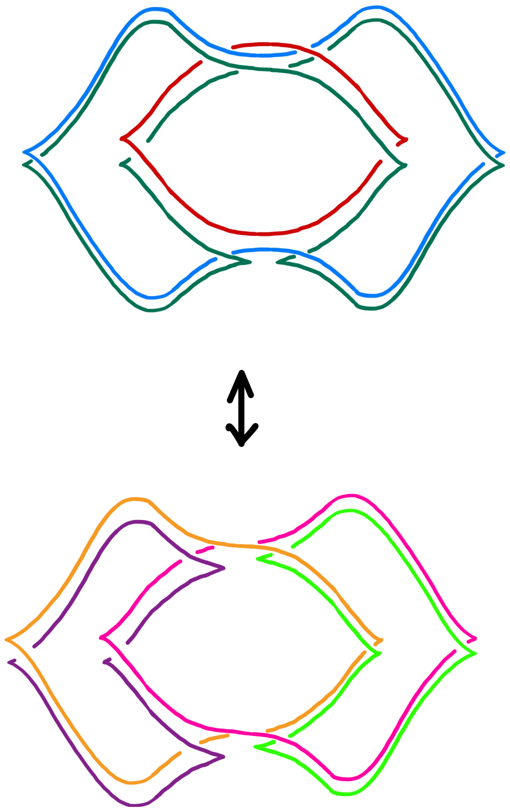
KIRBY'S THEOREM FOR CONTACT 3-MANIFOLDS



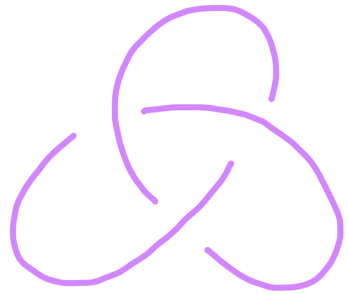
VERA VÉRTESI

JOINT WORK W/

M. KEGEL, E. STENHEDE & D. ZUDDAS

KIRBY'S THM FOR SMOOTH 3-MANIFOLDS

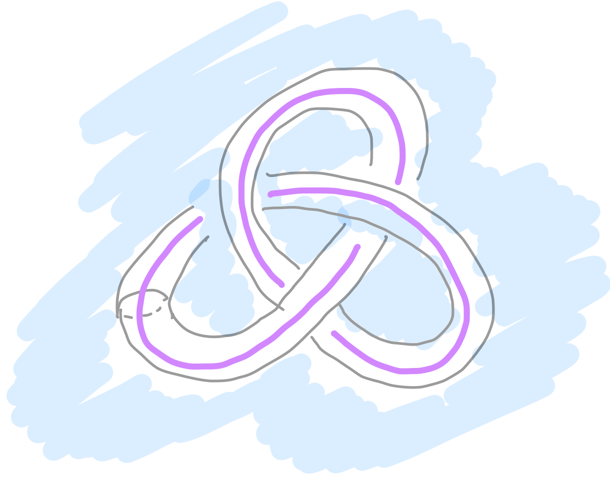
- **SURGERY ON KNOTS:**
  - GIVEN $K \hookrightarrow S^3$



