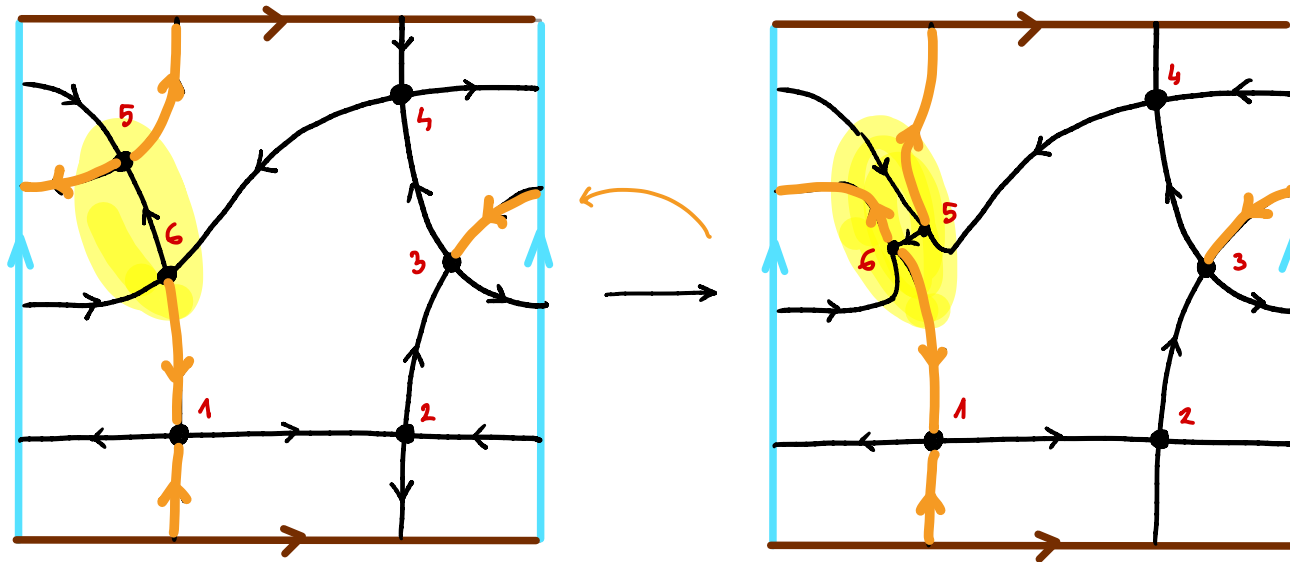


THE FEEDBACK ARC SET PROBLEM OF SURFACE GRAPHS

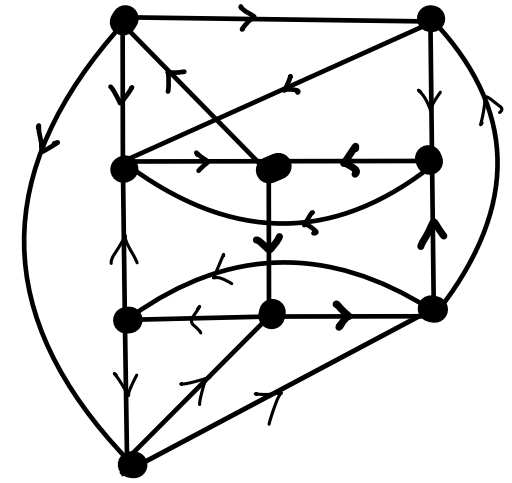


VERA VÉRTESI

Jan. 16.

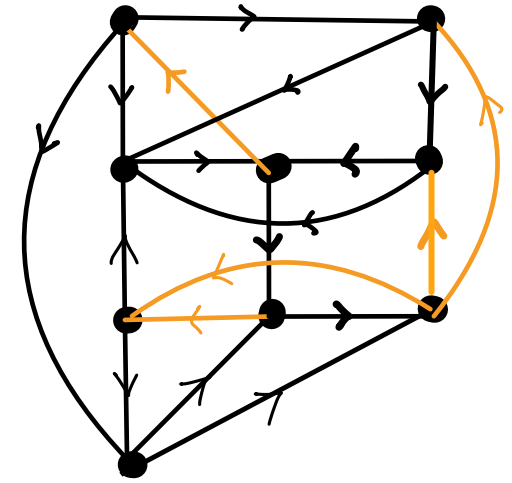
FEEDBACK ARC SET

DEF: A feedback arc set for a digraph $D(V, \vec{E})$ is $\vec{E}' \subseteq \vec{E}$ s.t. $D'(V, \vec{E} \setminus \vec{E}')$ is acyclic



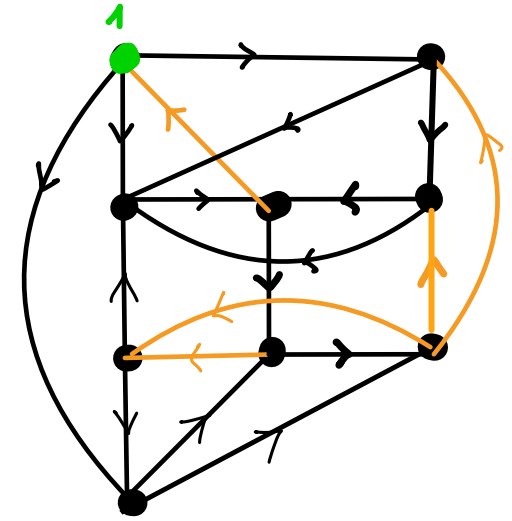
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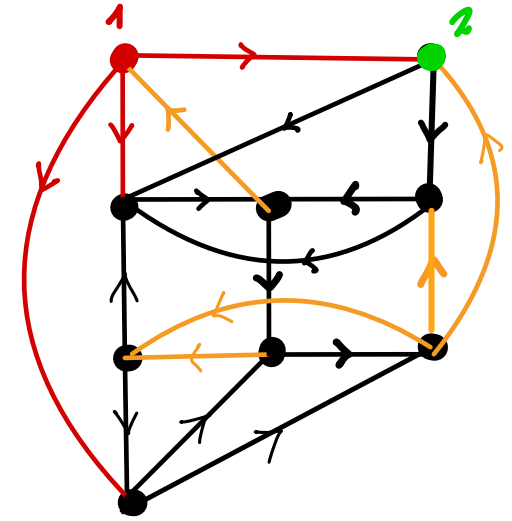
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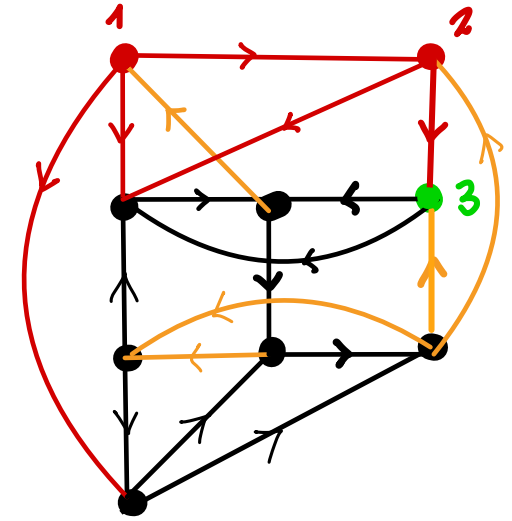
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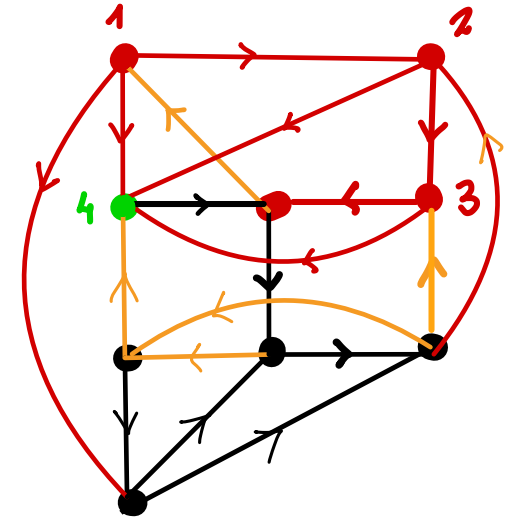
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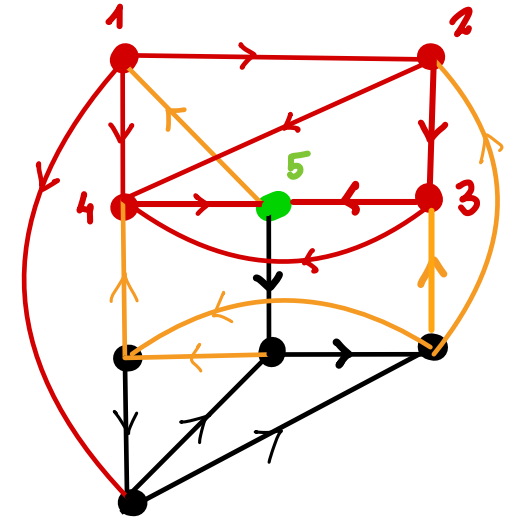
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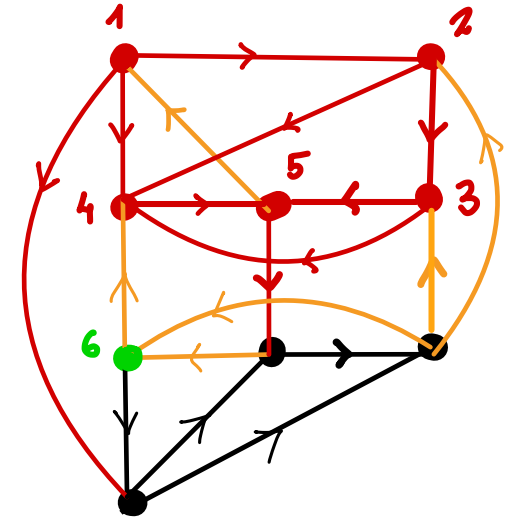
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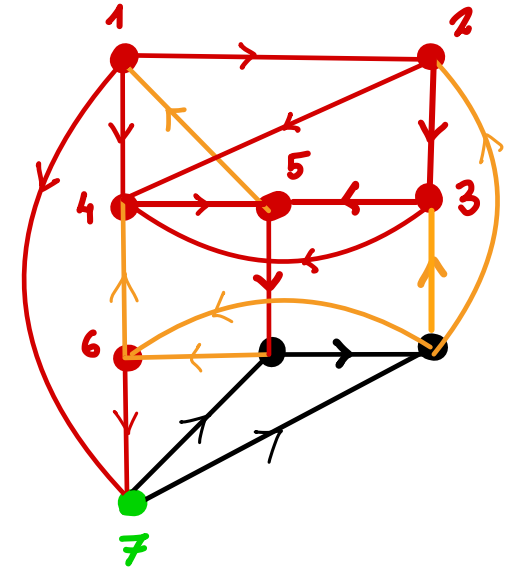
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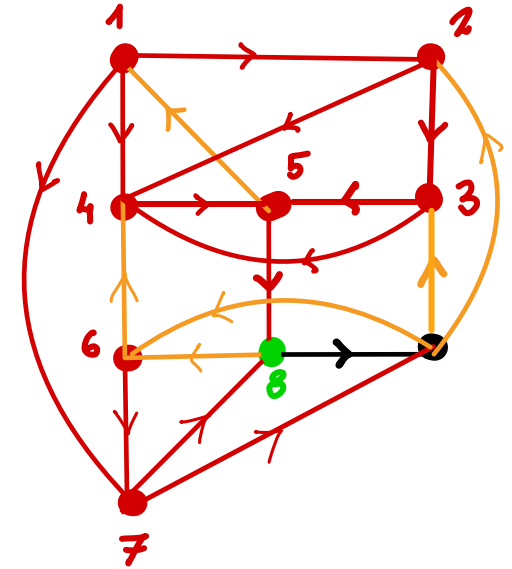
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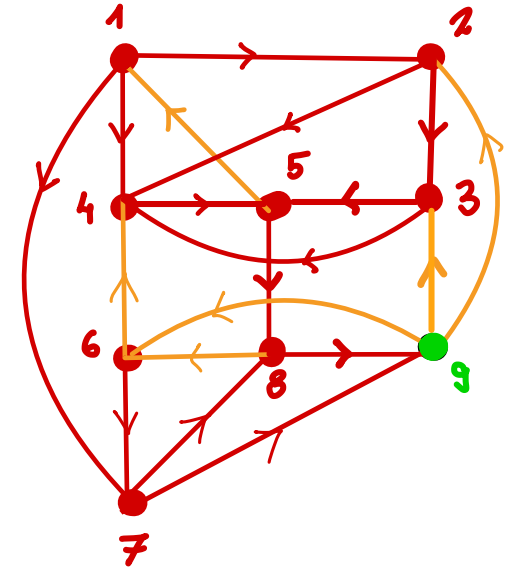
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FEEDBACK ARC SET

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FEEDBACK ARC SET

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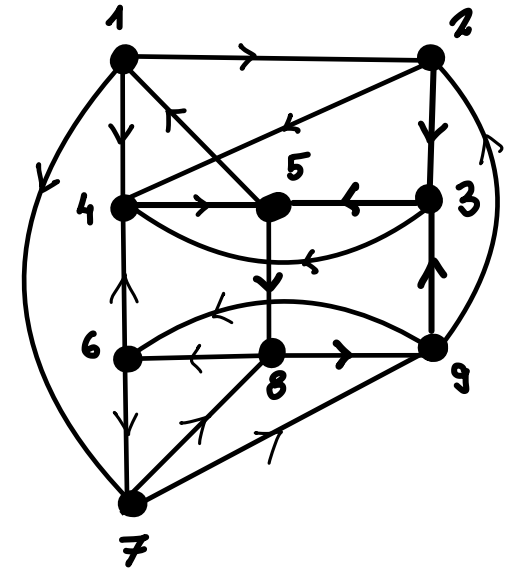
COMPUTATIONAL PROBLEM

GIVEN: $D(V, \vec{E})$ digraph & $k \in \mathbb{N}$

QUESTION: Is there a feedback arc set of size $\leq k$?

COMPUTATIONAL COMPLEXITY

- NP-complete (item 8 on Karp's Original list)
- fixed parameter tractable (FTP) : $\mathcal{O}(n^4 4^k k^3 k!)$

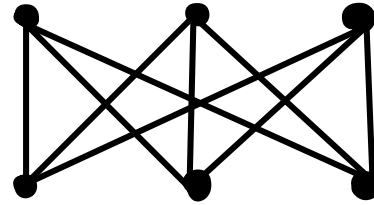


FEEDBACK ARC SET - SPECIAL INPUTS

if we restrict $D(V, \vec{E})$ to...

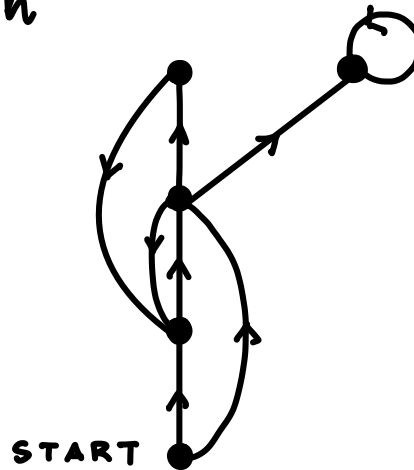
... directed planar graph : $\Theta(n^{5/2} \log n)$

... graphs w/ no $K_{3,3}$ minor :



... weakly acyclic digraph

... reducible flow graphs

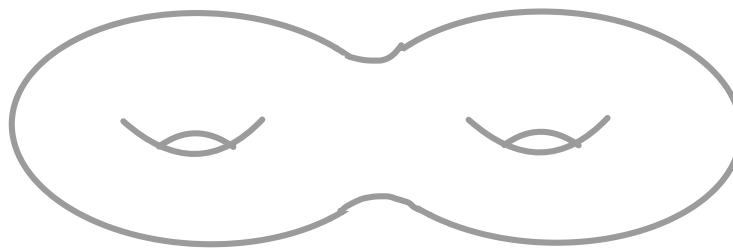
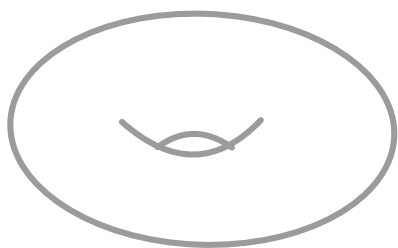


then there is polynomial algorithm

NEXT: specialised input important in my research

GRAPHS ON SURFACES

(ORIENTED) SURFACES



...

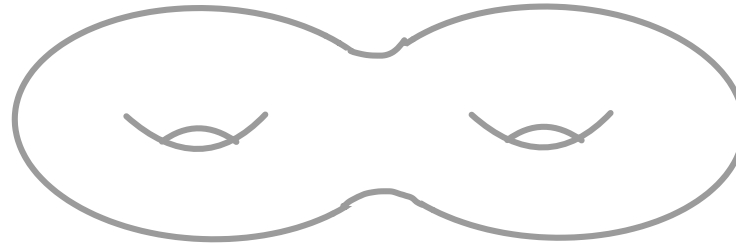
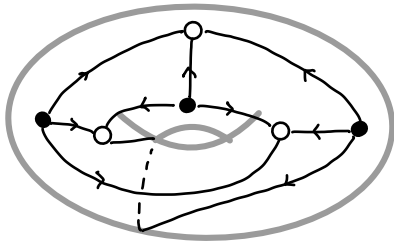
$$D(V, \vec{E}) \hookrightarrow \Sigma$$

V - points on Σ

E - arcs w/ no intersections

GRAPHS ON SURFACES

(ORIENTED) SURFACES



...

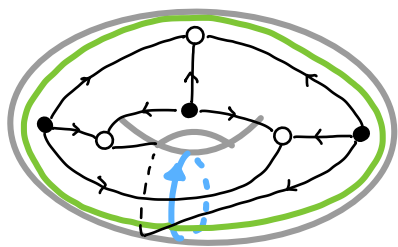
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GRAPHS ON SURFACES

(ORIENTED) SURFACES



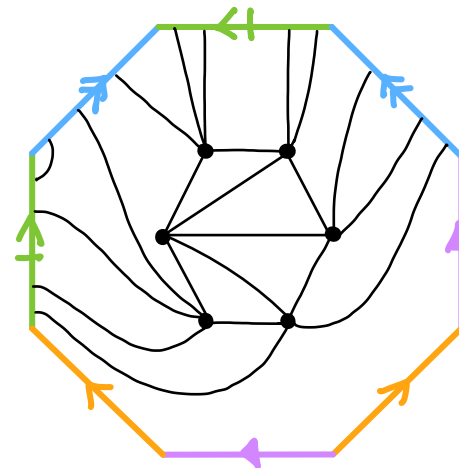
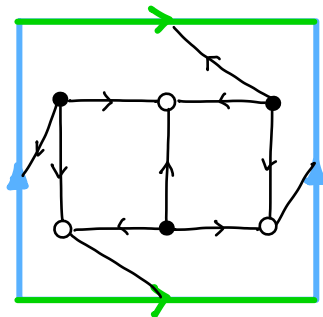
...

$$D(V, \vec{E}) \hookrightarrow \Sigma$$

V - points on Σ

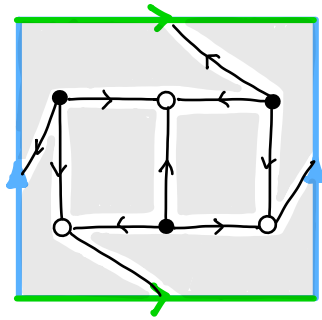
E - arcs w/ no intersections

cut open Σ :



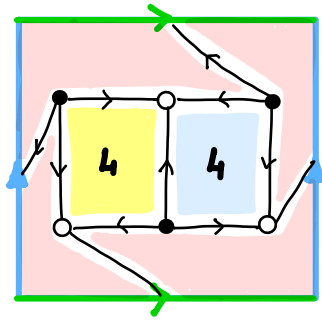
GRAPHS ON SURFACES

DEF: 2-cell embedding: each component of $\mathbb{Z} \setminus D$ is a polygon



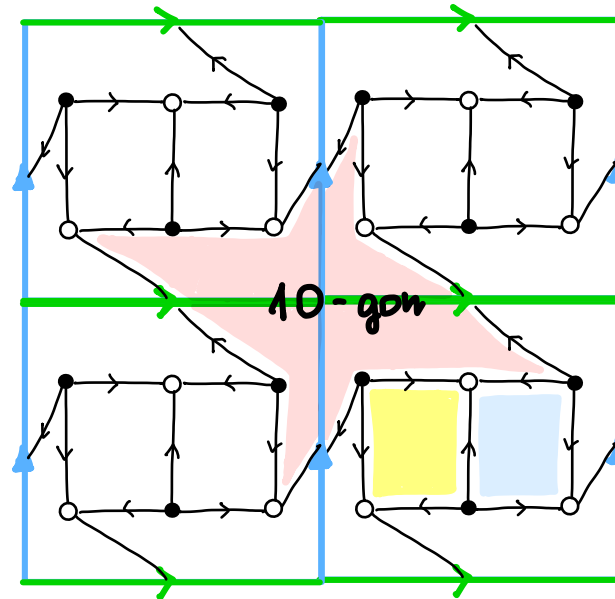
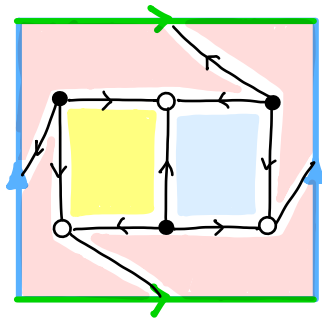
GRAPHS ON SURFACES

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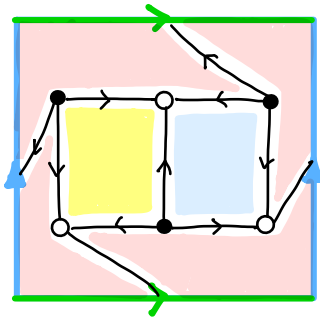
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GRAPHS ON SURFACES

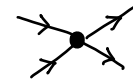
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face F

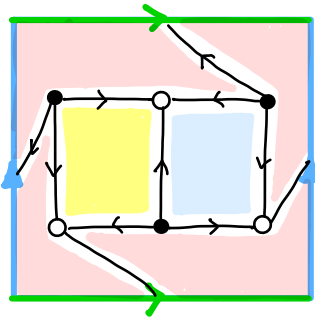
ASSUMPTIONS:

- $D(V, \vec{E})$ is Eulerian & 4-regular ($\forall v \in V: \deg_{in}(v) = \deg_{out}(v) = 2$)



GRAPHS ON SURFACES

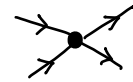
DEF: 2-cell embedding: each component of $\Sigma \setminus D$ is a polygon



face F

ASSUMPTIONS:

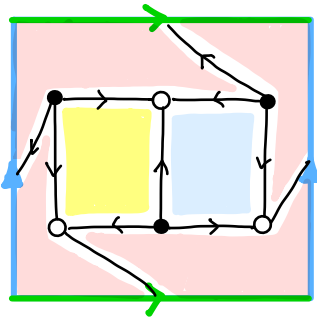
- $D(V, \vec{E})$ is Eulerian & 4-regular ($\forall v \in V: \deg_{\text{in}}(v) = \deg_{\text{out}}(v) = 2$)



- $D(V, \vec{E}) \hookrightarrow \Sigma$ is a 2-cell embedding s.t. for each face F
 - $\rightarrow$ the vertices on ∂F are all distinct

GRAPHS ON SURFACES

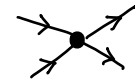
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face F

ASSUMPTIONS:

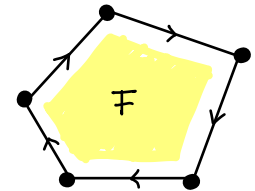
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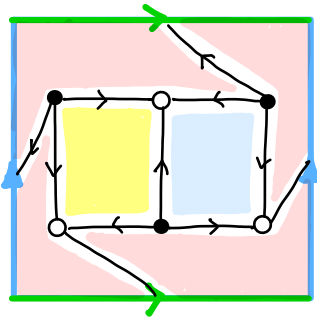
→ the vertices on ∂F are all distinct

