Contact Structures

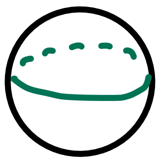
THM (Eliashberg): D^3 admits a unique tight contact structure with connected dividing curve on S^2:



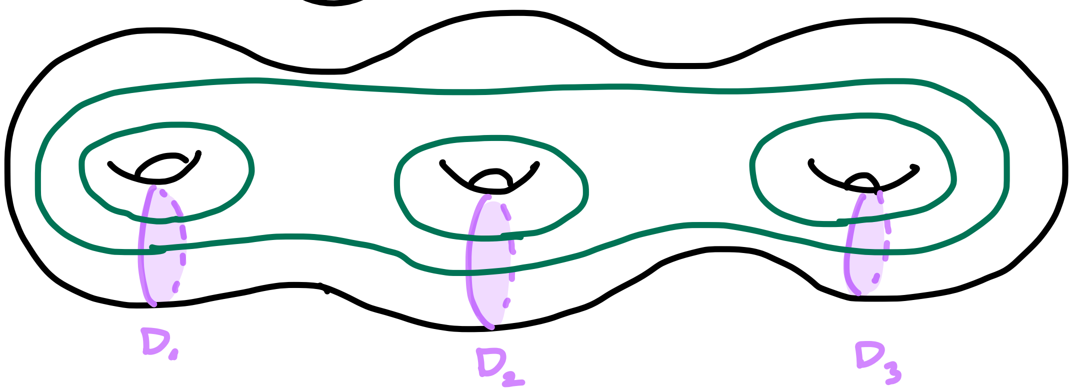
E.G.:



admits a unique tight contact structure

- What did we use in the proof?
- That $|\partial D \cap \Gamma| = 2$ thus the dividing curve on D was well defined
- & that we got  after cutting and rounding

SIMILARLY:



admits a unique tight contact structure

DEF: Product disc decomposable

TO BE CONTINUED ...

LECTURE 2: DESCRIBING CONTACT STRUCTURES

- CONTACT CELL DECOMPOSITIONS
- CONVEX SURFACE THEORY - BYPASSES
- CONTACT HEEGAARD SPLITTINGS (PROOF OF EXISTENCE)
- OPEN BOOK DECOMPOSITIONS
- OPEN BOOK DECOMPOSITIONS & CONTACT HEEGAARD SPLITTINGS