KIRBY'S THM FOR SMOOTH 3-MANIFOLDS

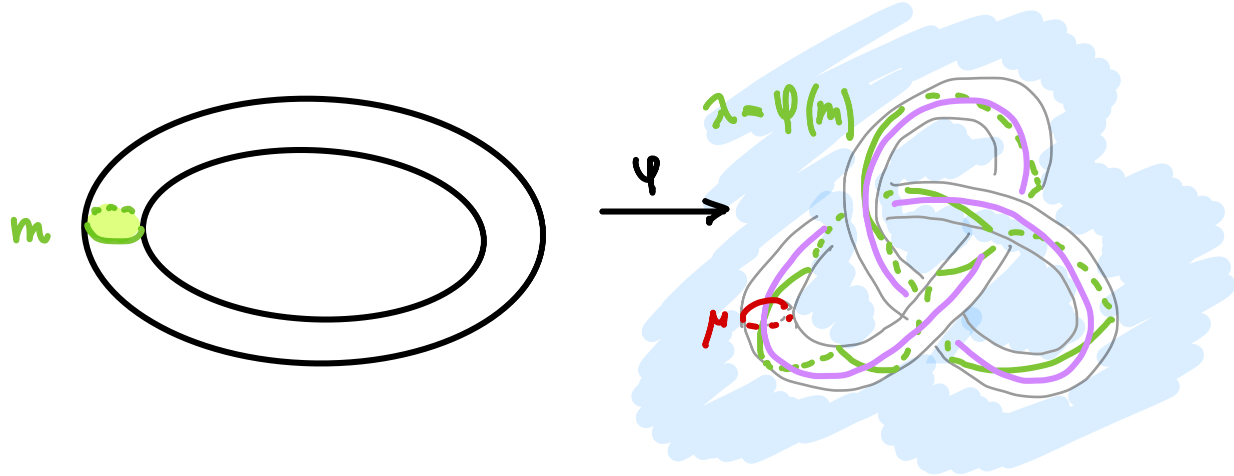
- **SURGERY ON KNOTS:**

  - GIVEN $K \hookrightarrow S^3$
  - $X := S^3 - N(K)$



Kirby's Thm for Smooth 3-Manifolds

- **<u>Surgery on knots:</u>**
  - Given $K \hookrightarrow S^3$
  - $X := S^3 - N(K)$
  - Glue a $D^2 \times S^1$ differently along $\varphi : \partial(D^2 \times S^1) \longrightarrow \partial X$



$\rightsquigarrow \quad M = S^3 - N(K) \cup_\varphi (D^2 \times S^1) = S^3_\lambda(K)$

M is determined by $\lambda = \varphi(m)$: a <u>framing</u> of K

We often compare λ to the <u>Seifert framing</u> ℓ of K

$$\lambda = \ell + n\mu \quad \rightsquigarrow \quad S^3_n(K) = S^3_\lambda(K)$$

Kirby's Thm for Smooth 3-manifolds

• **Surgery on knots:**

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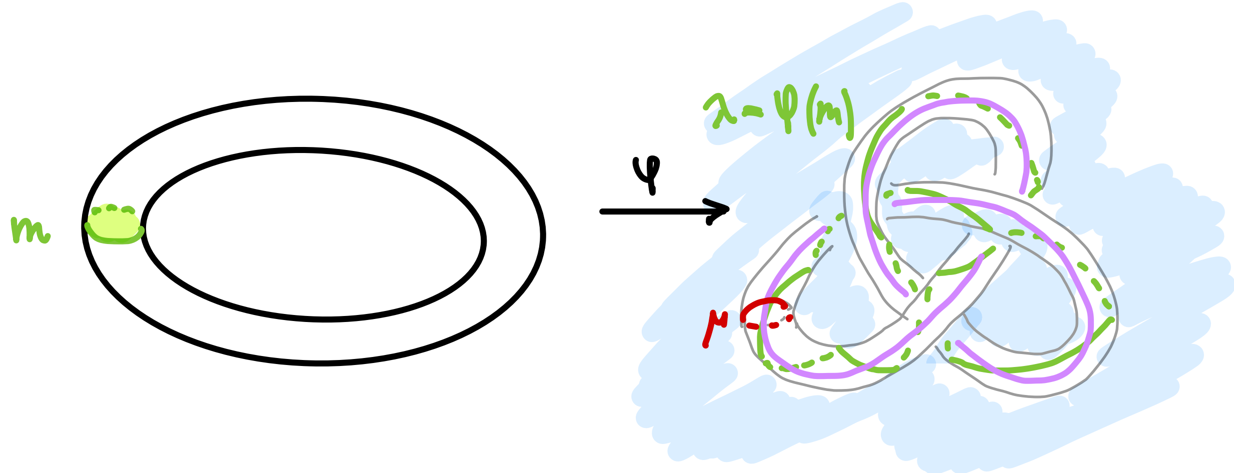

$$\rightsquigarrow \quad M = S^3 - N(K) \cup_\varphi (D^2 \times S^1) = S^3_\lambda(K)$$

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We often compare λ to the **Seifert framing** ℓ of K

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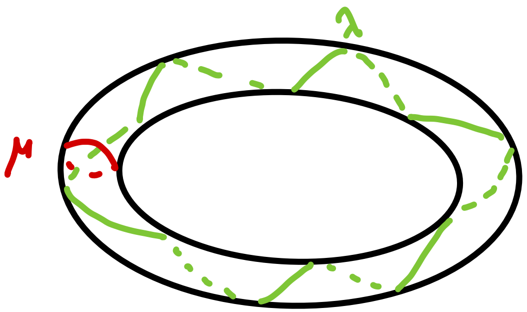
E.g.: • $S^3_p(\mathcal{U}) = L(1,-p)$