→ the edges on ∂F form an oriented circle



GRAPHS ON SURFACES

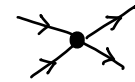
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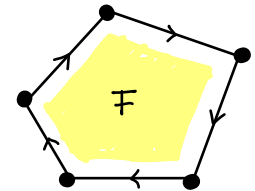
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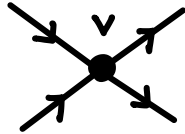
→ the vertices on ∂F are all distinct

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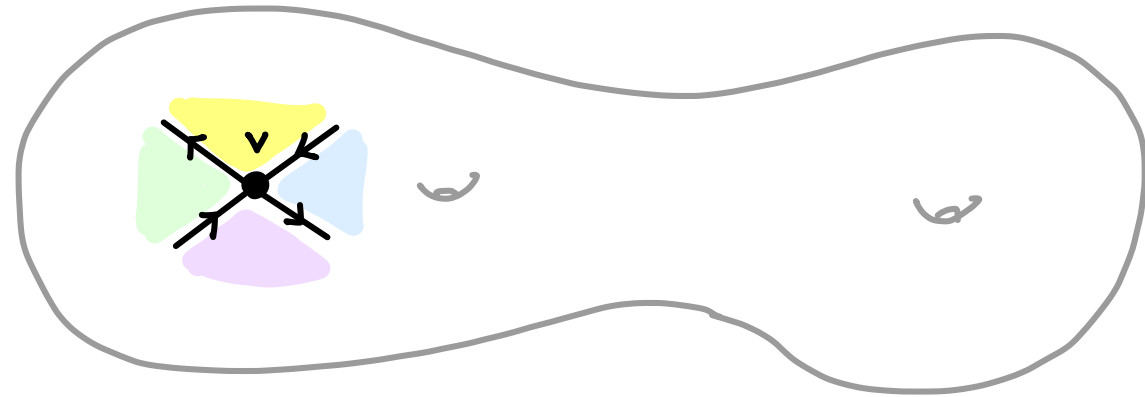


GRAPHS ON SURFACES

v in the abstract digraph



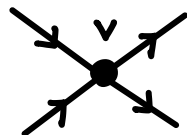
v after embedding in Σ



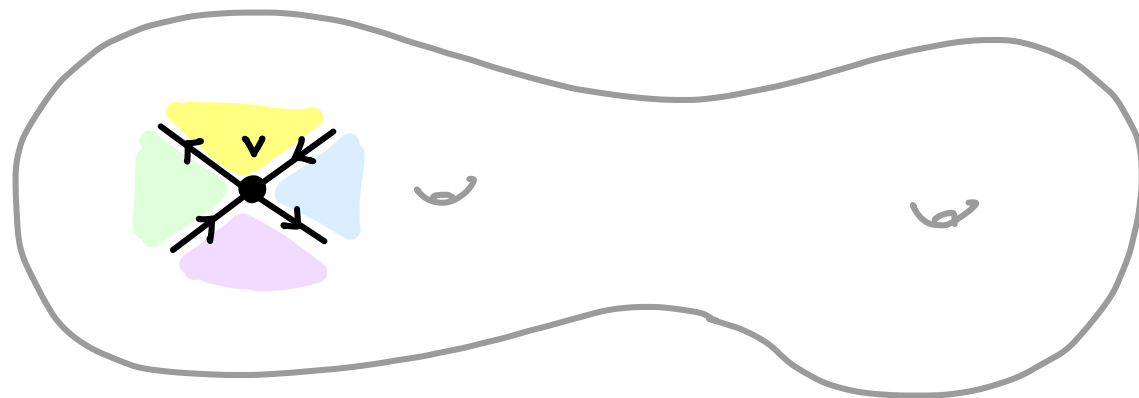
So: 2-cell embedding $\longrightarrow$ local order of edges
near every vertex

GRAPHS ON SURFACES

v in the abstract digraph



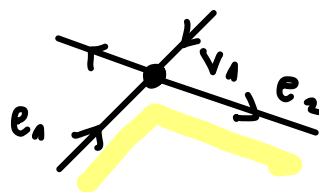
v after embedding in Σ



So: 2-cell embedding $\longrightarrow$ local order of edges
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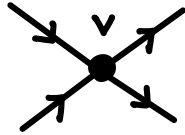
• can build up all faces:

$\rightarrow$ boundaries of faces

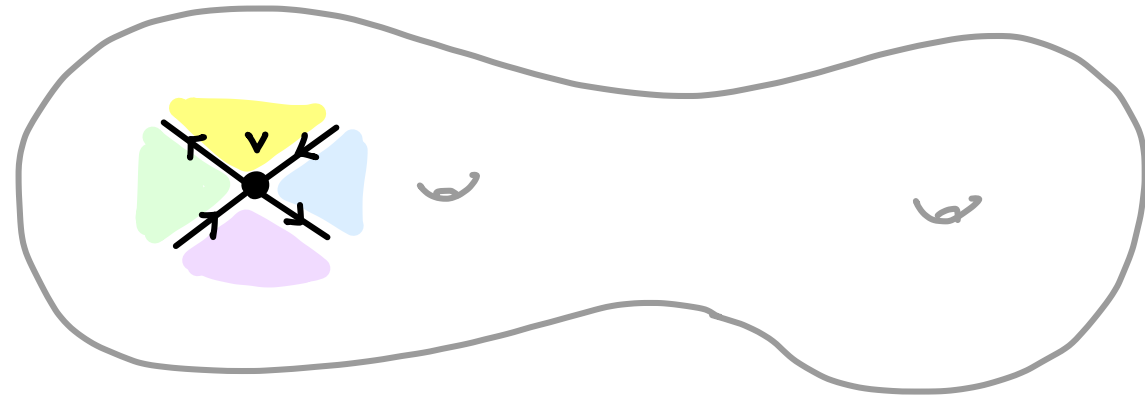


GRAPHS ON SURFACES

v in the abstract digraph

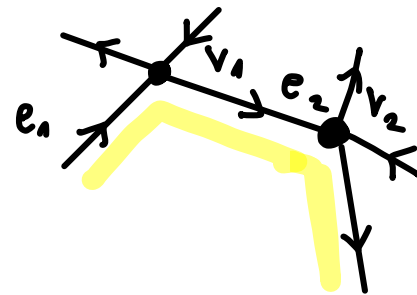


v after embedding in Σ



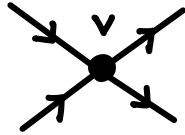
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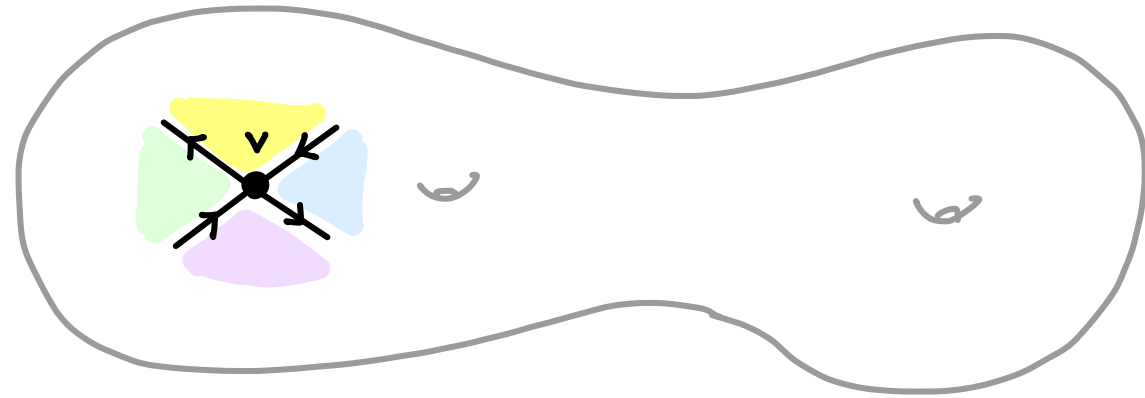


GRAPHS ON SURFACES

v in the abstract digraph

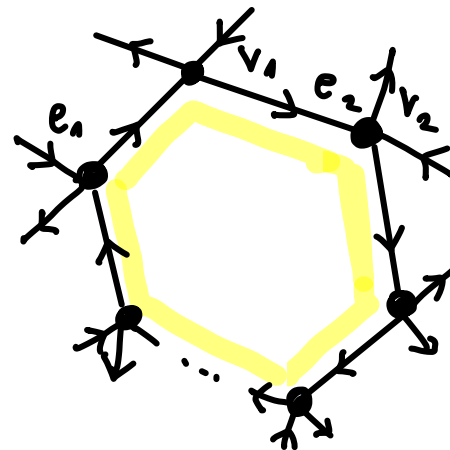


v after embedding in Σ



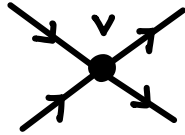
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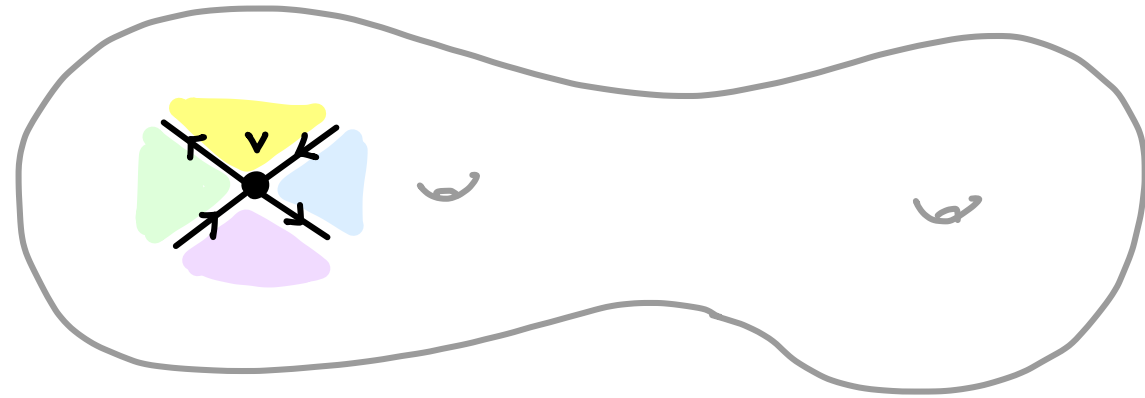


GRAPHS ON SURFACES

v in the abstract digraph



v after embedding in Σ

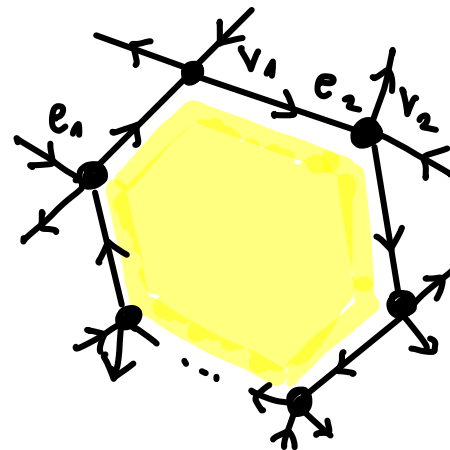


So: 2-cell embedding $\longrightarrow$ local order of edges
near every vertex

- can build up all faces:

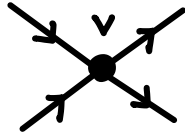
- $\rightarrow$ boundaries of faces

- $\rightarrow$ each face is
a polygon

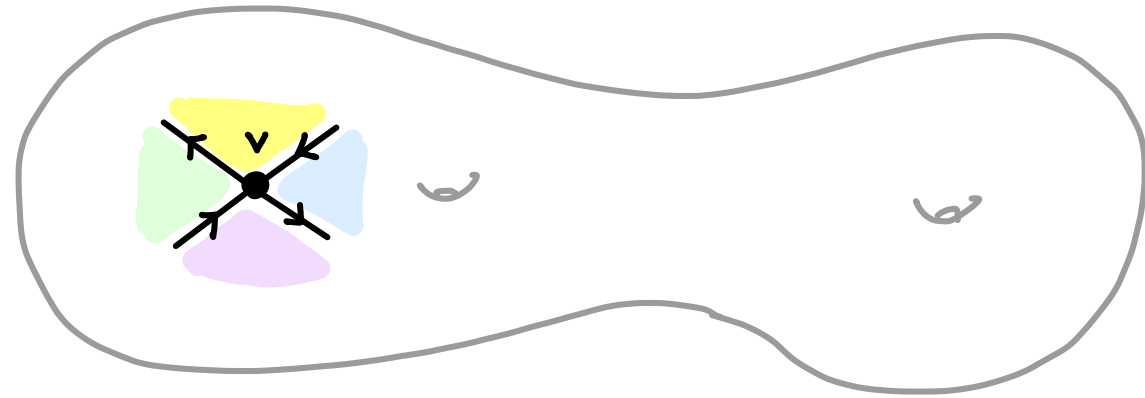


GRAPHS ON SURFACES

v in the abstract digraph



v after embedding in Σ



So: 2-cell embedding $\longrightarrow$ local order of edges
near every vertex

• can build up all faces:

$\rightarrow$ boundaries of faces

$\rightarrow$ each face is
a polygon



$\Rightarrow$ glues up
to a
surface Σ

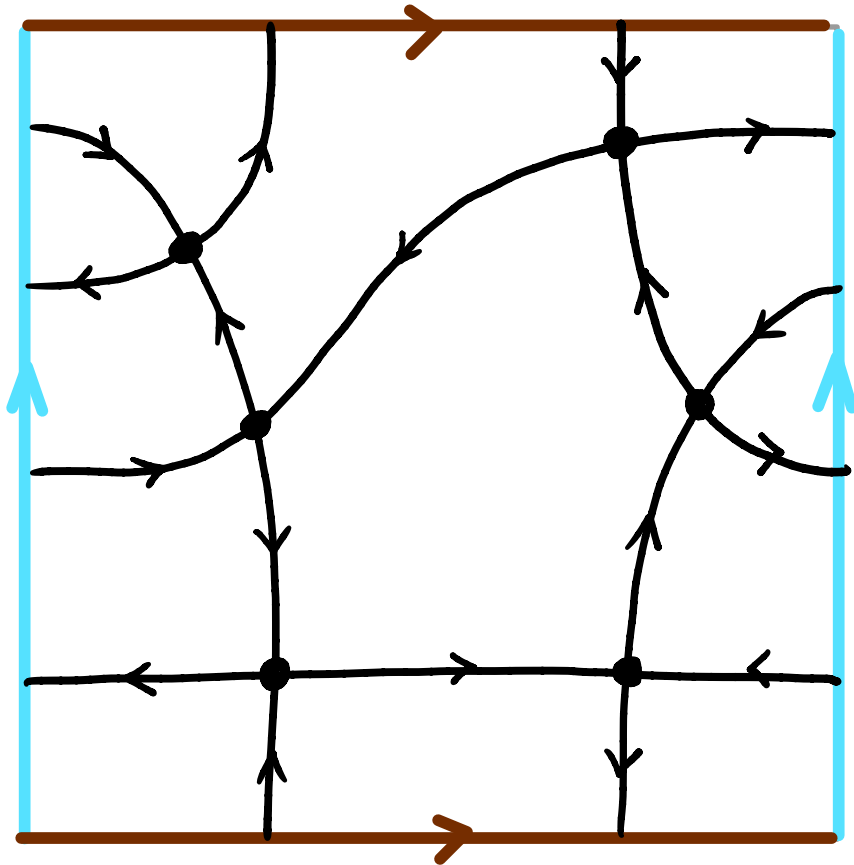
MAIN QUESTION

INPUT: an Eulerian & 4-regular digraph $D(V, \vec{E})$

& a 2-cell embedding $D(V, \vec{E}) \hookrightarrow \Sigma$ s.t. for each face F

→ the vertices on ∂F are all distinct

→ the edges on ∂F form an oriented circle



REMARKS:

- near each vertex D looks like

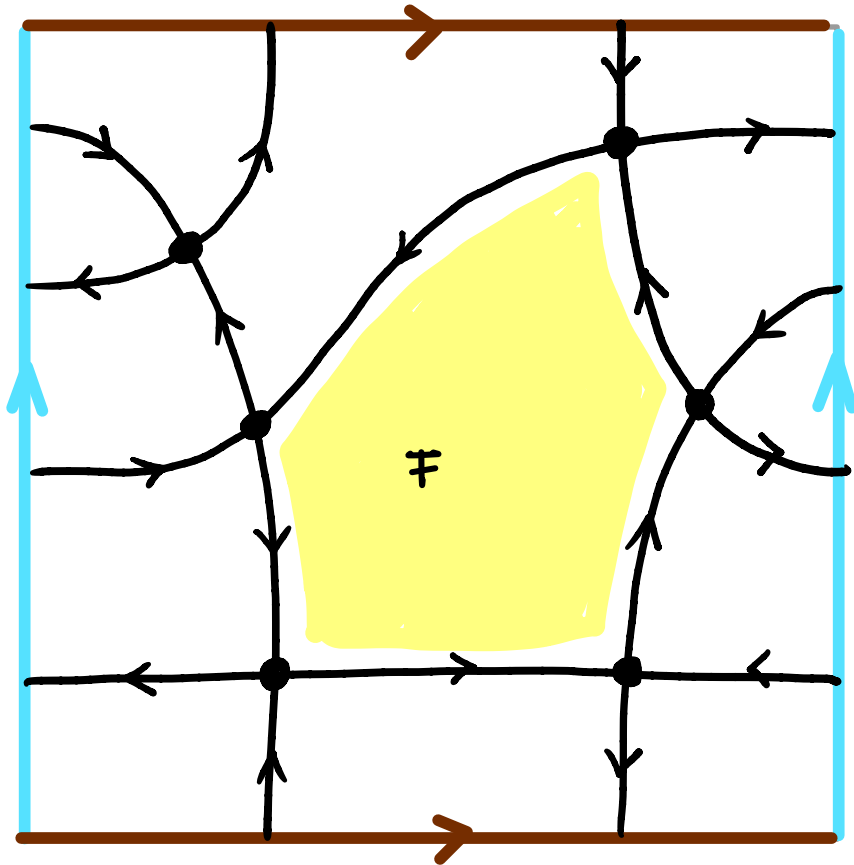
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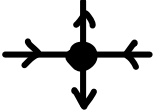
& a 2-cell embedding $D(V, \vec{E}) \hookrightarrow \Sigma$ s.t. for each face F

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REMARKS:

- near each vertex D looks like 
- ∂F oriented circle

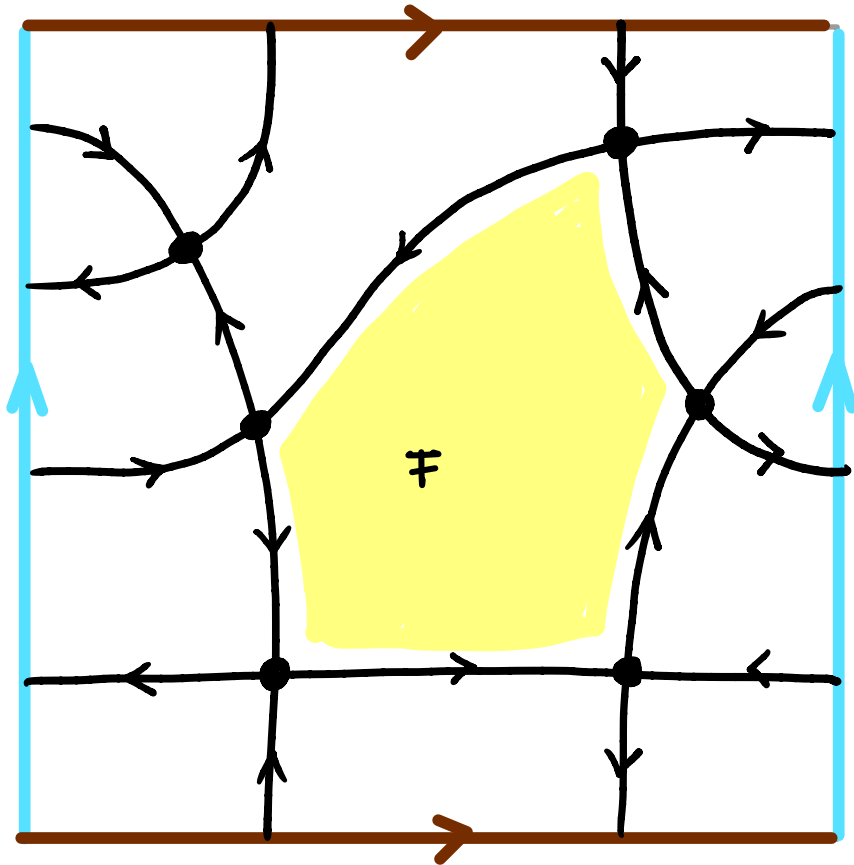
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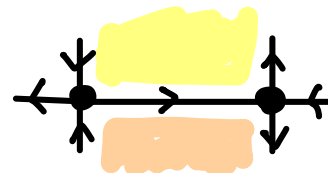
REMARKS:

• near each vertex D looks like

• ∂F oriented circle

⇒ any feedback arc set $\vec{E}' \subseteq \vec{E}$ must contain an edge from each ∂F

• an edge is contained in 2 faces



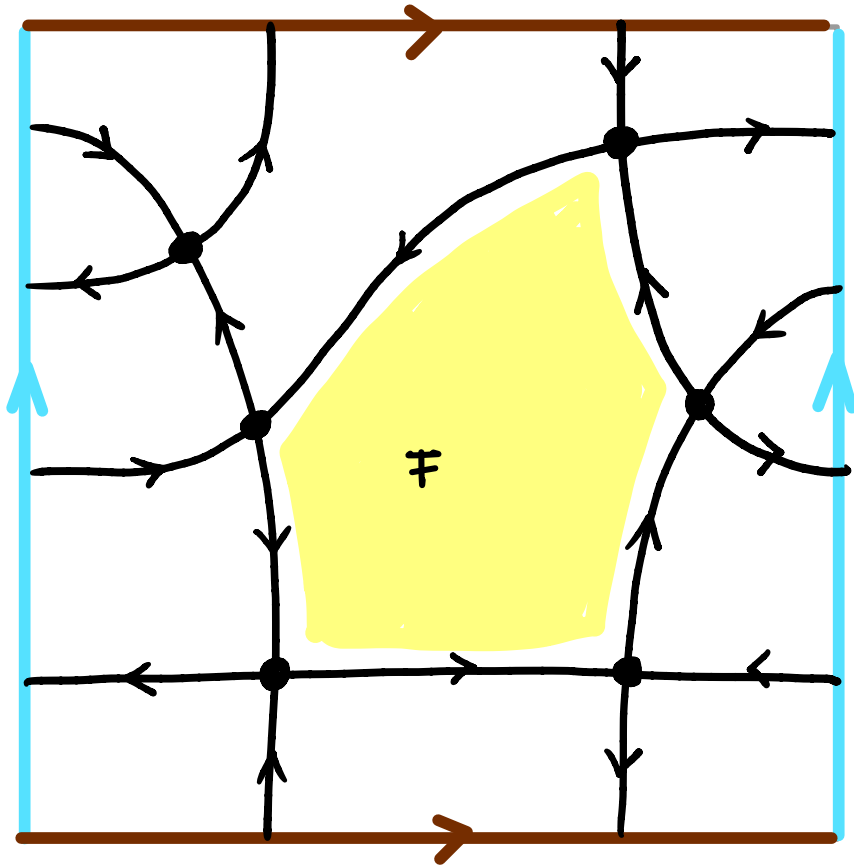
MAIN QUESTION

INPUT: an Eulerian & 4-regular digraph $D(V, \vec{E})$

& a 2-cell embedding $D(V, \vec{E}) \hookrightarrow \Sigma$ s.t. for each face F

→ the vertices on ∂F are all distinct

→ the edges on ∂F form an oriented circle



REMARKS:

• near each vertex D looks like $\rightarrow \bullet \leftarrow$

• ∂F oriented circle

$\Rightarrow$ any feedback arc set $\vec{E}' \subseteq \vec{E}$ must contain an edge from each ∂F

• an edge is contained in 2 faces

$$\Rightarrow |\vec{E}'| \geq \frac{\# \text{ faces}}{2}$$

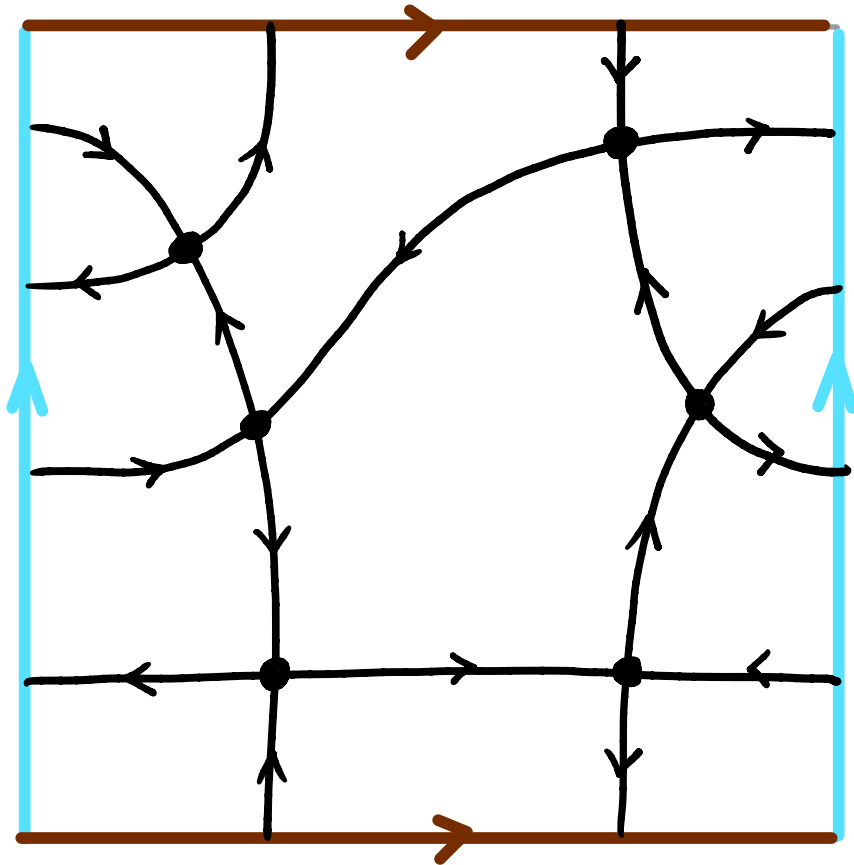
MAIN QUESTION

INPUT: an Eulerian & 4-regular digraph $D(V, \vec{E})$

& a 2-cell embedding $D(V, \vec{E}) \hookrightarrow \Sigma$ s.t. for each face F

→ the vertices on ∂F are all distinct

→ the edges on ∂F form an oriented circle



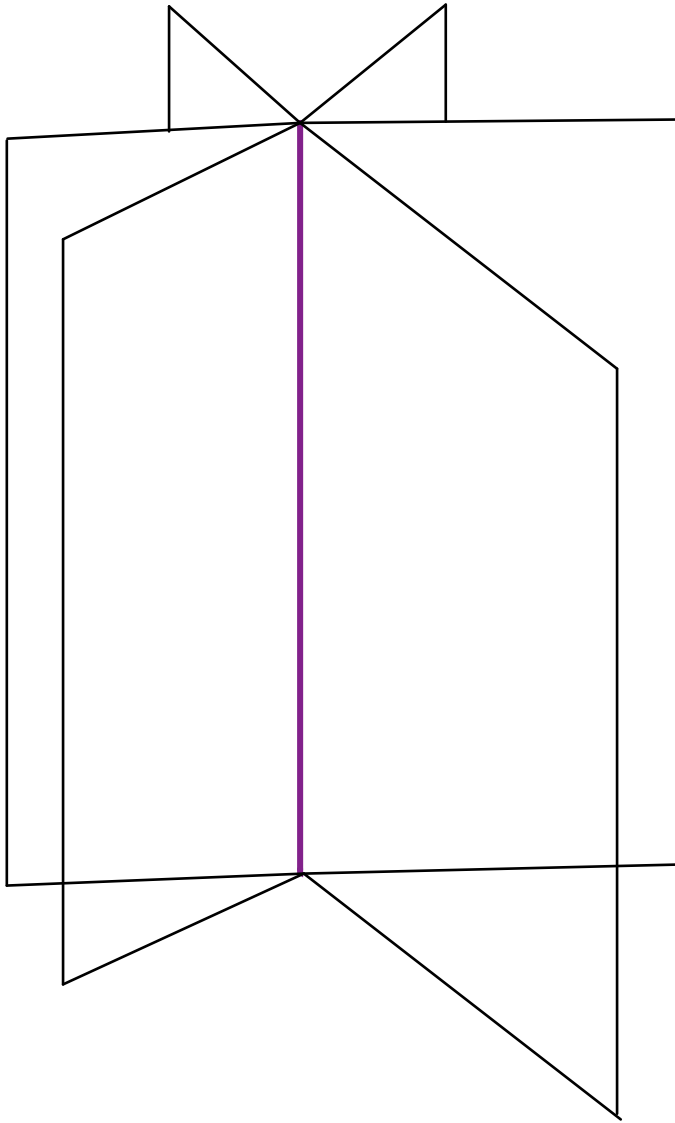
QUESTION: Is there a feedback arc set
of size $\frac{\# \text{ faces}}{2}$?

PLAN FOR REST OF THE TALK

- origin of the problem
 - braid foliations
 - open book foliations
- idea for solution
- further directions

BRAID FOLIATIONS

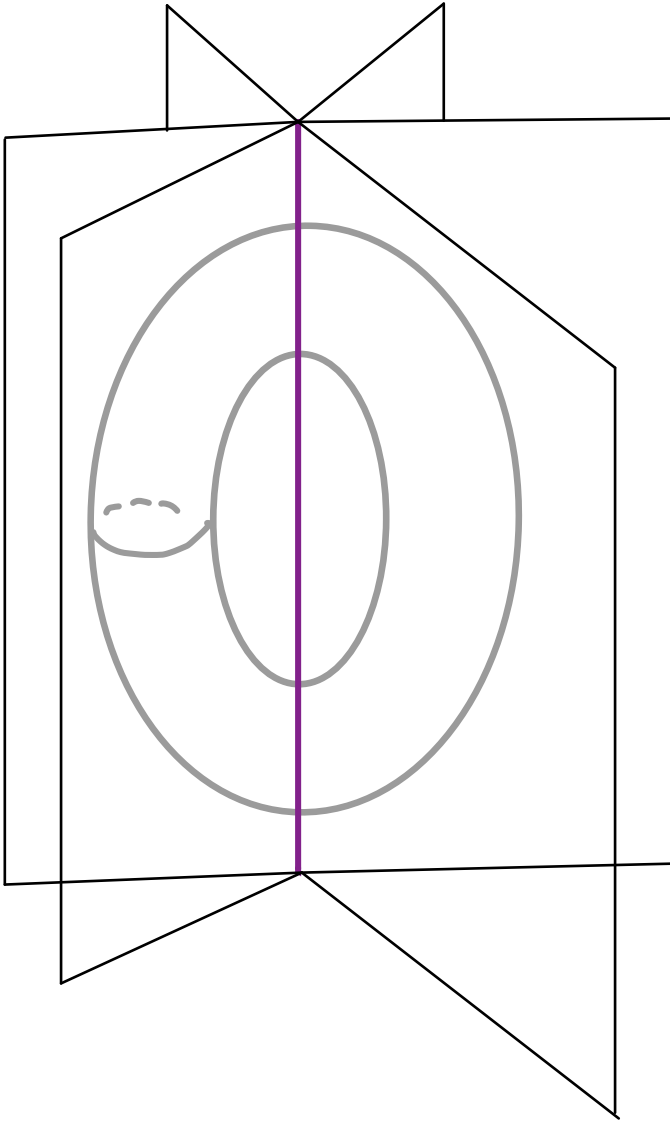
$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)



- $H_t := \{ \vartheta = t \}$ half planes
- + they intersect in the z -axis

BRAID FOLIATIONS

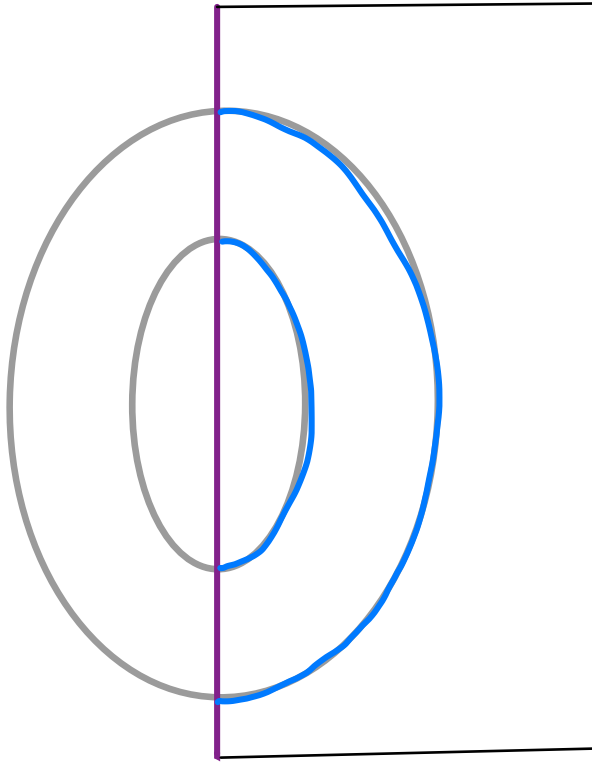
$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)



- $H_t := \{ \vartheta = t \}$ half planes
- they intersect in the z -axis
- take a surface Σ in $\mathbb{R}^3$

BRAID FOLIATIONS

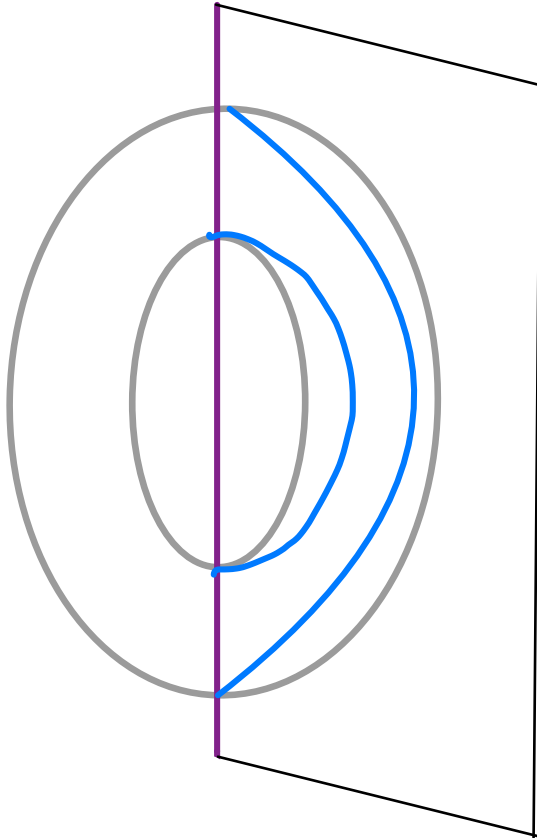
$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)



- $H_t := \{ \vartheta = t \}$ half planes
 - they intersect in the z -axis
 - take a surface Σ in $\mathbb{R}^3$
- & look at its intersection w/ H_t
for $t = 0$

BRAID FOLIATIONS

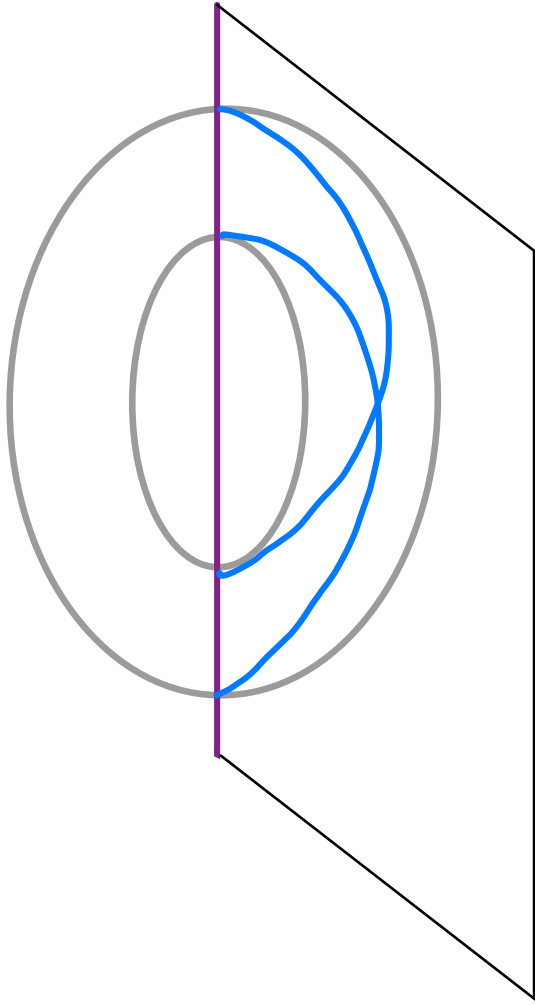
$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)



- $H_t := \{ \vartheta = t \}$ half planes
 - they intersect in the z -axis
 - take a surface Σ in $\mathbb{R}^3$
- & look at its intersection w/ H_t
- for $t = -\frac{\pi}{8}$

BRAID FOLIATIONS

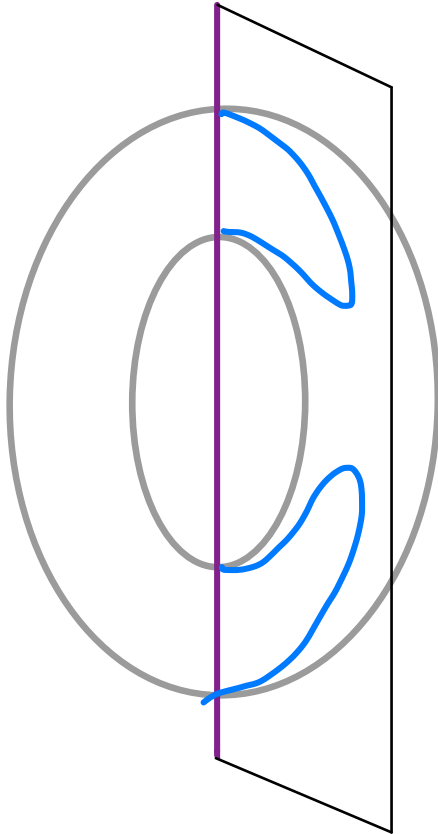
$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)



- $H_t := \{ \vartheta = t \}$ half planes
 - they intersect in the z -axis
 - take a surface Σ in $\mathbb{R}^3$
- & look at its intersection w/ H_t
for $t = -\frac{\pi}{4}$

BRAID FOLIATIONS

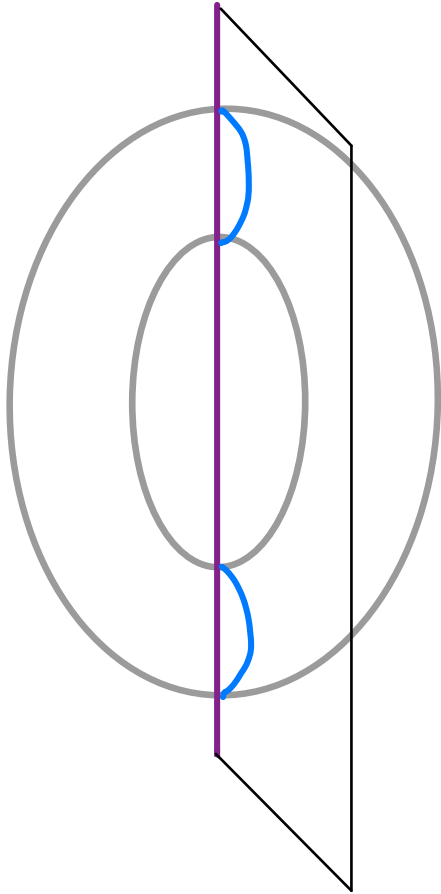
$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)



- $H_t := \{ \vartheta = t \}$ half planes
 - they intersect in the z -axis
 - take a surface Σ in $\mathbb{R}^3$
- & look at its intersection w/ H_t
- for $t = -\frac{3\pi}{8}$

BRAID FOLIATIONS

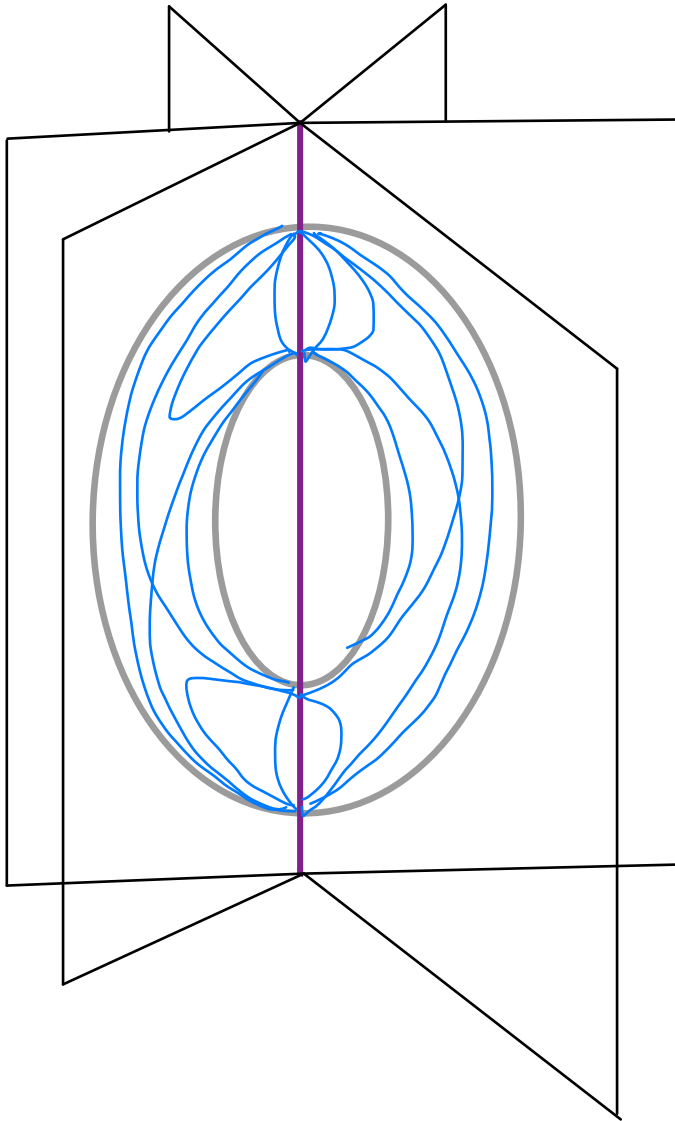
$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)



- $H_t := \{ \vartheta = t \}$ half planes
 - they intersect in the z -axis
 - take a surface Σ in $\mathbb{R}^3$
- & look at its intersection w/ H_t
- for $t = -\frac{\pi}{2}$

BRAID FOLIATIONS

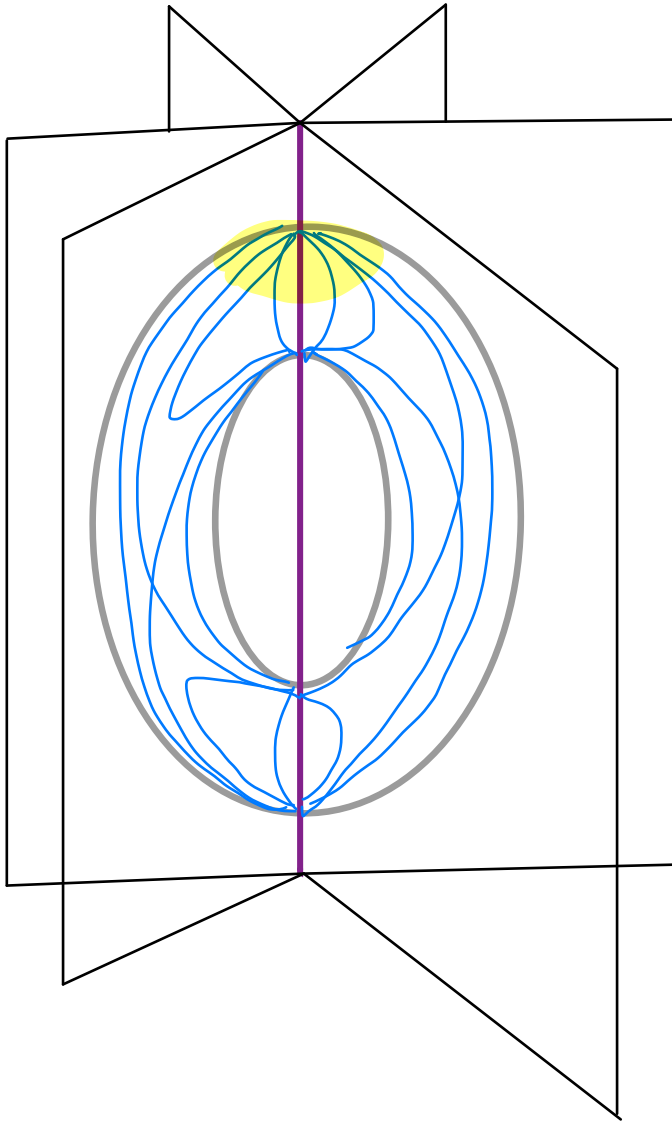
$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)



- $H_t := \{ \vartheta = t \}$ half planes
 - + they intersect in the z -axis
 - take a surface Σ in $\mathbb{R}^3$
 - & look at its intersection w/ H_t
- $\leadsto$ gives a singular foliation $\mathcal{F}$ on Σ

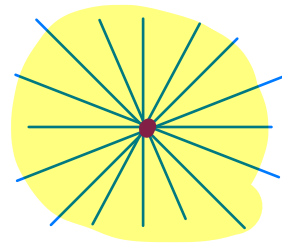
BRAID FOLIATIONS

$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)



- $H_t := \{ \vartheta = t \}$ half planes
- + they intersect in the z -axis
- take a surface Σ in $\mathbb{R}^3$
- & look at its intersection w/ H_t