• $S^3_{\pm}(\mathcal{U}) = S^3$

KIRBY'S THM FOR SMOOTH 3-MANIFOLDS

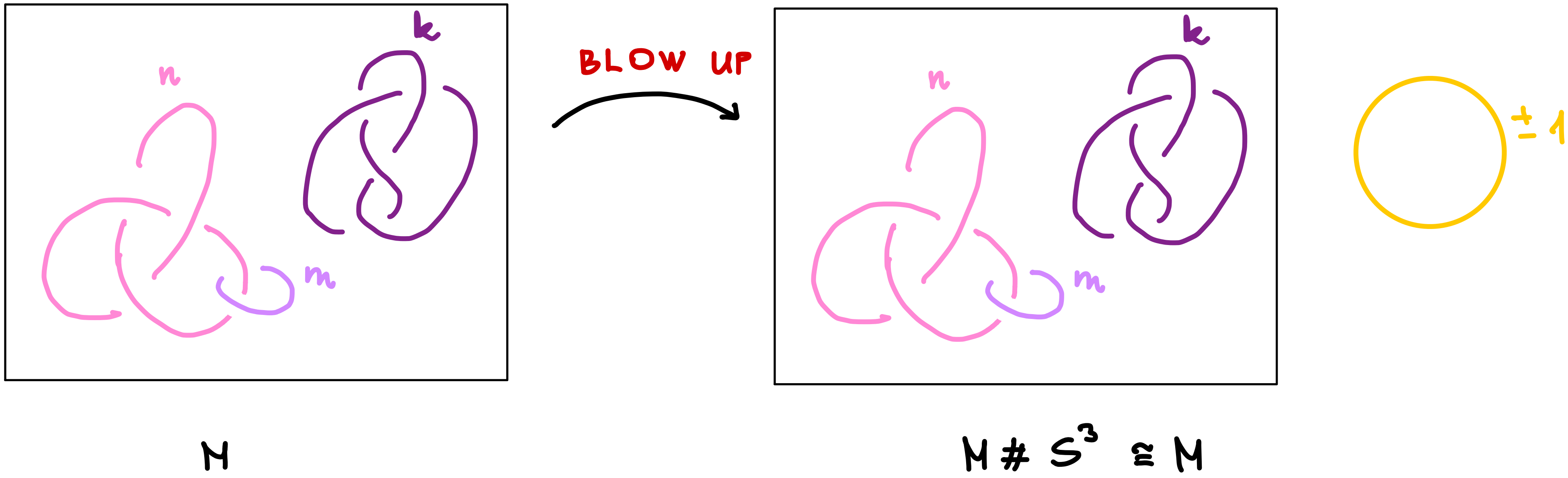
WE CAN DO SURGERY ON FRAMED LINKS $\mathbb{L} \hookrightarrow S^3$

IN FACT ANY 3-MFD CAN BE OBTAINED THIS WAY:

THM (LICKORISH '62, WALLACE '60): ANY CLOSED, CONNECTED, ORIENTED, SMOOTH 3-MANIFOLD M^3 IS OBTAINED VIA SURGERY ALONG A FRAMED LINK $\mathbb{L} \subset S^3$

KIRBY'S THM FOR SMOOTH 3-MANIFOLDS

HOW MANY DIFFERENT WAYS CAN WE GET THE SAME M ?



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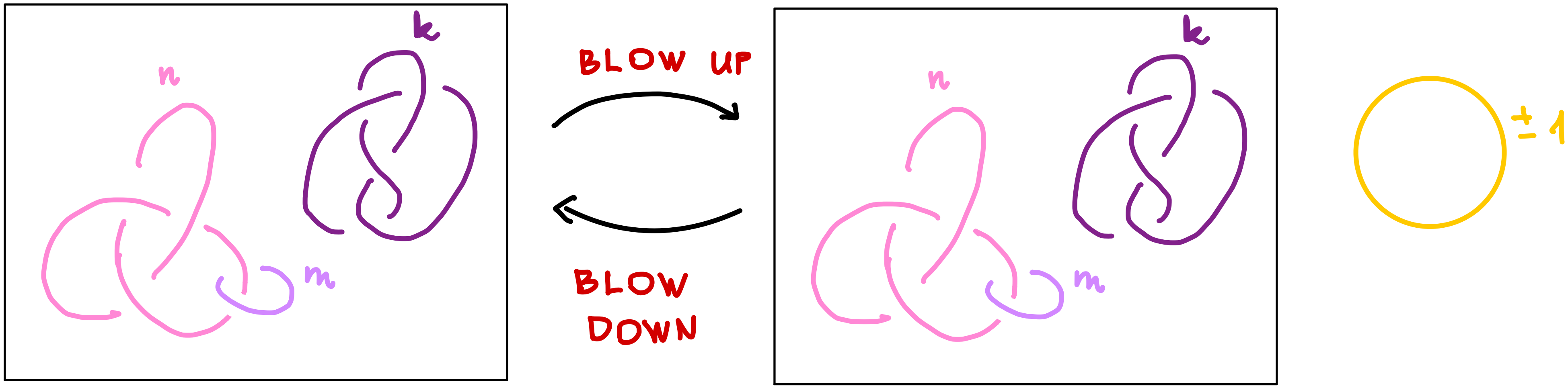