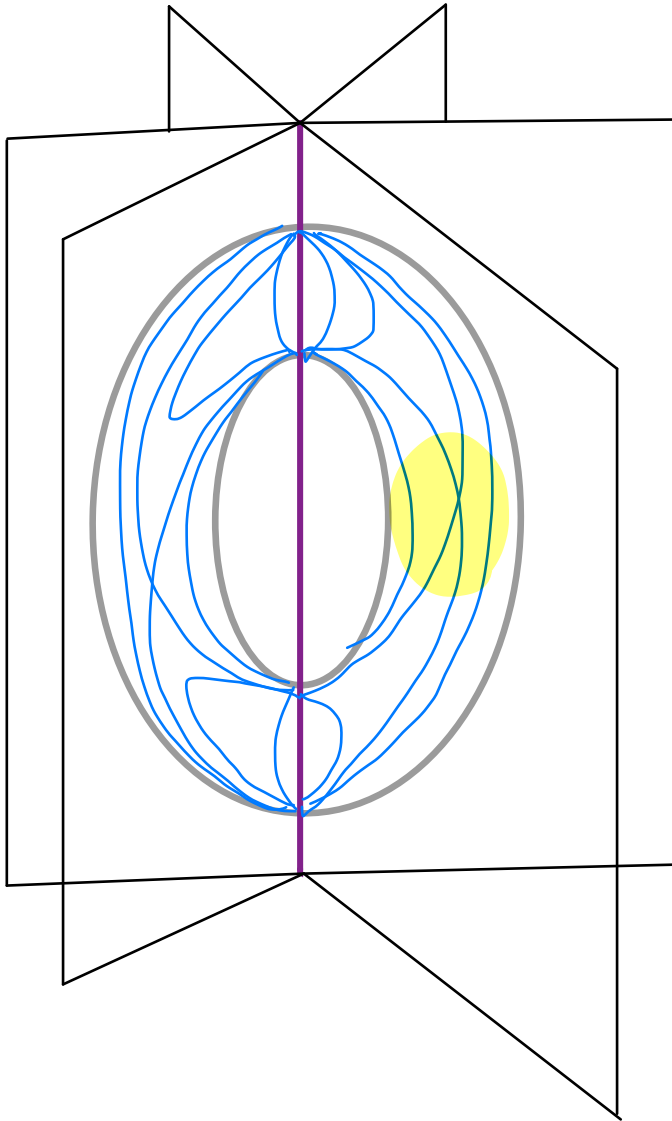
$\leadsto$ gives a singular foliation $\mathcal{F}$ on Σ
w/ singularities :



node

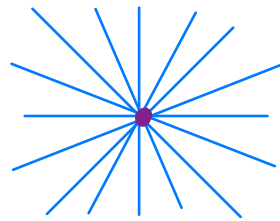
BRAID FOLIATIONS

$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)

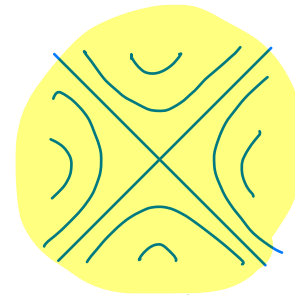


- $H_t := \{ \vartheta = t \}$ half planes
- + they intersect in the z -axis
- take a surface Σ in $\mathbb{R}^3$
- & look at its intersection w/ H_t

$\leadsto$ gives a singular foliation $\mathcal{F}$ on Σ
w/ singularities :



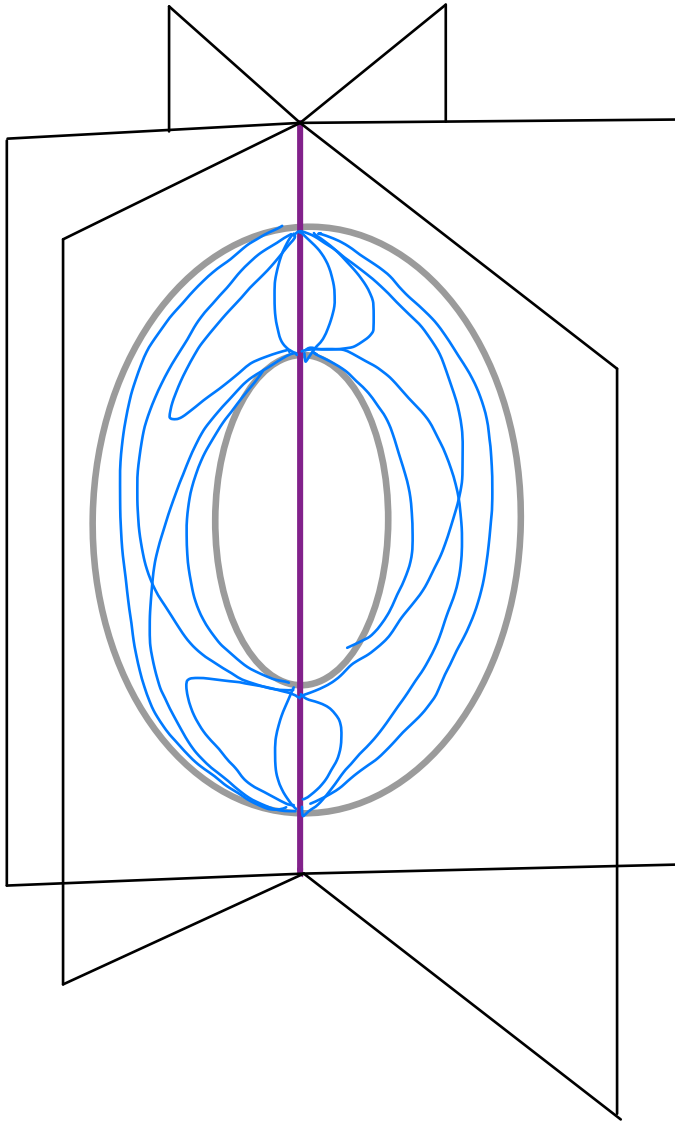
node



saddle

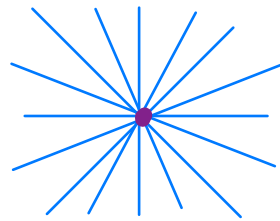
BRAID FOLIATIONS

$\mathbb{R}^3$ w/ cylindrical coordinates (z, r, ϑ)

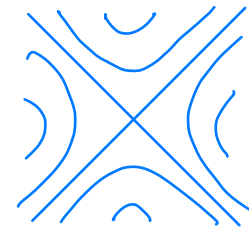


- $H_t := \{ \vartheta = t \}$ half planes
- they intersect in the z -axis
- take a surface Σ in $\mathbb{R}^3$
- & look at its intersection w/ H_t

$\leadsto$ gives a singular foliation $\mathcal{F}$ on Σ
w/ singularities:



node



saddle

(can remove other type of singularities via isotopy)

WHAT BRAID FOLIATIONS ARE USED FOR?

Bennequin 1980: the standard contact structure on $\mathbb{R}^3$
is tight

Binman - Menasco 1990s: - Marlov Theorem w/o stabilisation
- construction of transversally
nonsimple knots

LaFountain - Menasco, Dynnikov - Prasolov 2019:
generalised Jones conjecture

Dynnikov - Prasolov 2023: algorithmic classification
of Legendrian knots

GENERALISED JONES CONJECTURE

Suppose two braid - words $u = \prod \sigma_{i_j}^{n_j}$ $v = \prod \sigma_{k_j}^{m_j}$

- represent the same knot &

- have minimal braid index $\stackrel{?}{\Rightarrow} \sum n_j = \sum m_j$

e.g.:



$$u = \sigma_1^{-4} \sigma_2^2 \sigma_1^{-1} \sigma_2^3$$



$$v = \sigma_1^{-4} \sigma_2^3 \sigma_1^{-1} \sigma_2^2$$

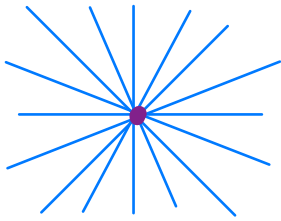
these • represent the same knot ✓

- have minimal braid index ✓

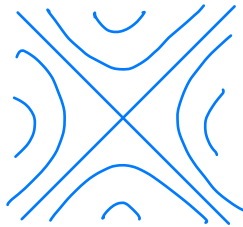
& indeed $-4 + 2 - 1 + 3 = -4 + 3 - 1 + 2$

RECOGNISING BRAID FOLIATIONS

INPUT: a singular foliation $\mathcal{F}$ on Σ w/ singularities:



node

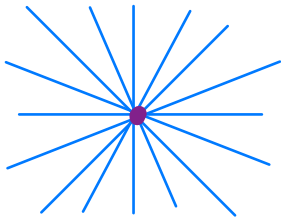


saddle

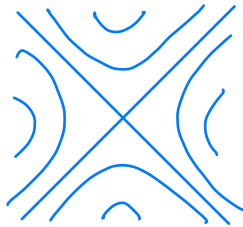
QUESTION: Can $\mathcal{F}$ be obtained as a braid foliation on some embedding of Σ in $\mathbb{R}^3$?

RECOGNISING BRAID FOLIATIONS

INPUT: a singular foliation $\mathcal{F}$ on Σ w/ singularities:



node



saddle

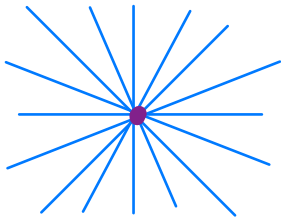
QUESTION: Can $\mathcal{F}$ be obtained as a braid foliation on some embedding of Σ in $\mathbb{R}^3$?

hard to reformulate combinatorially

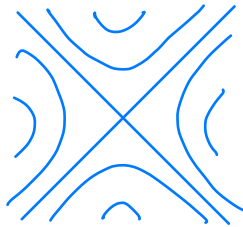
$\Rightarrow$ ask a more general question

RECOGNISING BRAID FOLIATIONS

INPUT: a singular foliation $\mathcal{F}$ on Σ w/ singularities:



node



saddle

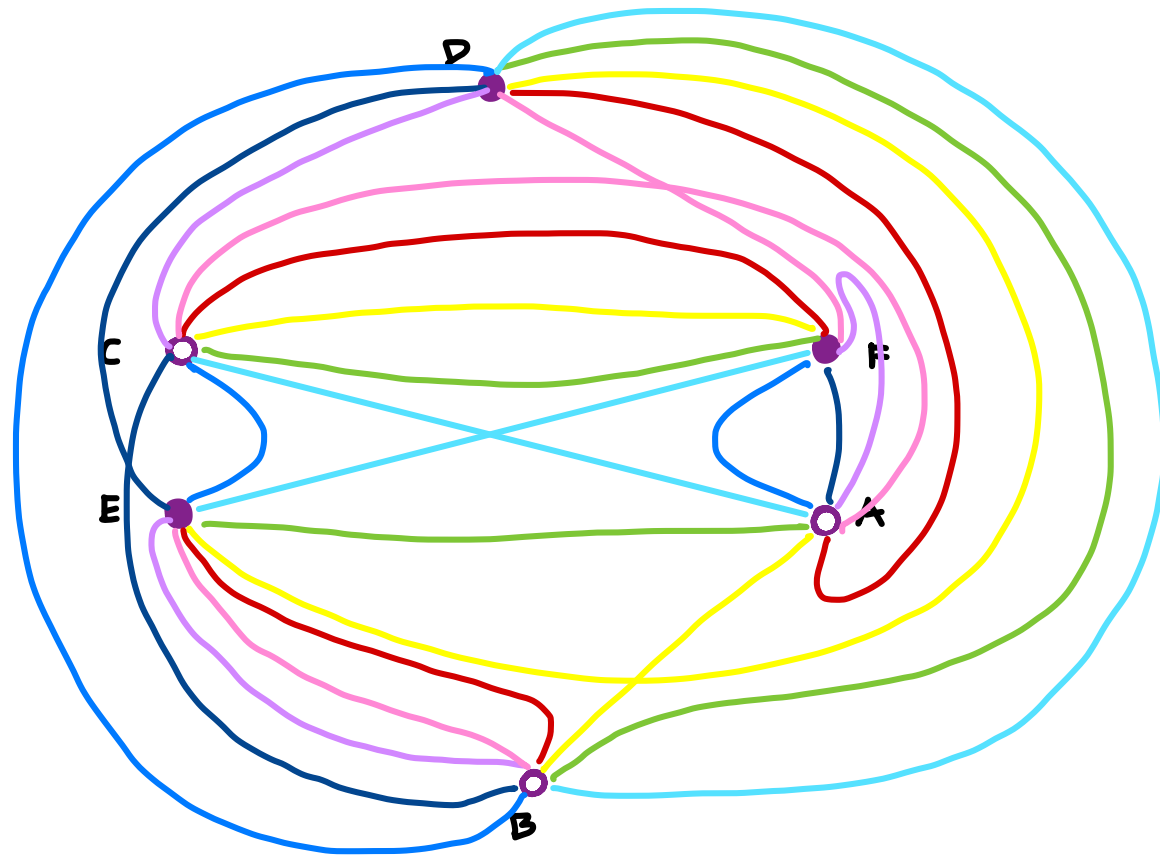
open book foliation

QUESTION: Can $\mathcal{F}$ be obtained as a ~~braid foliation~~
on some embedding of Σ in $\mathbb{R}^3$?

hard to reformulate combinatorially

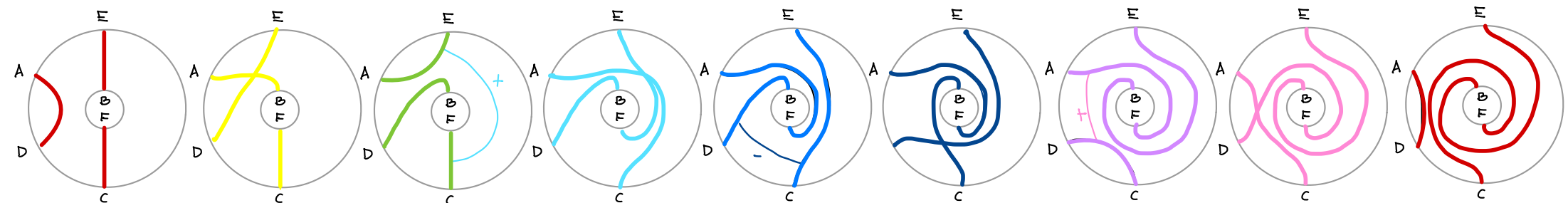
$\Rightarrow$ ask a more general question

AN OPEN BOOK FOLIATION THAT IS NOT A BRAID FOLIATION



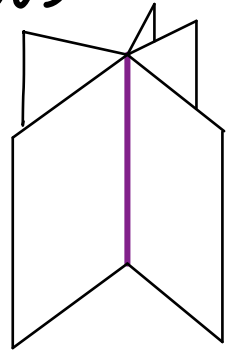
on S^2 (sphere)

an embedding of $\mathcal{F}$ onto the annulus



OPEN BOOK FOLIATIONS

- uses more general surfaces instead of half-planes
- defined on arbitrary 3-manifolds



Ito - Kawamuro 2009 - 2010: tightness criteria
for contact structures

Hayden 2021: characterisation of quasipositive links

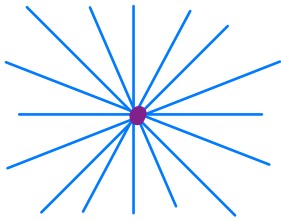
Licata - V 2022: combinatorial description of contact
structures on 3-manifolds w/ boundary

Alishasi - Földvári - Hendricks - Licata - Petřava - V 2023:

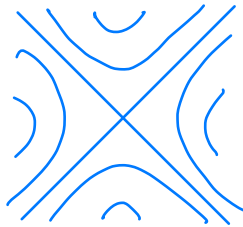
quable invariant for contact structures in
Heegaard Floer homology

RECOGNISING OPEN BOOK FOLIATIONS

INPUT: a singular foliation $\mathcal{F}$ on Σ w/ singularities:



node



saddle

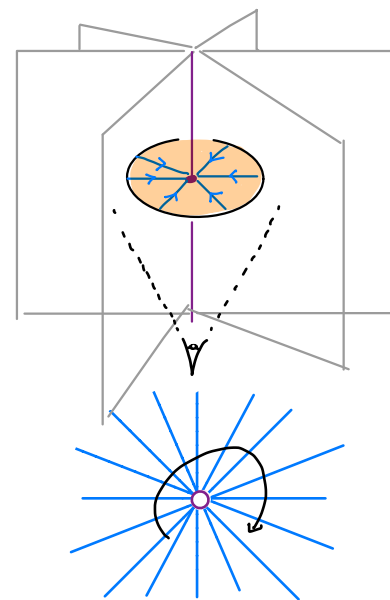
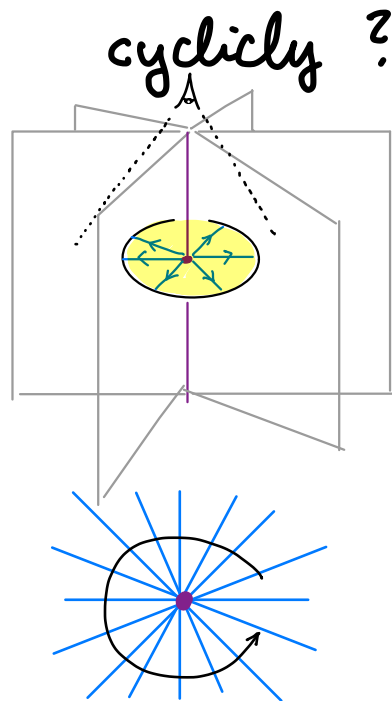
QUESTION: Can $\mathcal{F}$ be obtained as a **open book foliation** on some embedding of Σ in $\mathbb{R}^3$?

RECOGNISING OPEN BOOK FOLIATIONS

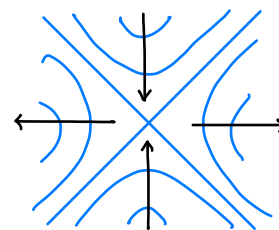
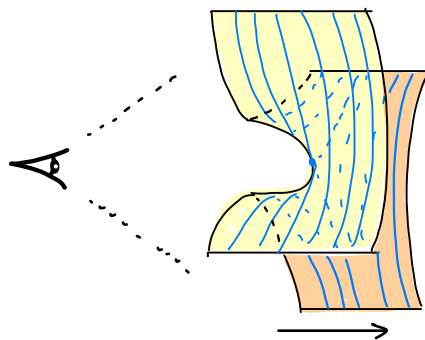
INPUT: a singular foliation $\mathcal{F}$ on Σ w/ node & saddle singularities

EQUIVALENT QUESTION: Can one index the leaves of $\mathcal{F}$ cyclicly?

two types of nodes



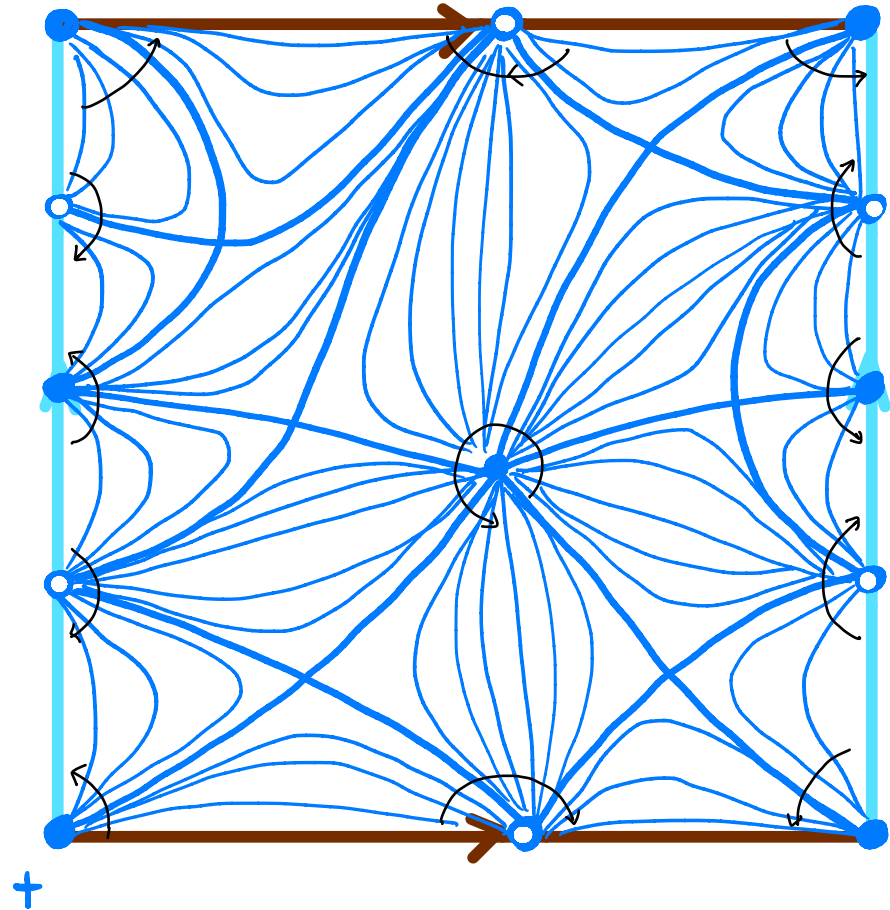
near a saddle



REFORMULATION

ANOTHER EQUIVALENT QUESTION: Can one order the
saddles cyclicly?

given: $\mathcal{F}$ on Σ

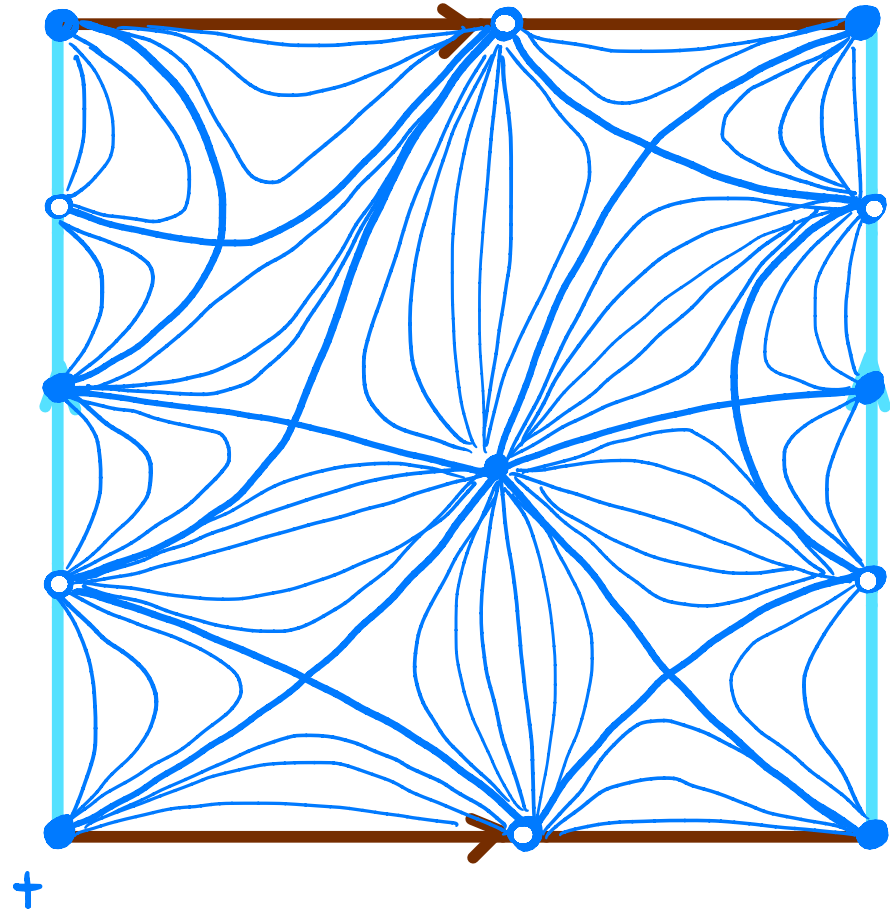


REFORMULATION

ANOTHER EQUIVALENT QUESTION: Can one order the
saddles cyclicly?

given: $\mathcal{F}$ on Σ

define: a digraph $D(V, \vec{E})$
on Σ



REFORMULATION

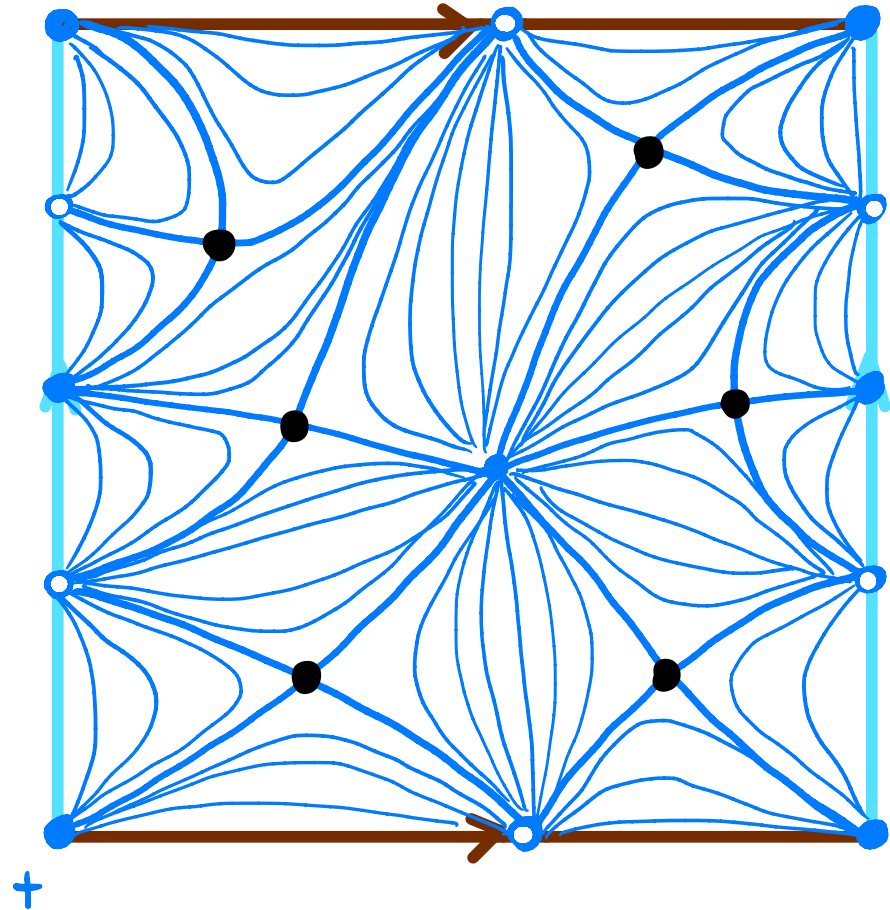
ANOTHER EQUIVALENT QUESTION: Can one order the
saddles cyclicly?

given: $\mathcal{F}$ on Σ

define: a digraph $D(V, \vec{E})$

on Σ

- $V =$ saddles



REFORMULATION

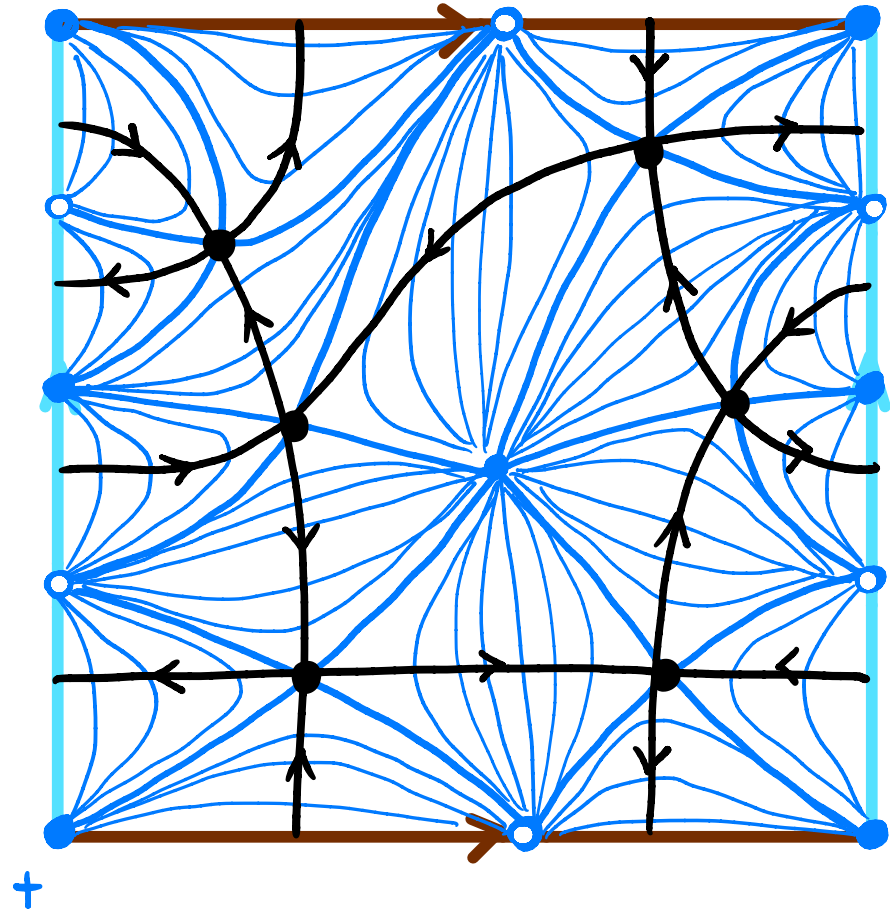
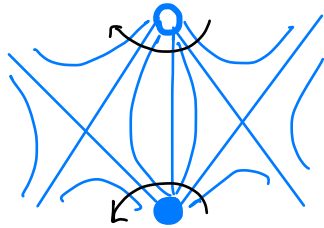
ANOTHER EQUIVALENT QUESTION: Can one order the
saddles cyclicly?

given: $\mathbb{F}$ on Σ

define: a digraph $D(V, \vec{E})$
on Σ

- $V =$ saddles

- edges:



REFORMULATION

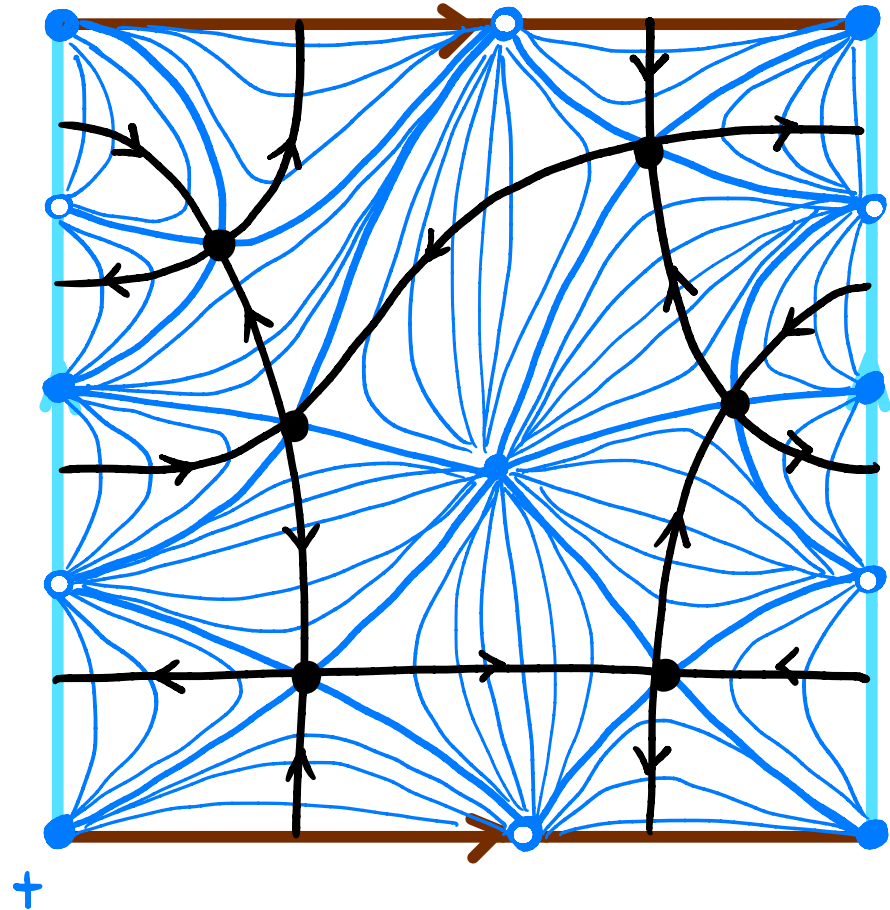
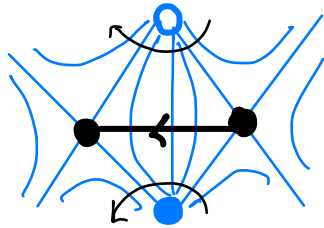
ANOTHER EQUIVALENT QUESTION: Can one order the
saddles cyclicly?

given: $\mathbb{F}$ on Σ

define: a digraph $D(V, \vec{E})$
on Σ

- V = saddles

- edges:



REFORMULATION

ANOTHER EQUIVALENT QUESTION: Can one order the
saddles cyclicly?

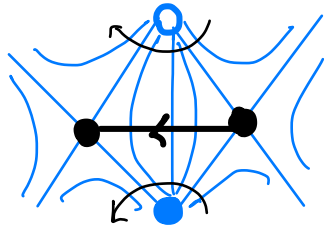
given: $\mathbb{F}$ on Σ

define: a digraph $D(V, \vec{E})$

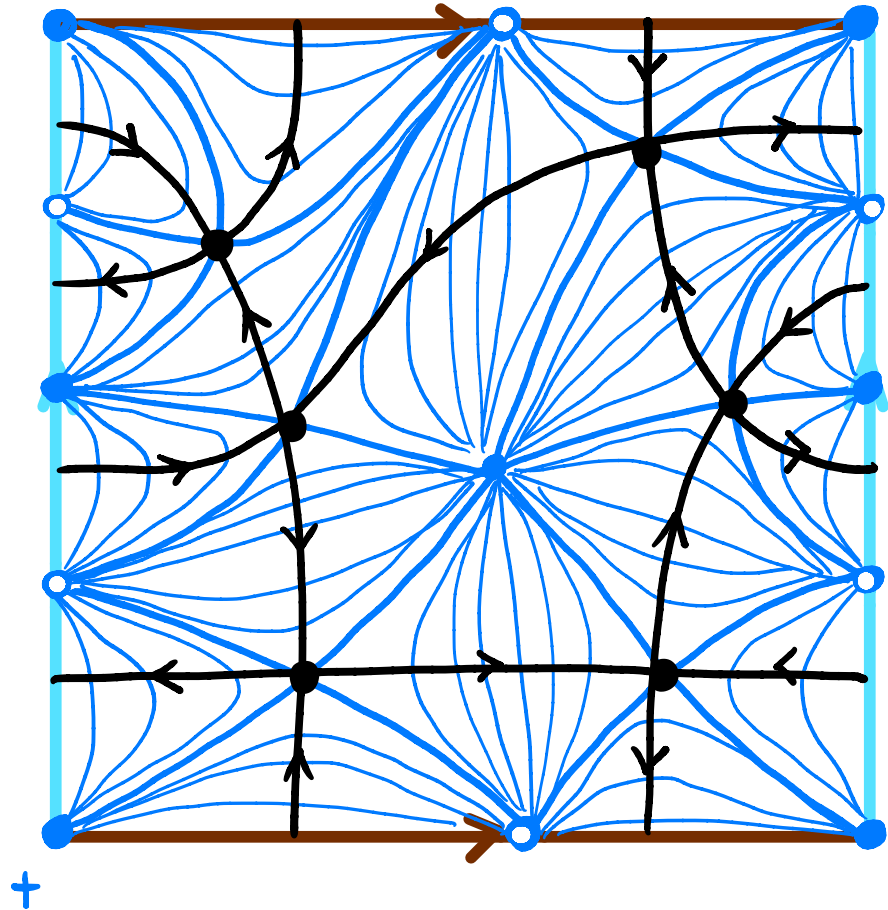
on Σ

- $V =$ saddles

- edges:



$\Rightarrow$ Eulerian & 4-regular digraph



REFORMULATION

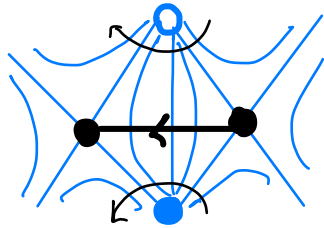
ANOTHER EQUIVALENT QUESTION: Can one order the
saddles cyclicly?

given: $\mathbb{F}$ on Σ

define: a digraph $D(V, \vec{E})$
on Σ

- V = saddles

- edges:

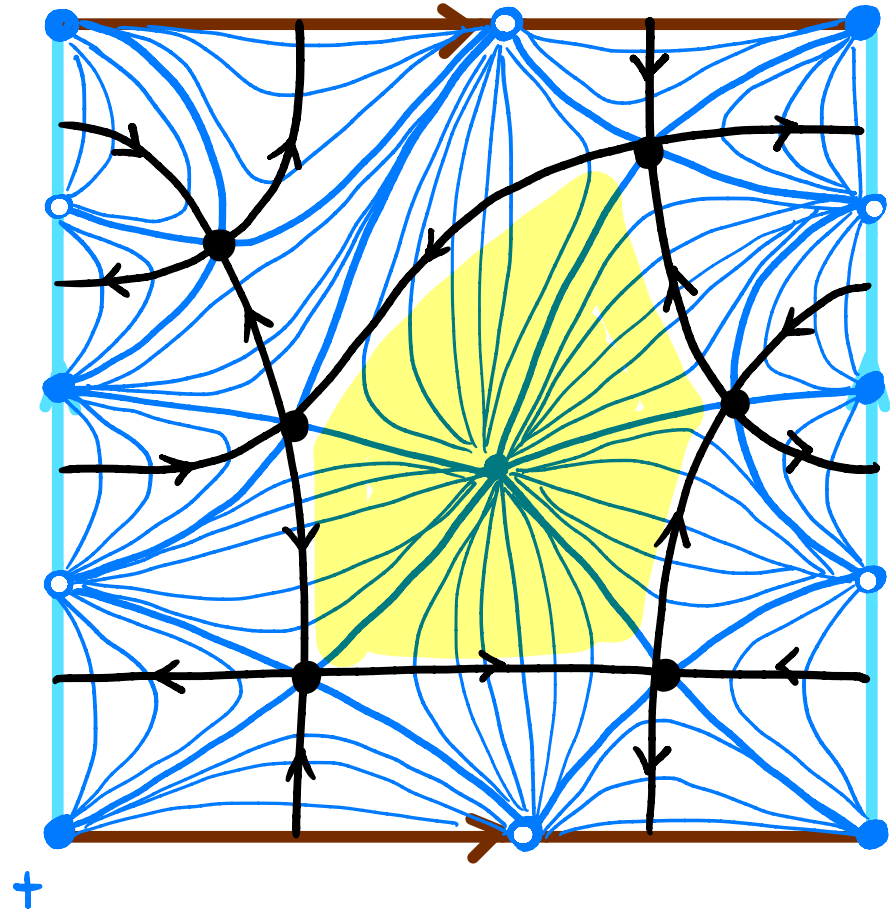


$\Rightarrow$ Eulerian & 4-regular digraph

w/ a 2-cell-embedding



nodes



REFORMULATION

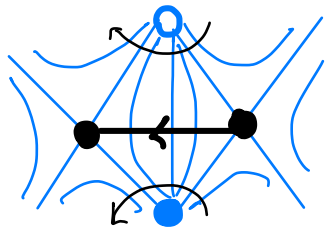
ANOTHER EQUIVALENT QUESTION: Can one order the
saddles cyclicly?

given: $\mathbb{F}$ on Σ

define: a digraph $D(V, \vec{E})$
on Σ

- V = saddles

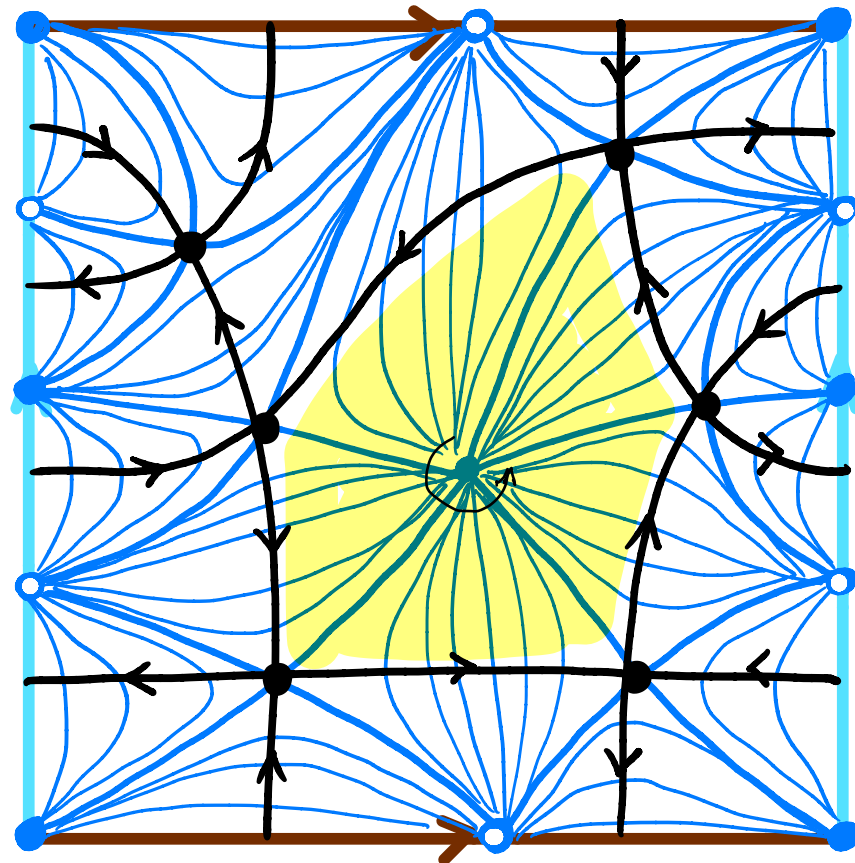
- edges:



$\Rightarrow$ Eulerian & 4-regular digraph

w/ a 2-cell-embedding s.t. for each face F

the edges on ∂F form an oriented circle &



REFORMULATION

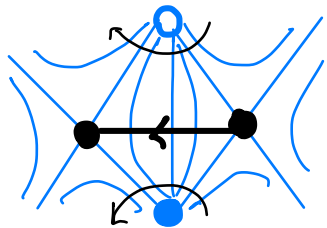
ANOTHER EQUIVALENT QUESTION: Can one order the
saddles cyclicly?

given: $\mathbb{F}$ on Σ

define: a digraph $D(V, \vec{E})$
on Σ

- V = saddles

- edges:

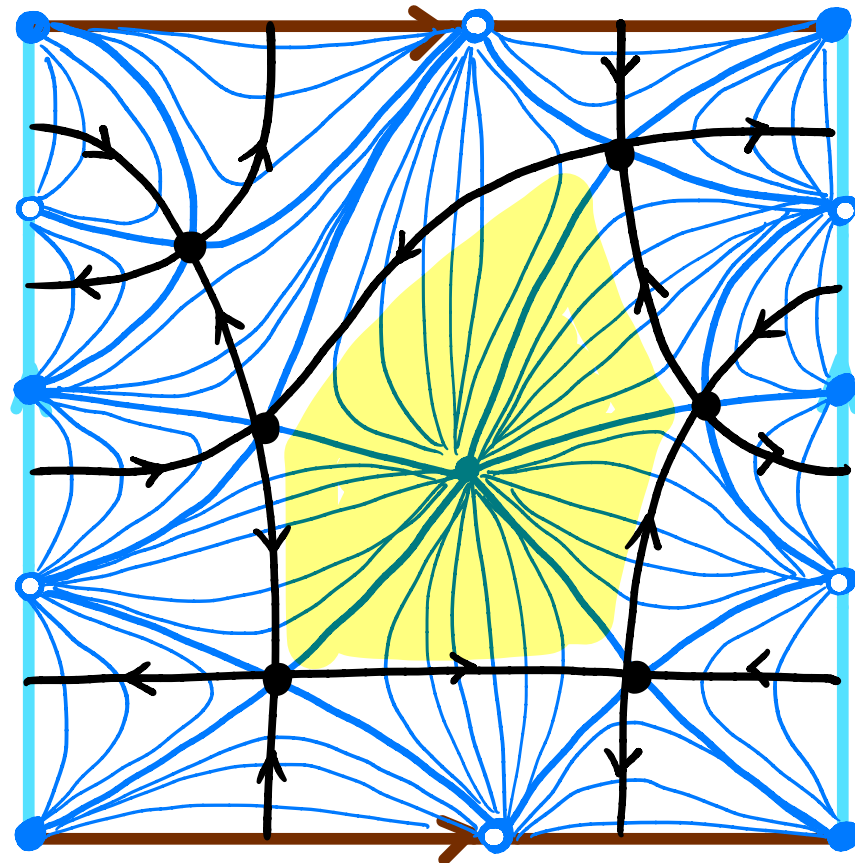


$\Rightarrow$ Eulerian & 4-regular digraph

w/ a 2-cell-embedding s.t for each face F

the edges on ∂F form an oriented circle &

the vertices on ∂F are all distinct



REFORMULATION - CONTINUED

$D(V, \vec{E})$ Eulerian & 4-regular digraph

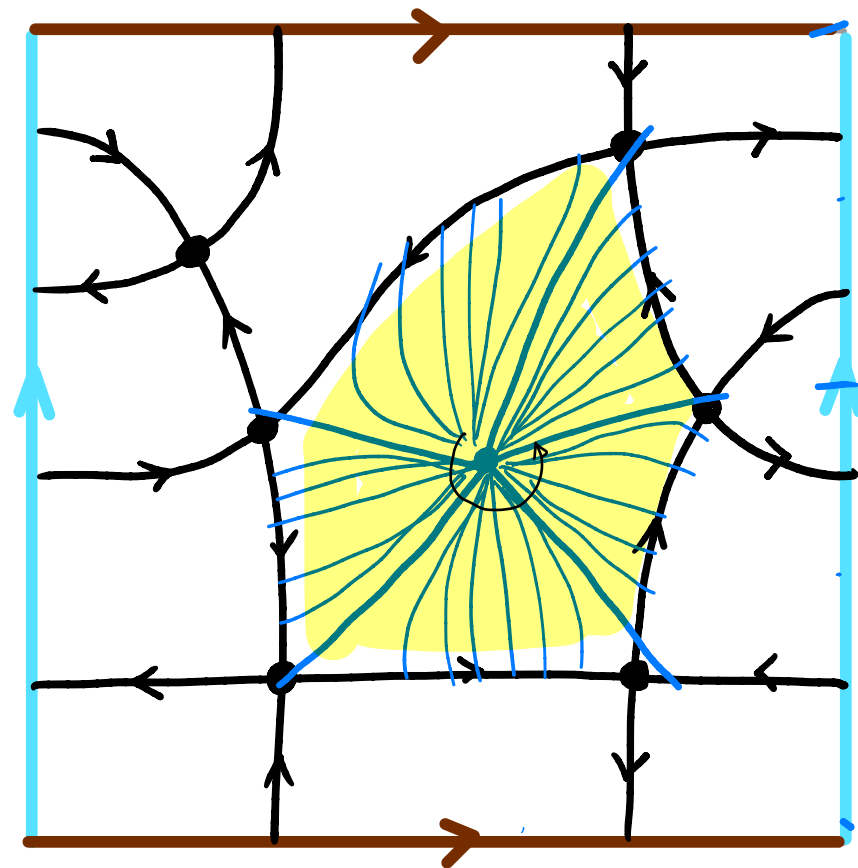
w/ a 2-cell-embedding into Σ

s.t for each face F

the edges on ∂F form an oriented circ.

the vertices on ∂F are all distinct

QUESTION: Does the vertices have
a cyclic order that
restricts to a cyclic order on each ∂F ?



REFORMULATION - CONTINUED

$D(V, \vec{E})$ Eulerian & 4-regular digraph

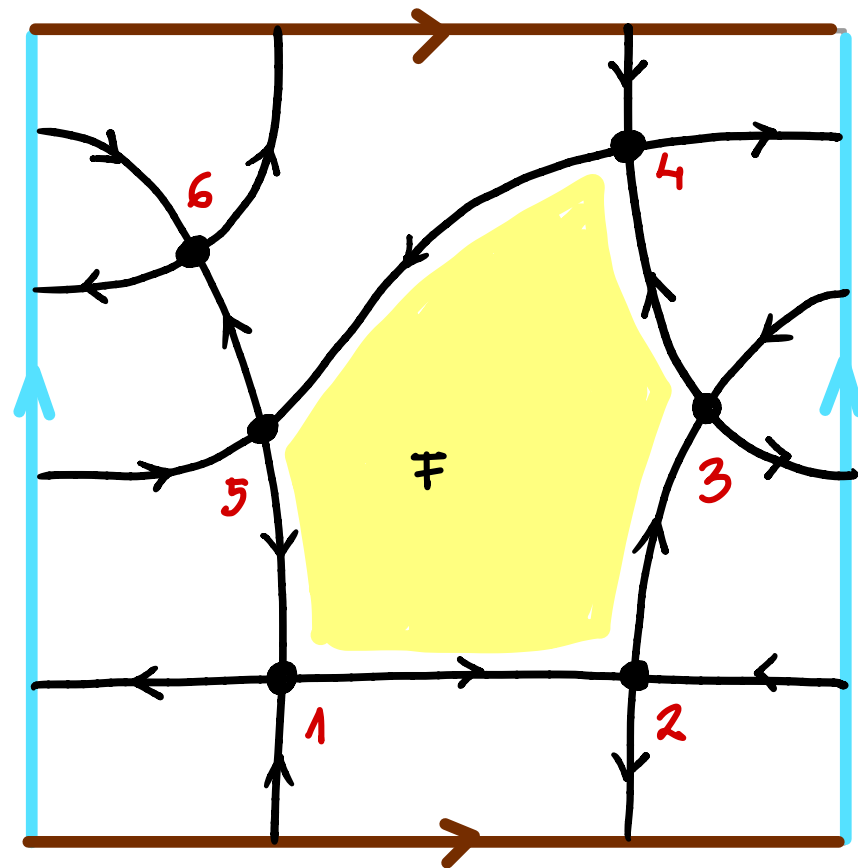
w/ a 2-cell-embedding into Σ

s.t for each face F

the edges on ∂F form an oriented circ.

the vertices on ∂F are all distinct

QUESTION: Does the vertices have
a cyclic order that
restricts to a cyclic order on each ∂F ?



REFORMULATION - CONTINUED

$D(V, \vec{E})$ Eulerian & 4-regular digraph

w/ a 2-cell-embedding into Σ

s.t for each face F

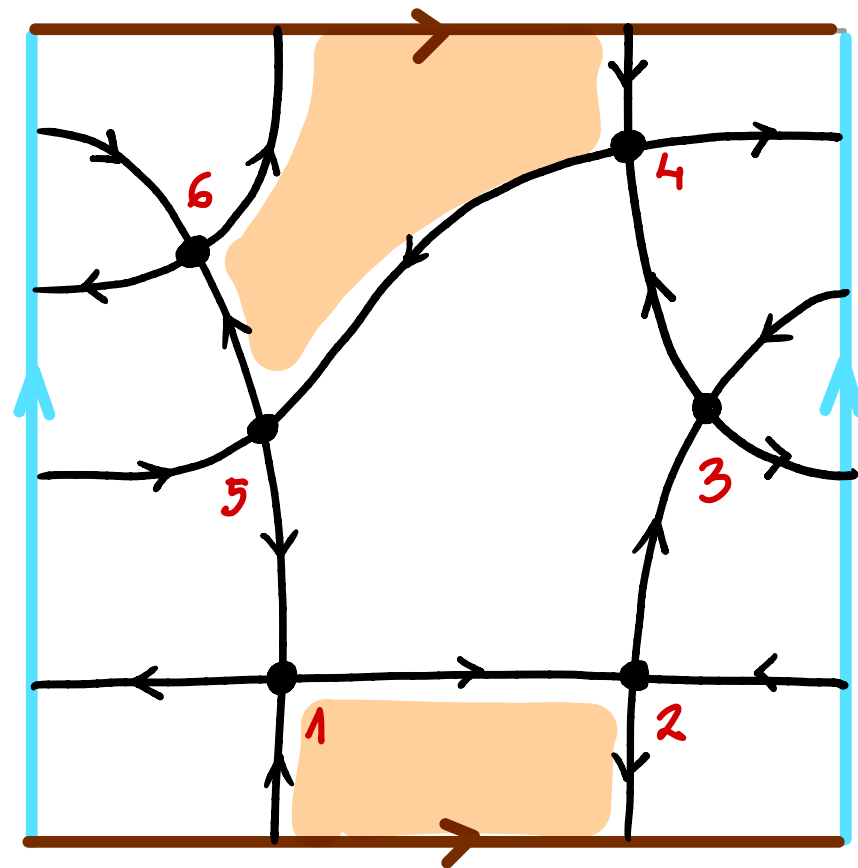
the edges on ∂F form an oriented circ.

the vertices on ∂F are all distinct

QUESTION: Does the vertices have

a cyclic order that

restricts to a cyclic order on each ∂F ?



REFORMULATION - CONTINUED

$D(V, \vec{E})$ Eulerian & 4-regular digraph

w/ a 2-cell-embedding into Σ

s.t for each face F

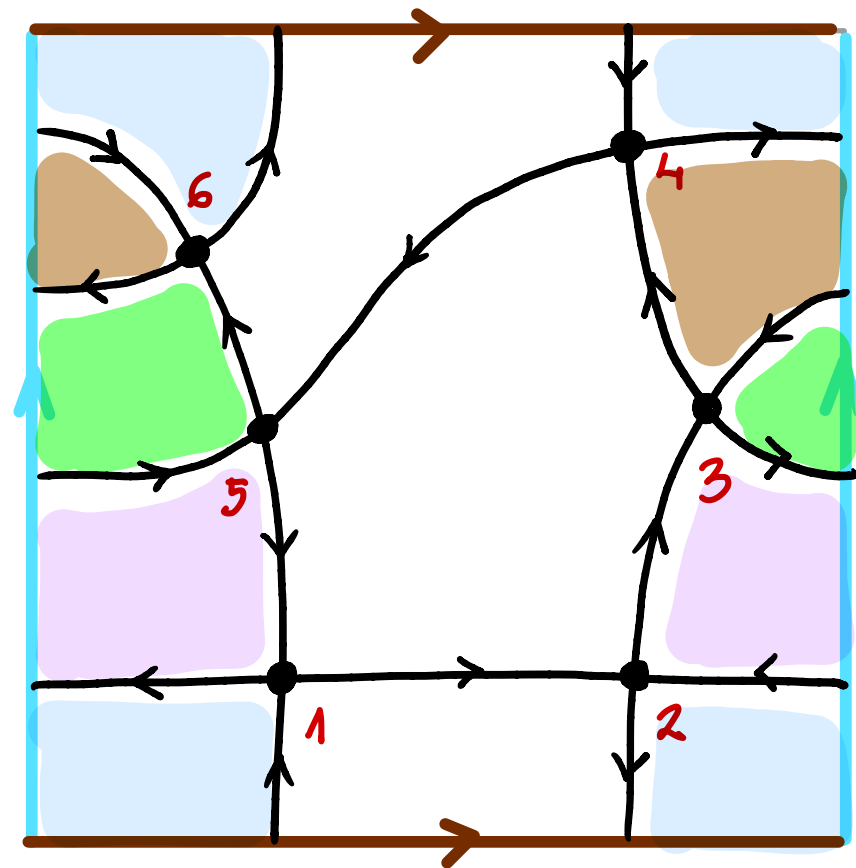
the edges on ∂F form an oriented circ.

the vertices on ∂F are all distinct

QUESTION: Does the vertices have

a cyclic order that

restricts to a cyclic order on each ∂F ?



REFORMULATION - CONTINUED

$D(V, \vec{E})$ Eulerian & 4-regular digraph

w/ a 2-cell-embedding into Σ

s.t for each face F

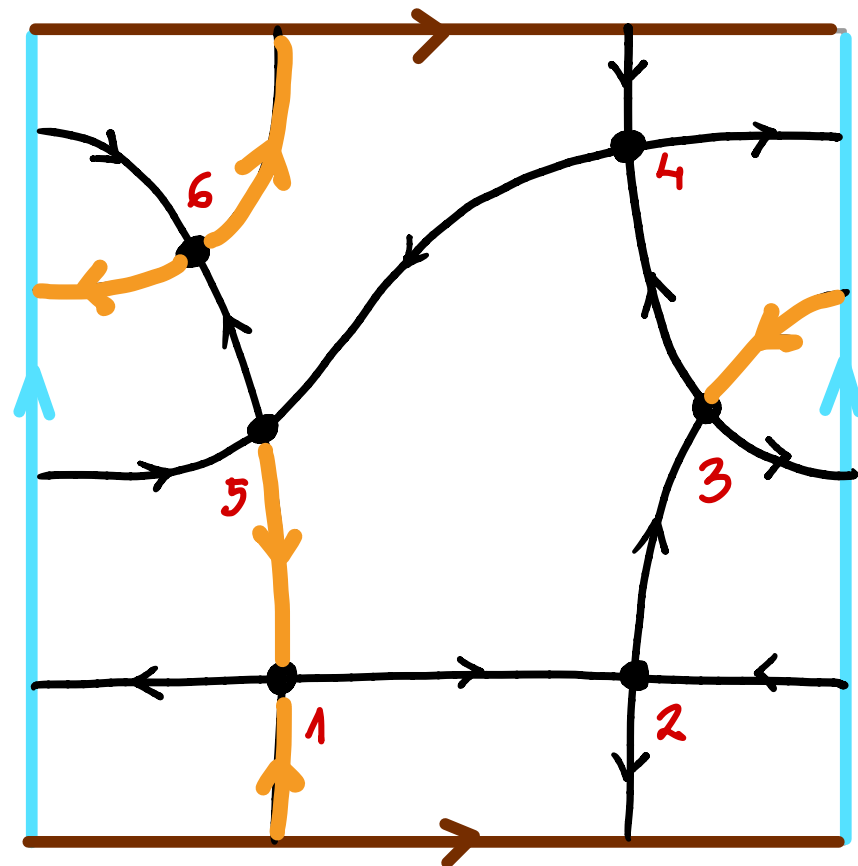
the edges on ∂F form an oriented circ.

the vertices on ∂F are all distinct

QUESTION: Does the vertices have

a cyclic order that

restricts to a cyclic order on each ∂F ?



→ feedback arc set: $\vec{E}' = \{\text{backward edges}\}$

one on each $\partial F \Rightarrow |\vec{E}'| = \frac{\# \text{ faces}}{2}$

REFORMULATION - CONTINUED

$D(V, \vec{E})$ Eulerian & 4-regular digraph

w/ a 2-cell-embedding into Σ

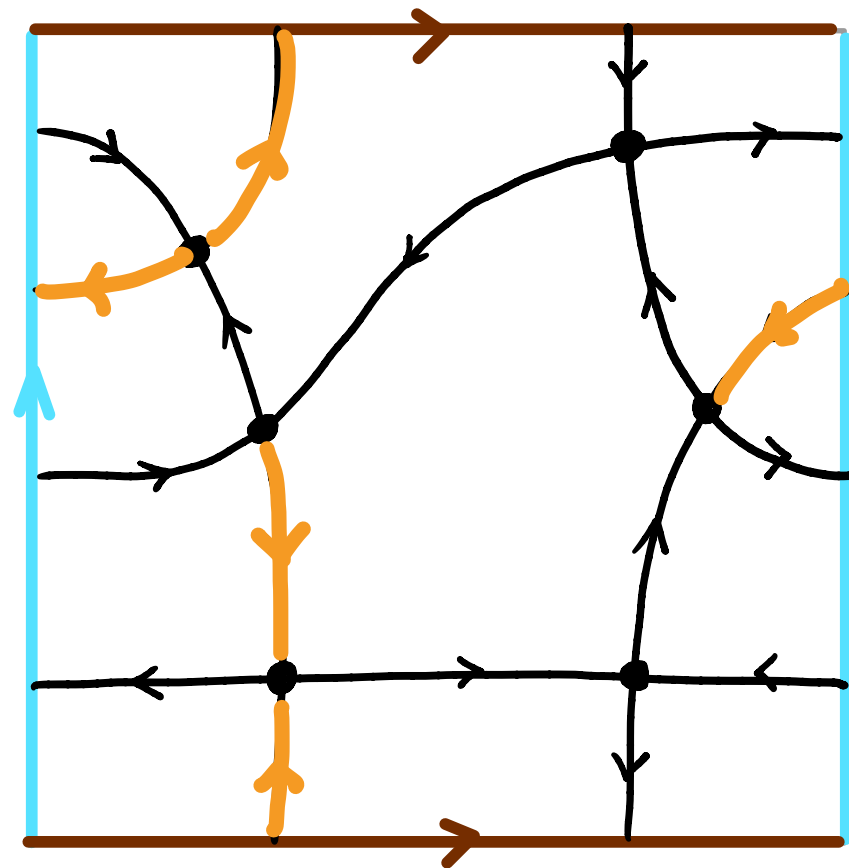
s.t for each face F

the edges on ∂F form an oriented circ.

the vertices on ∂F are all distinct

CONVERSELY suppose we've found a

feedback arc set: $\vec{E}'$ of size $\frac{\# \text{ faces}}{2}$



REFORMULATION - CONTINUED

$D(V, \vec{E})$ Eulerian & 4-regular digraph

w/ a 2-cell-embedding into Σ

s.t for each face F

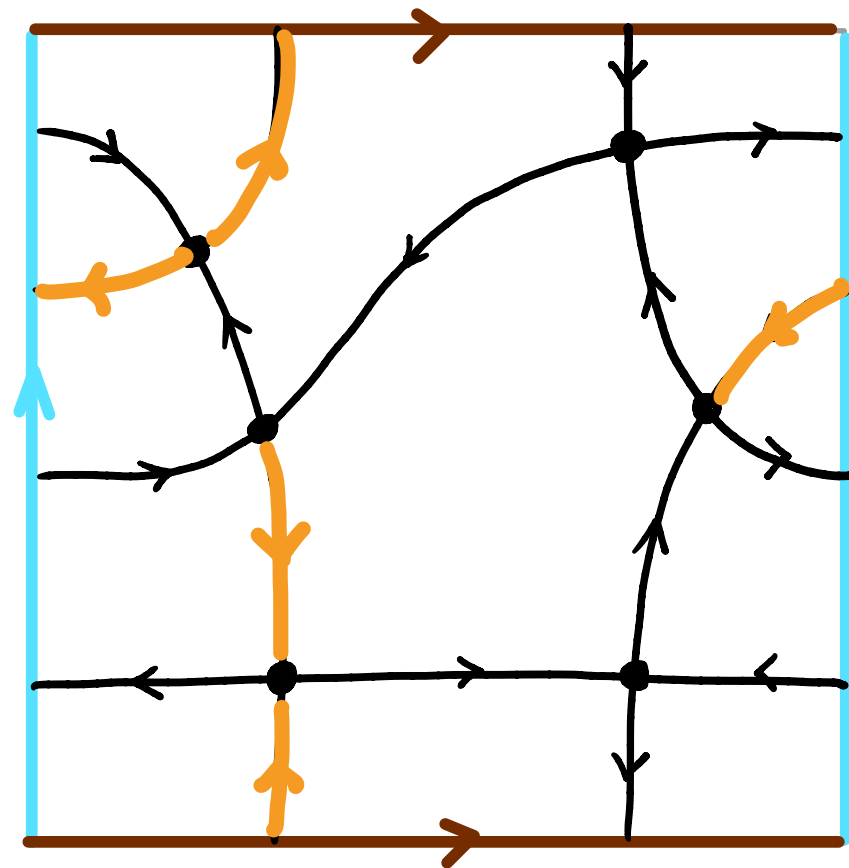
the edges on ∂F form an oriented circ.

the vertices on ∂F are all distinct

CONVERSELY suppose we've found a

feedback arc set: $\vec{E}'$ of size $\frac{\# \text{ faces}}{2}$

$\Rightarrow$ must have at least one edge on each ∂F



REFORMULATION - CONTINUED

$D(V, \vec{E})$ Eulerian & 4-regular digraph

w/ a 2-cell-embedding into Σ

s.t for each face F

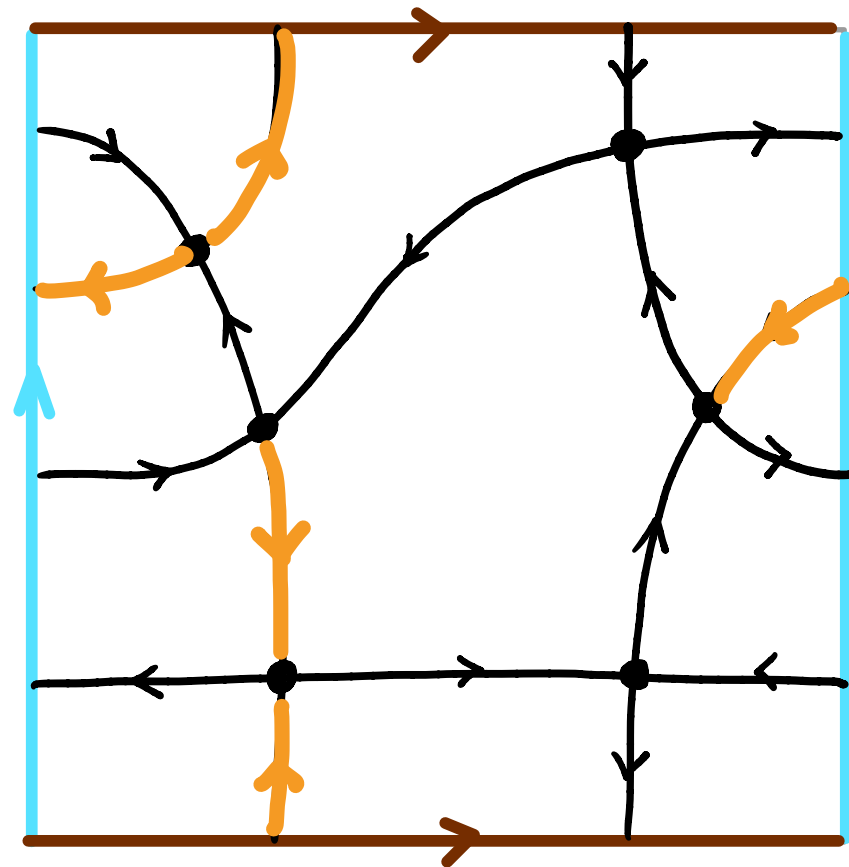
the edges on ∂F form an oriented circ.

the vertices on ∂F are all distinct

CONVERSELY suppose we've found a

feedback arc set: $\vec{E}'$ of size $\frac{\# \text{ faces}}{2}$

$\Rightarrow$ must have ~~at least~~ ^{exactly} one edge on each ∂F



REFORMULATION - CONTINUED

$D(V, \vec{E})$ Eulerian & 4-regular digraph

w/ a 2-cell-embedding into Σ

s.t for each face F

the edges on ∂F form an oriented circ.

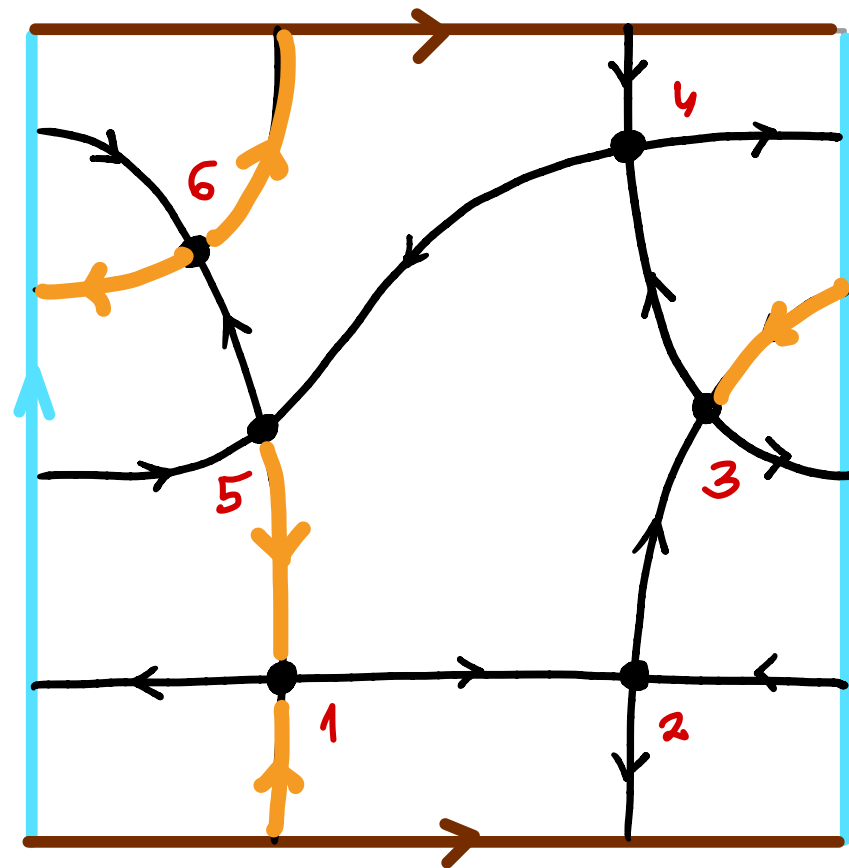
the vertices on ∂F are all distinct

CONVERSELY suppose we've found a

feedback arc set: $\vec{E}'$ of size $\frac{\# \text{ faces}}{2}$

$\Rightarrow$ must have exactly one edge on each ∂F

- linear order $\leadsto$ a cyclic order that restricts to a cyclic order on each ∂F ?



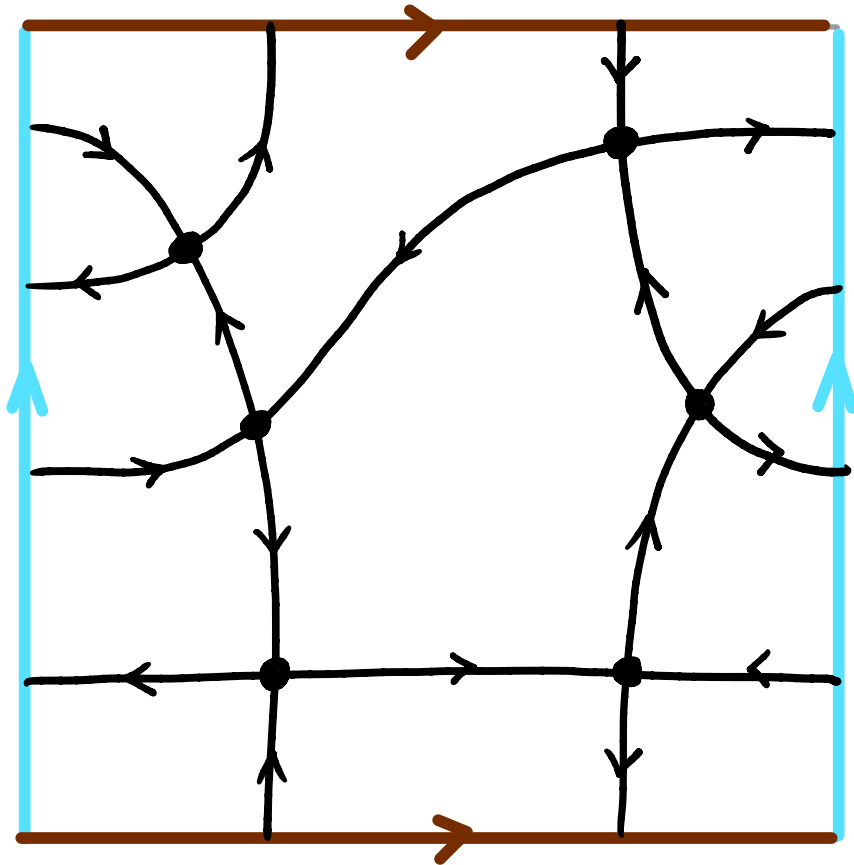
MAIN QUESTION

INPUT: an Eulerian & 4-regular digraph $D(V, \vec{E})$

& a 2-cell embedding $D(V, \vec{E}) \hookrightarrow \Sigma$ s.t. for each face F

→ the vertices on ∂F are all distinct

→ the edges on ∂F form an oriented circle

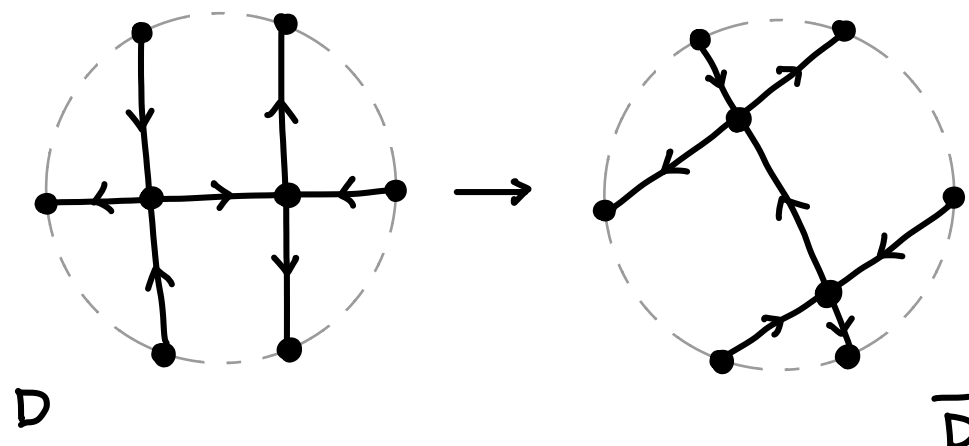


QUESTION: Is there a feedback arc set
of size $\frac{\# \text{ faces}}{2}$?

STRATEGY

D is an INPUT of the problem

local change:



LEMMA: Assume D' is also an "INPUT". Then

D has a feedback arc set of size $\frac{\# \text{ faces}}{2}$



D' has a feedback arc set of size $\frac{\# \text{ faces}}{2}$

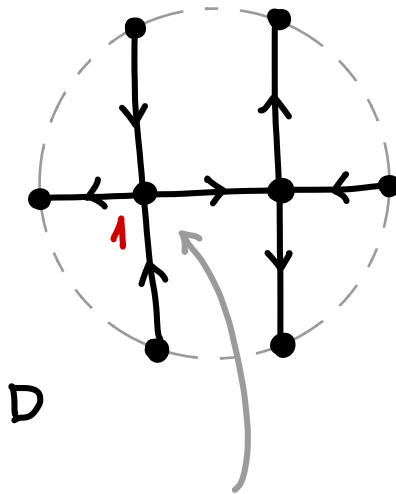
PROOF OF LEMMA:

Recall: feedback arc set of size $\frac{\# \text{ faces}}{2}$

$\updownarrow$
cyclic order of vertices that restricts to a
cyclic order on each ∂F

↖ "backward"
edges

- assume D has a cyclic order as above



we can assume

PROOF OF LEMMA:

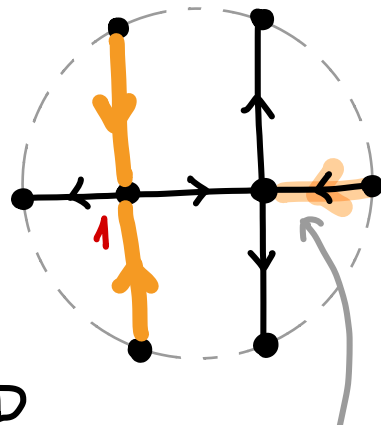
Recall: feedback arc set of size $\frac{\# \text{ faces}}{2}$

$\updownarrow$
cyclic order of vertices that restricts to a
cyclic order on each ∂F

↖ "backward"
edges

- assume D has a cyclic order as above

- the corresponding backward edges



maybe this too (or not)

PROOF OF LEMMA:

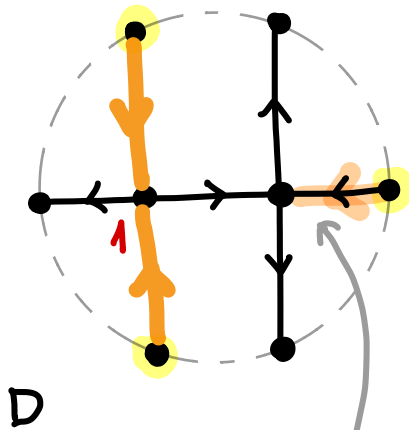
Recall: feedback arc set of size $\frac{\# \text{ faces}}{2}$

cyclic order of vertices that restricts to a
cyclic order on each ∂F

↖ "backward"
edges

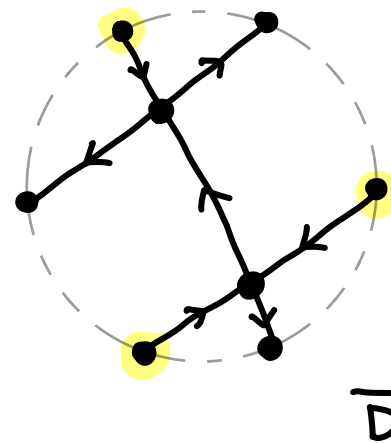
- assume D has a cyclic order as above

• the corresponding backward edges



maybe this too (or not)

local
change



PROOF OF LEMMA:

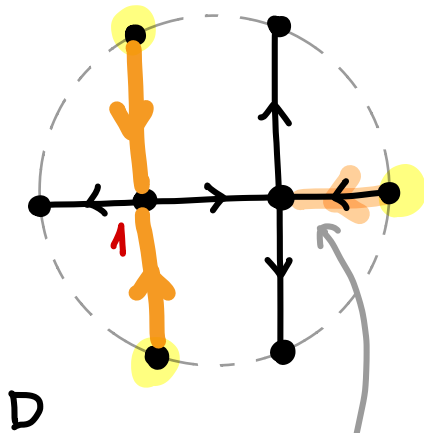
Recall: feedback arc set of size $\frac{\# \text{ faces}}{2}$

$\updownarrow$
cyclic order of vertices that restricts to a
cyclic order on each ∂F

↖ "backward"
edges

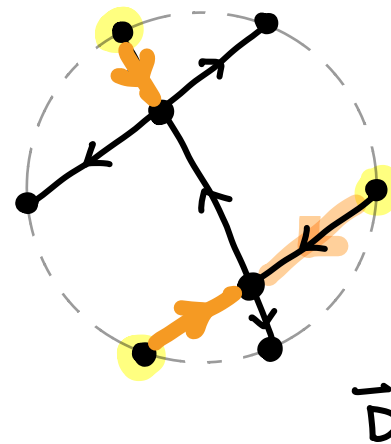
- assume D has a cyclic order as above

• the corresponding backward edges



D

local
change



D_1

maybe this too (or not)

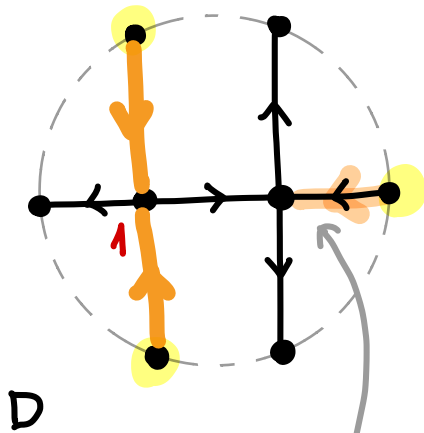
PROOF OF LEMMA:

Recall: feedback arc set of size $\frac{\# \text{ faces}}{2}$

$\updownarrow$
cyclic order of vertices that restricts to a
cyclic order on each ∂F

↖ "backward"
edges

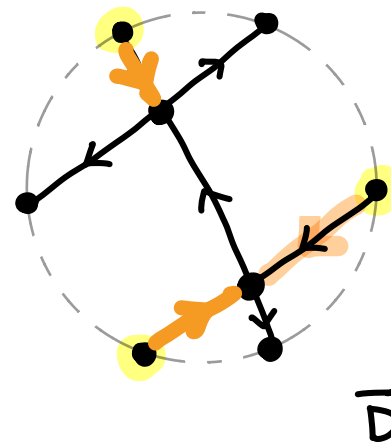
- assume D has a cyclic order as above



maybe this too (or not)

local
change

- the corresponding backward edges



- same vertices
are connected
on $\partial(\text{disc})$
in $E \setminus E'$ & $\bar{E} \setminus \bar{E}'$

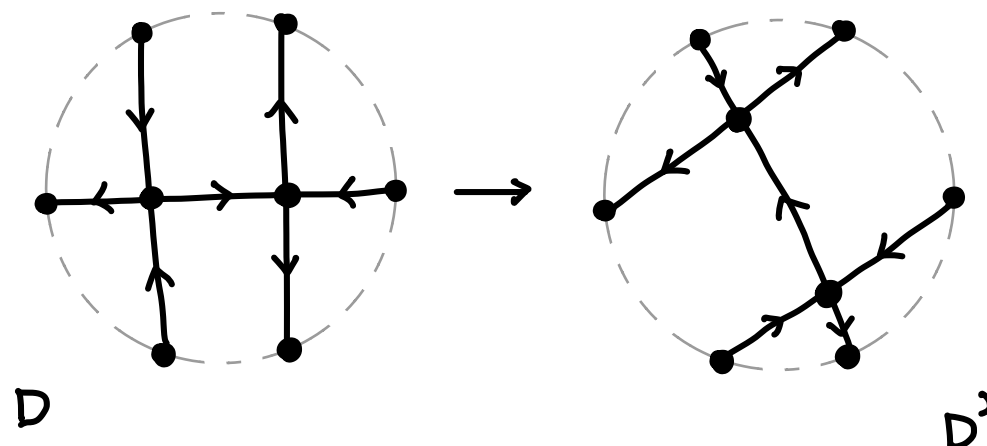
so $\bar{E}'$ gives a feedback arc set for $\bar{D}$ of size $\frac{\# \text{ faces}}{2}$ ✓



STRATEGY

D is an INPUT of the problem

local change:



LEMMA: Assume D' is also an "INPUT". Then

D has a feedback arc set of size $\frac{\# \text{ faces}}{2}$

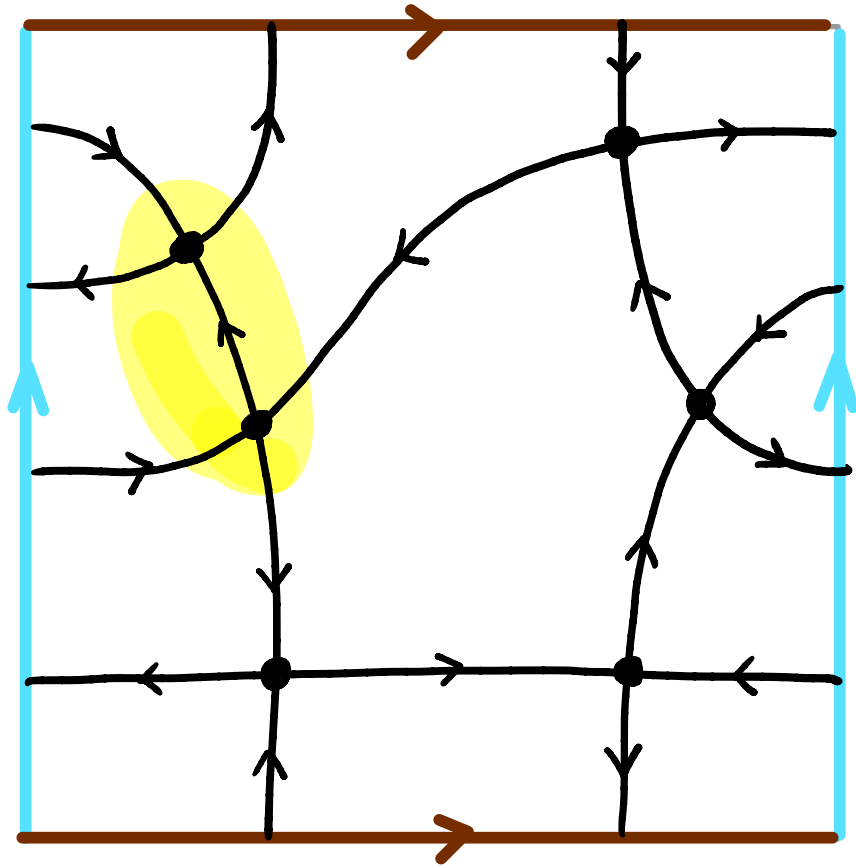


D' has a feedback arc set of size $\frac{\# \text{ faces}}{2}$

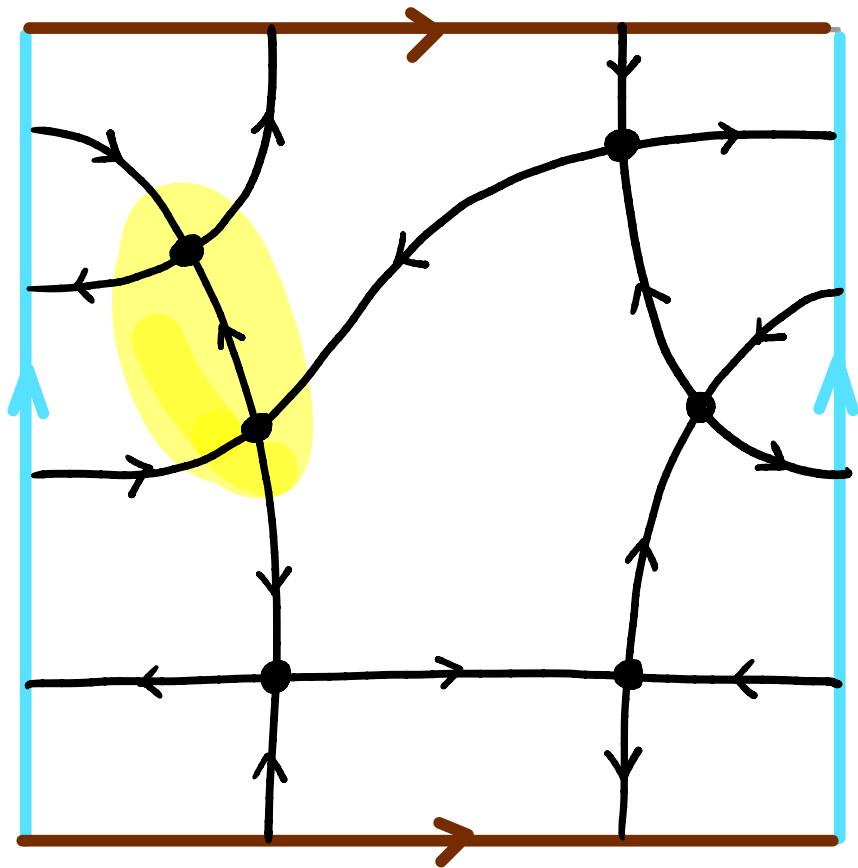
Change D into a digraph w/ a "big" face (w/ all v)

Then it's easy to tell.

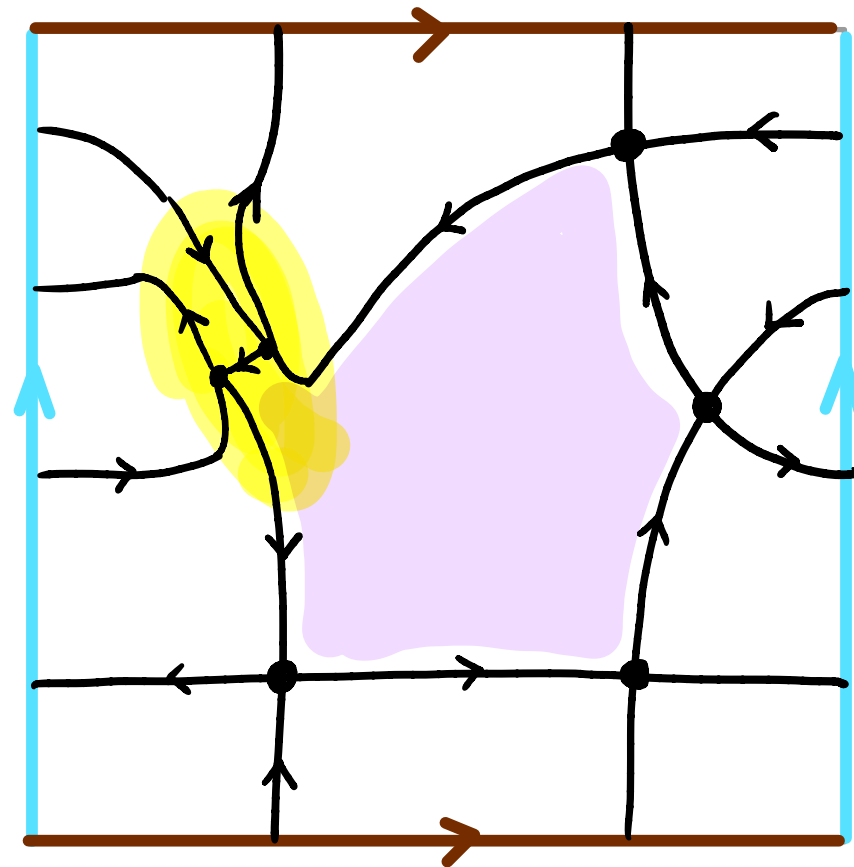
A GOOD EXAMPLE:



A GOOD EXAMPLE:

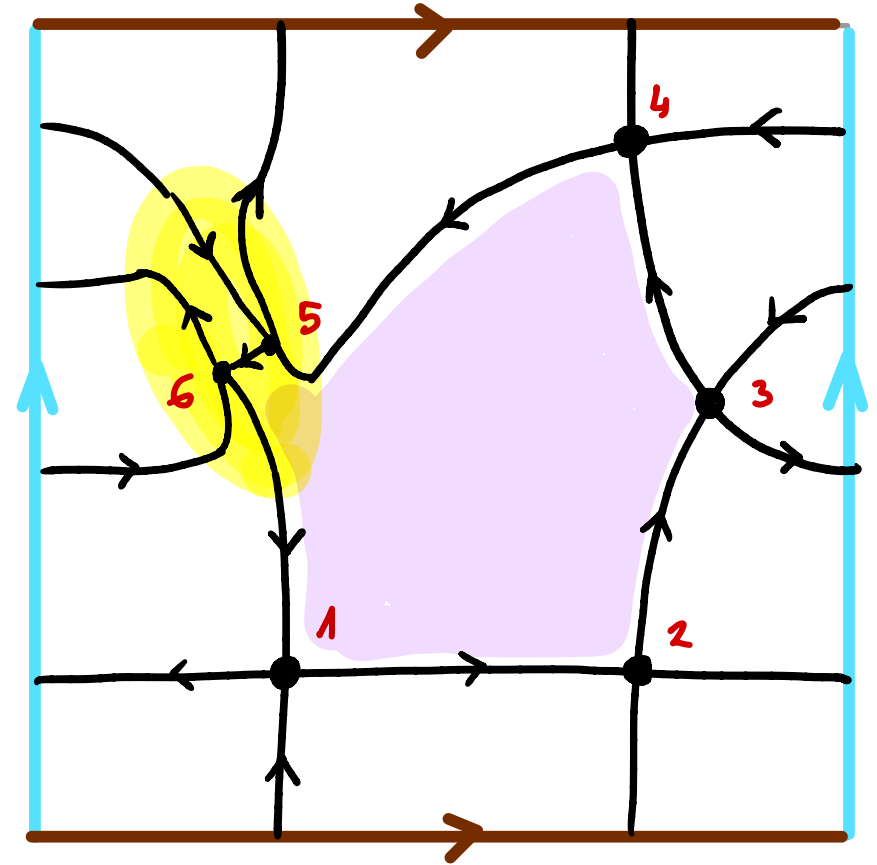
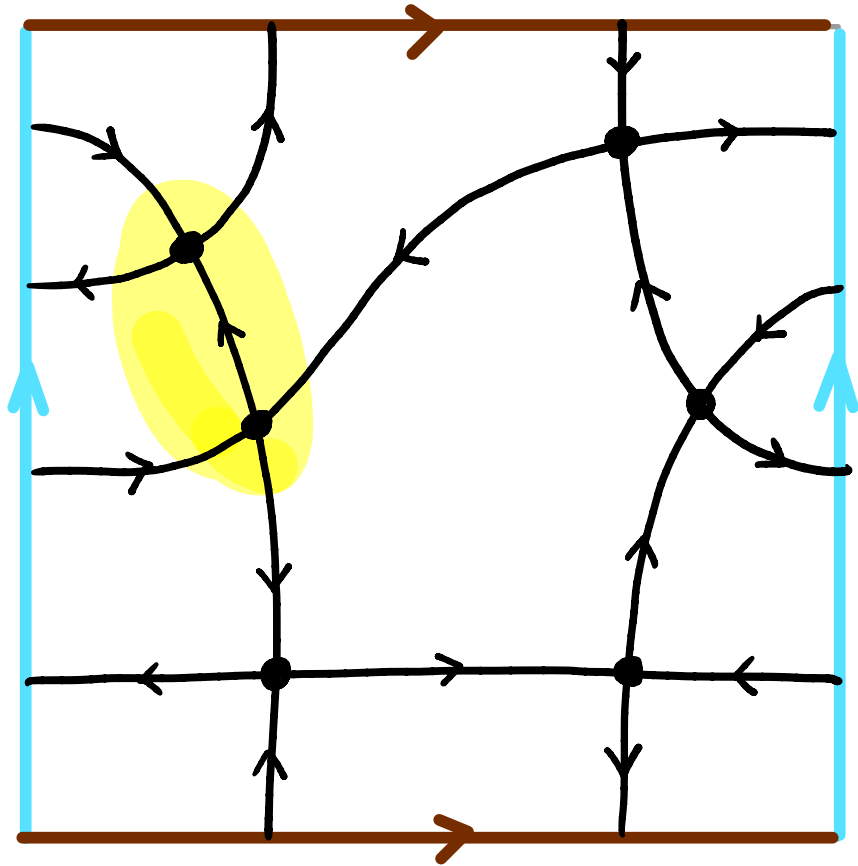


local
→
change



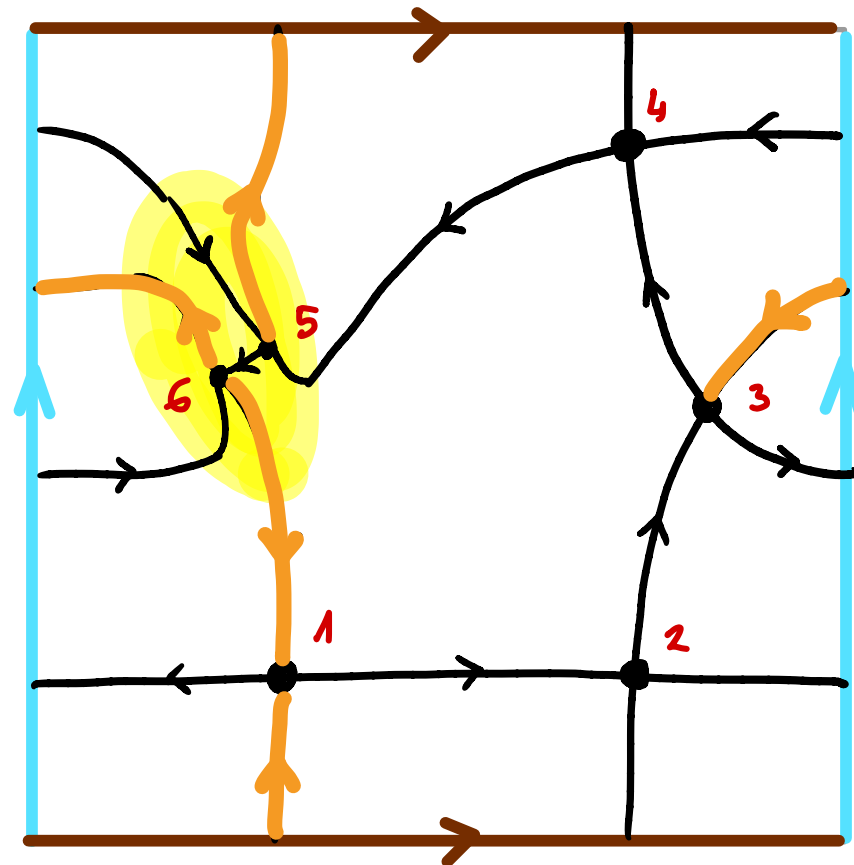
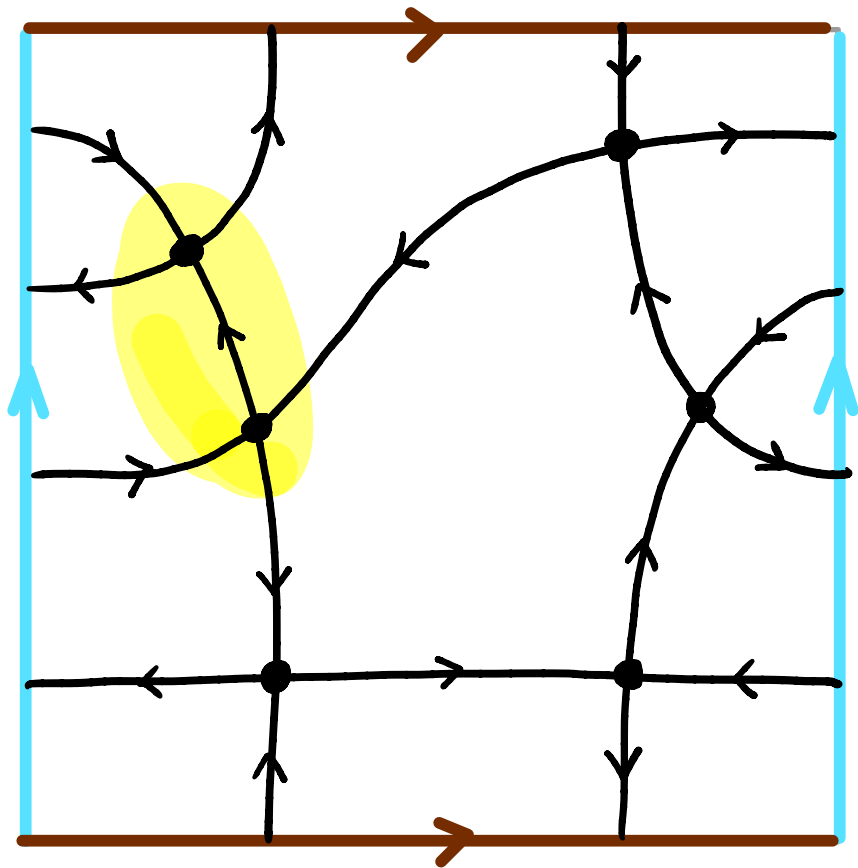
do local changes to get a "big circle"

A GOOD EXAMPLE:



the "big circle" gives a potential cyclic order
check if it works or not

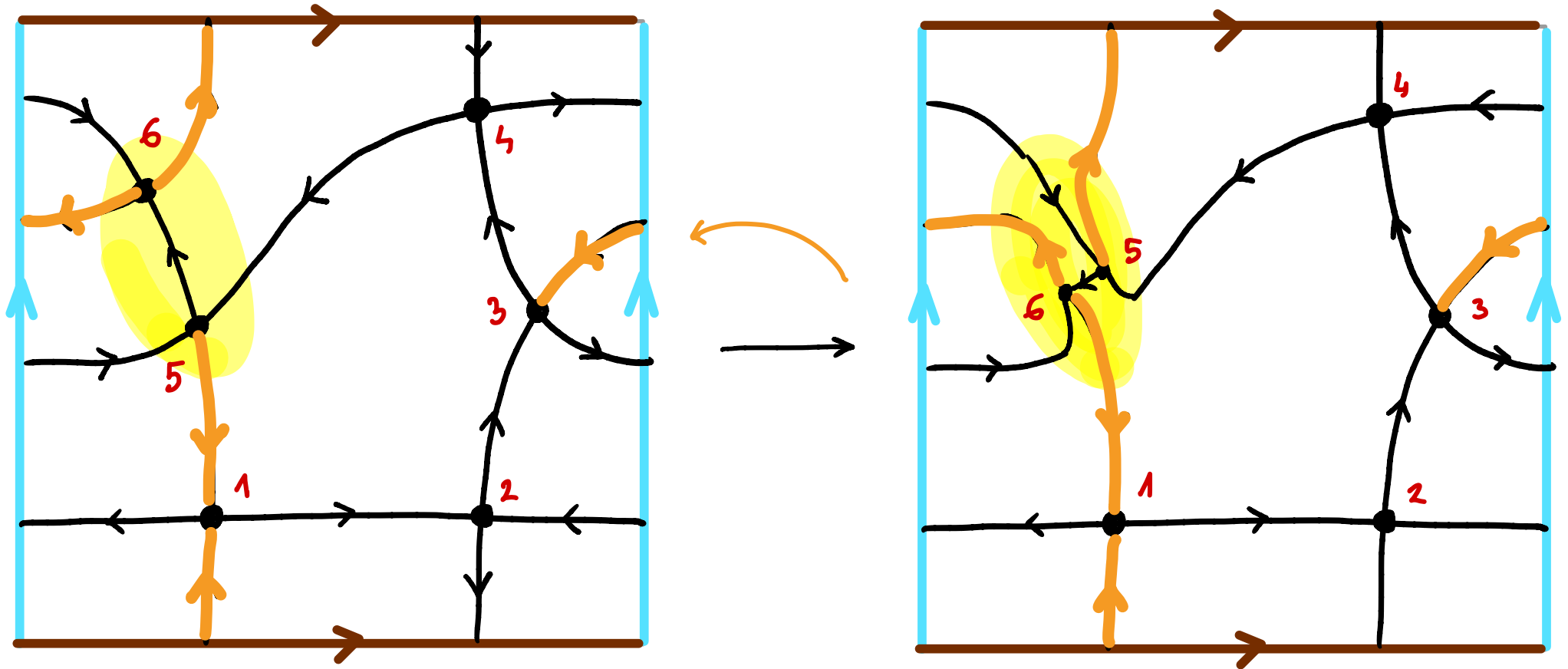
A GOOD EXAMPLE:



if it works construct a feedback arc set

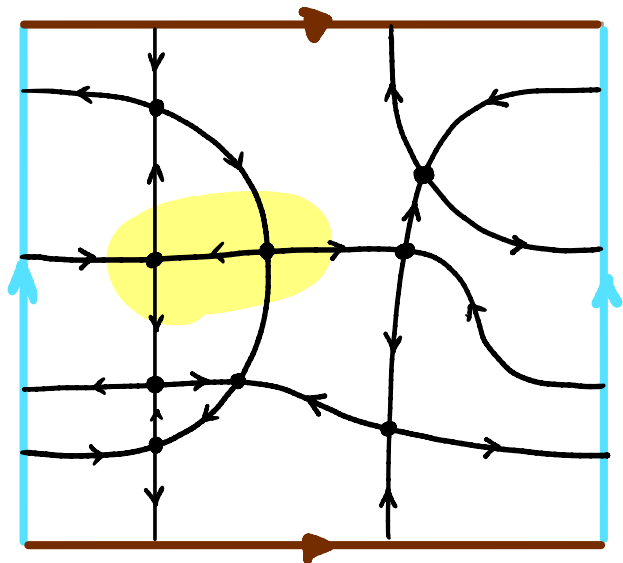
$\vec{E}' := \{ \text{backwards edges} \}$ on D'

A GOOD EXAMPLE:

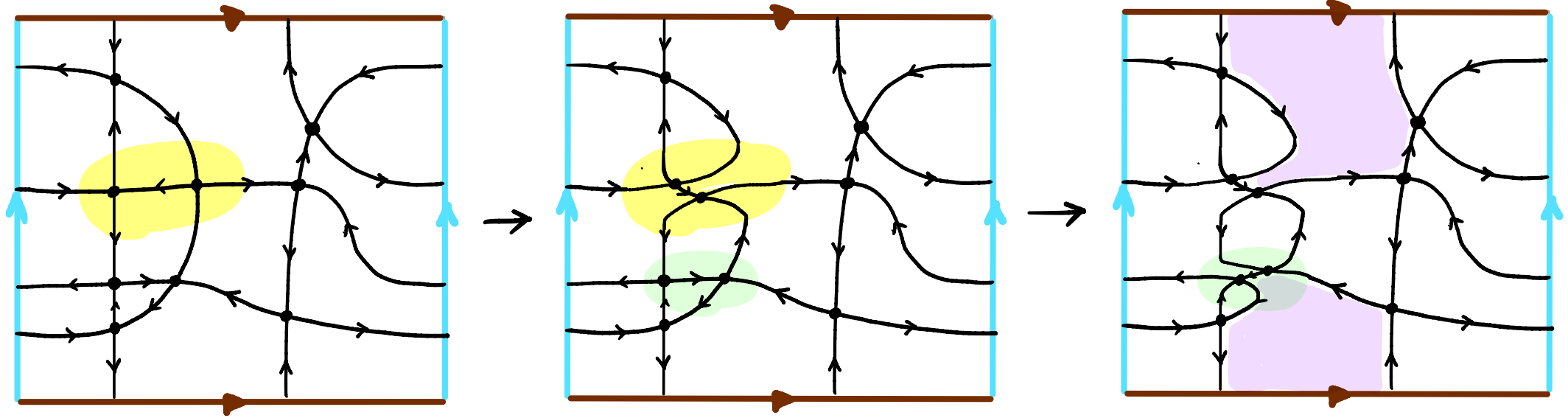


construct a feedback arc set on D too
& use it to get a cyclic order

A BAD EXAMPLE:

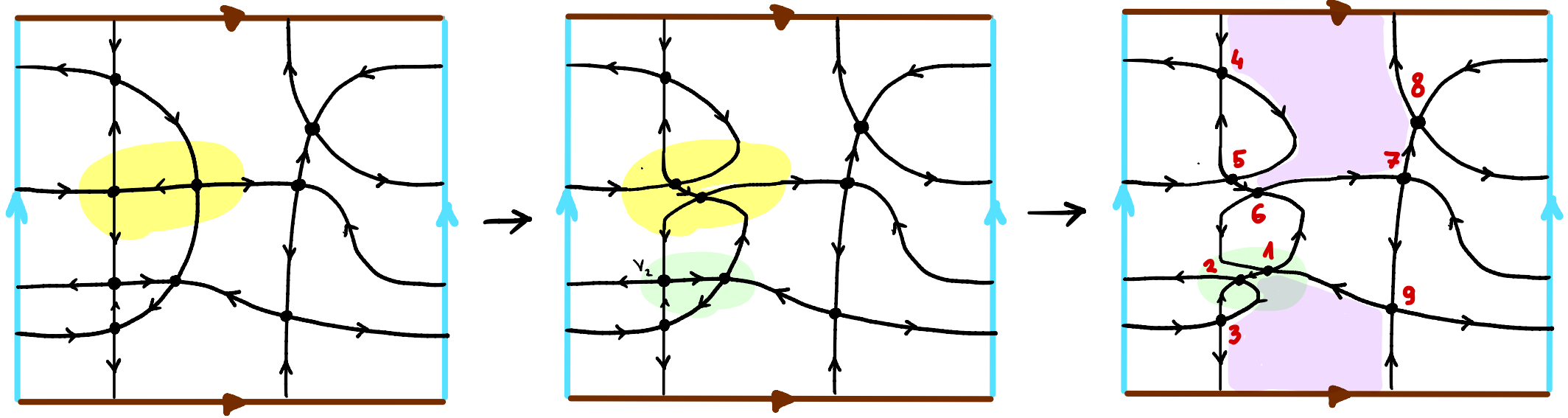


A BAD EXAMPLE:



do local changes to get a "big circle"

A BAD EXAMPLE:

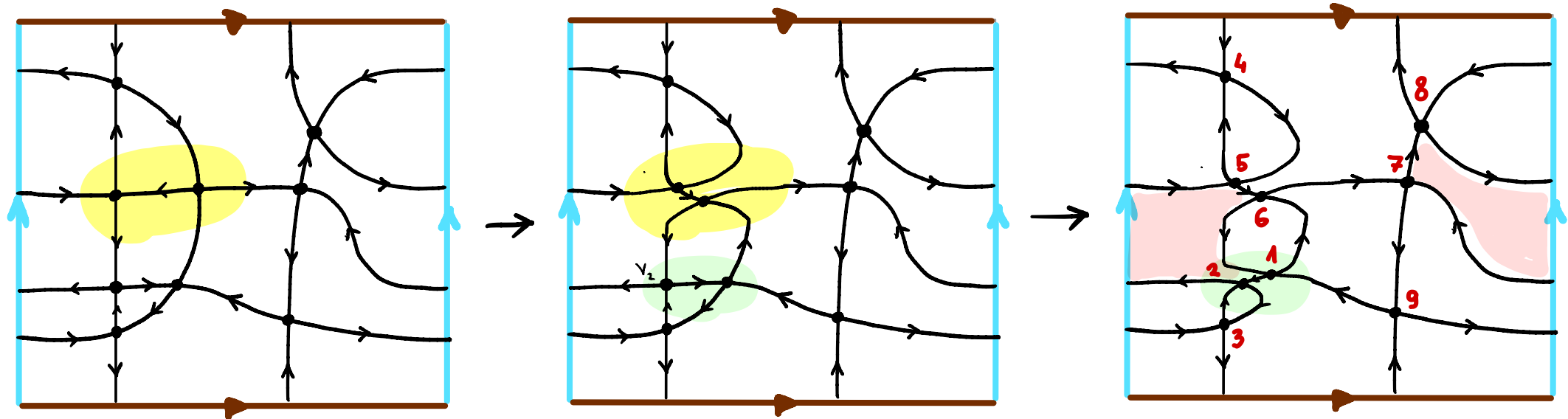


do local changes to get a "big circle"

the "big circle" gives a potential cyclic order

check if it works or not

A BAD EXAMPLE:



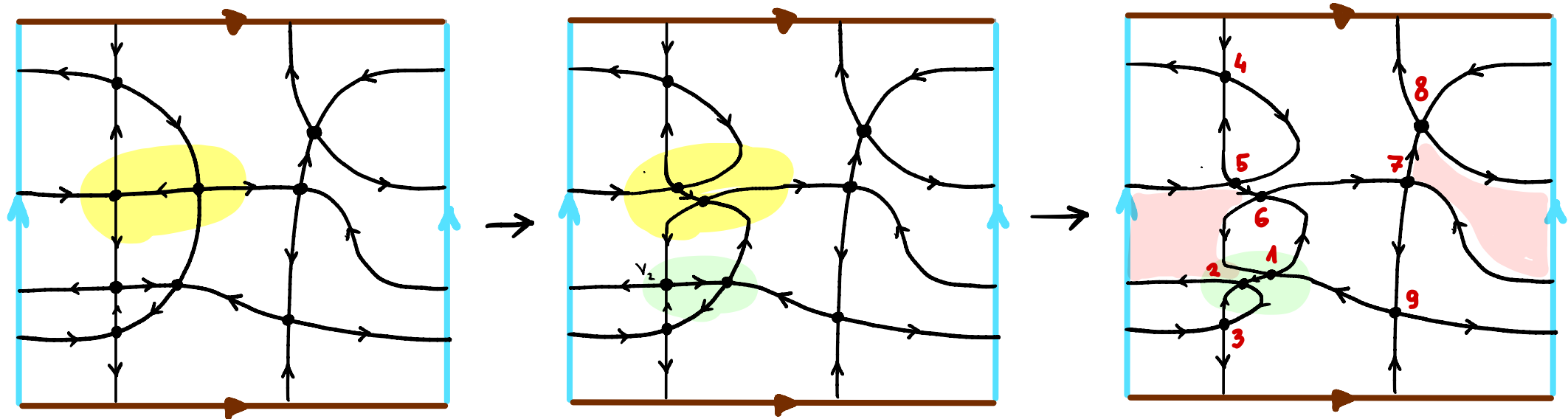
do local changes to get a "big circle"

the "big circle" gives a potential cyclic order

check if it works or not

! it doesn't work !

A BAD EXAMPLE:



do local changes to get a "big circle"

the "big circle" gives a potential cyclic order

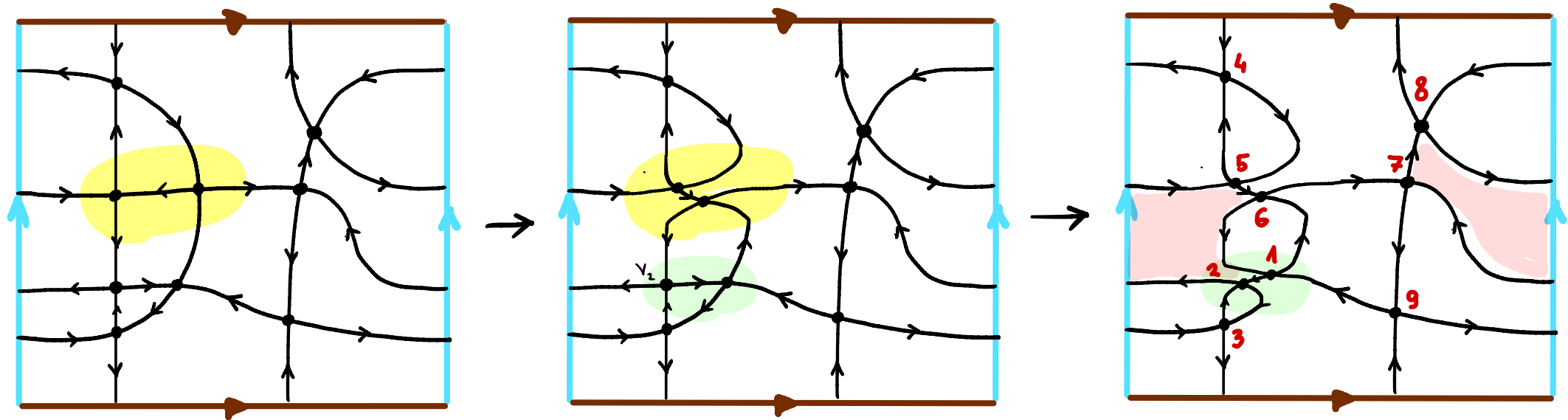
check if it works or not

! it doesn't work !

LEMMA

$\Rightarrow$ NO feedback arc set of size $\frac{\# \text{ faces}}{2}$ for D

A BAD EXAMPLE:



do local changes to get a "big circle"

the "big circle" gives a potential cyclic order

check if it works or not

! it doesn't work !

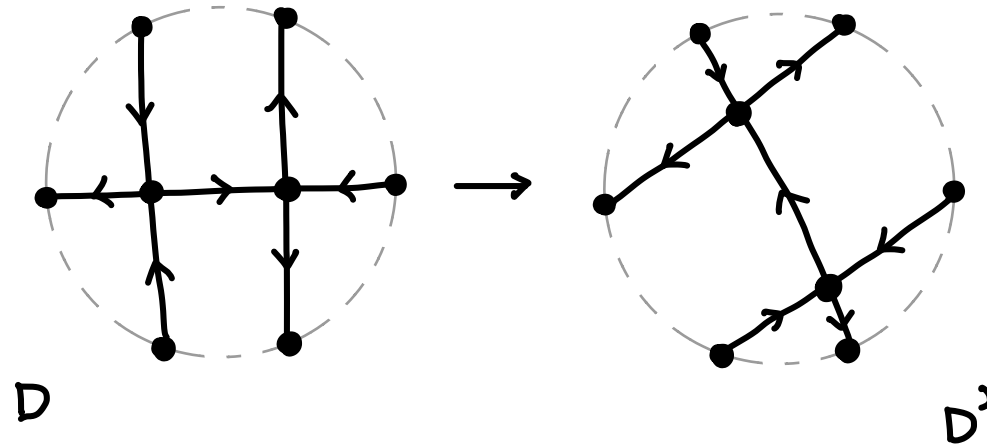
LEMMA

$\Rightarrow$ NO feedback arc set of size $\frac{\# \text{ faces}}{2}$ for D

~~STRATEGY~~

D is an INPUT of the problem

local change:



LEMMA: Assume D' is also an "INPUT". Then

D has a feedback arc set of size $\frac{\# \text{ faces}}{2}$



D' has a feedback arc set of size $\frac{\# \text{ faces}}{2}$

BUT!: we cannot always change D to have a big face ...

Kiss - V: still works for low genus (≤ 3 , possibly more)

FURTHER DIRECTIONS:

GIVEN: 2 Eulerian & 4-regular digraph $D_0(V_0, \vec{E}_0)$ & $D_1(V_1, \vec{E}_1)$

& 2-cell embeddings $D_0(V_0, \vec{E}_0) \hookrightarrow \Sigma$
 $D_1(V_1, \vec{E}_1) \nearrow$

w/ the usual properties

(for each face F
→ the vertices on ∂F are all distinct
→ the edges on ∂F form an oriented circle)

QUESTION: Elementary moves

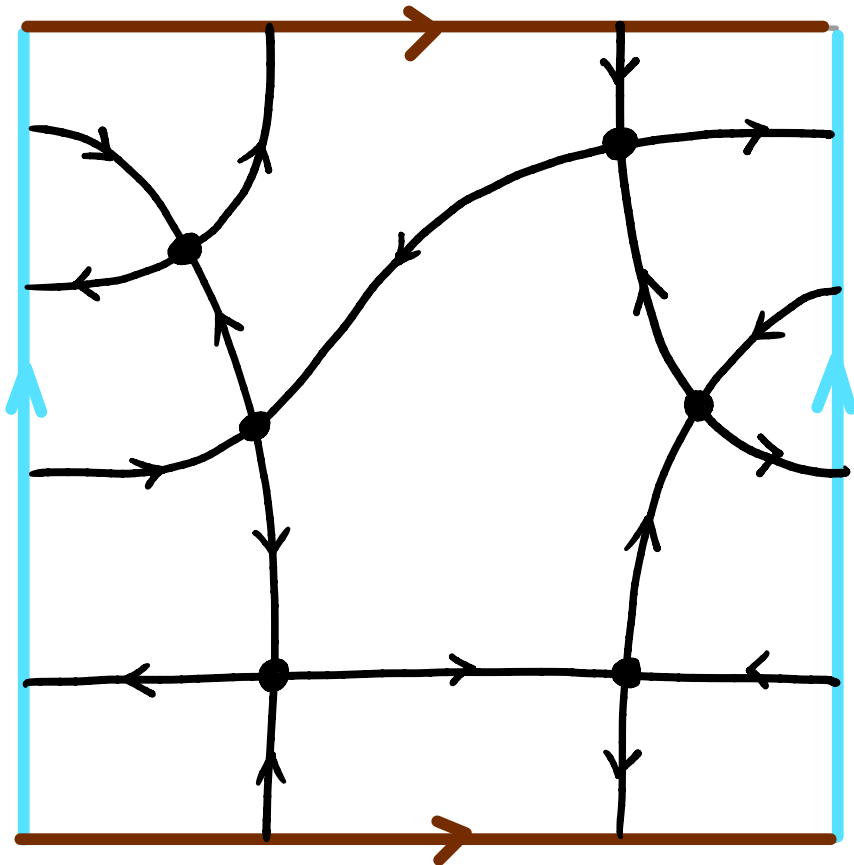
ANY NEW IDEAS ?

INPUT: an Eulerian & 4-regular digraph $D(V, \vec{E})$

& a 2-cell embedding $D(V, \vec{E}) \hookrightarrow \Sigma$ s.t. for each face F

→ the vertices on ∂F are all distinct

→ the edges on ∂F form an oriented circle



QUESTION: Is there a feedback arc set
of size $\frac{\# \text{ faces}}{2}$?

THANKS FOR
YOUR
ATTENTION!