M

$M \# S^3 \cong M$

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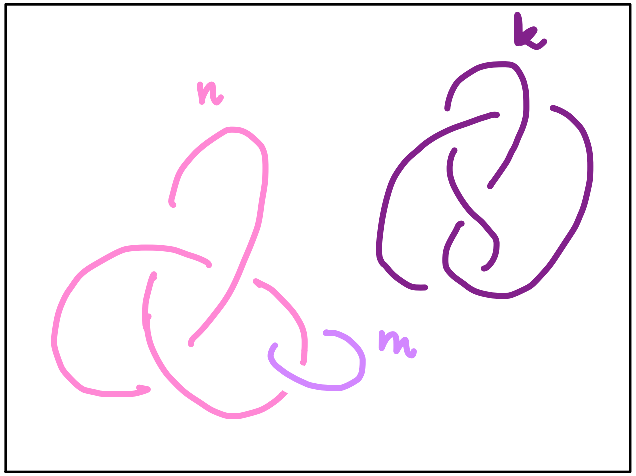
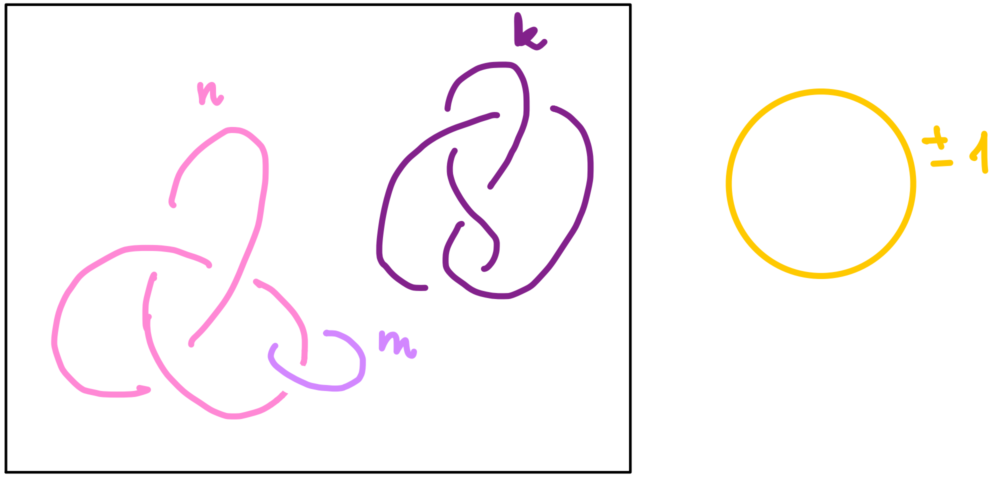


BLOW UP

BLOW DOWN



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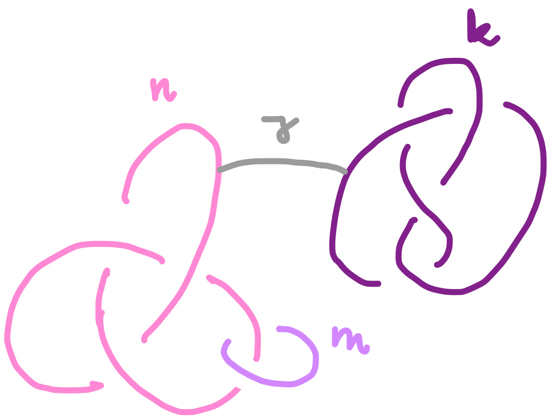

$\xrightarrow[\text{SLIDE}]{\text{HANDLE-}}$

KIRBY'S THM FOR SMOOTH 3-MANIFOLDS

THM (KIRBY '78): SURGERY ALONG THE FRAMED LINKS $\mathbb{L}$ & $\mathbb{L}' \subset S^3$ GIVE DIFFEOMORPHIC MANIFOLDS IFF $\mathbb{L}$ & $\mathbb{L}'$ ARE RELATED VIA

- BLOW UPS & BLOW DOWNS
- HANDLESLIDES

THESE MOVES ARE NOT LOCAL

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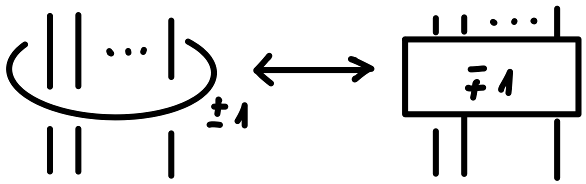

ARE ENOUGH.

STILL NOT FINITE

KIRBY'S THM FOR SMOOTH 3-MANIFOLDS

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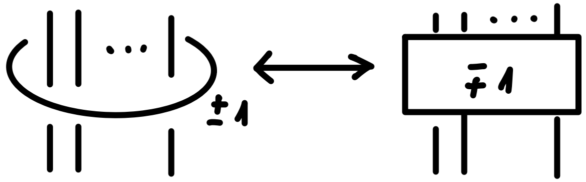
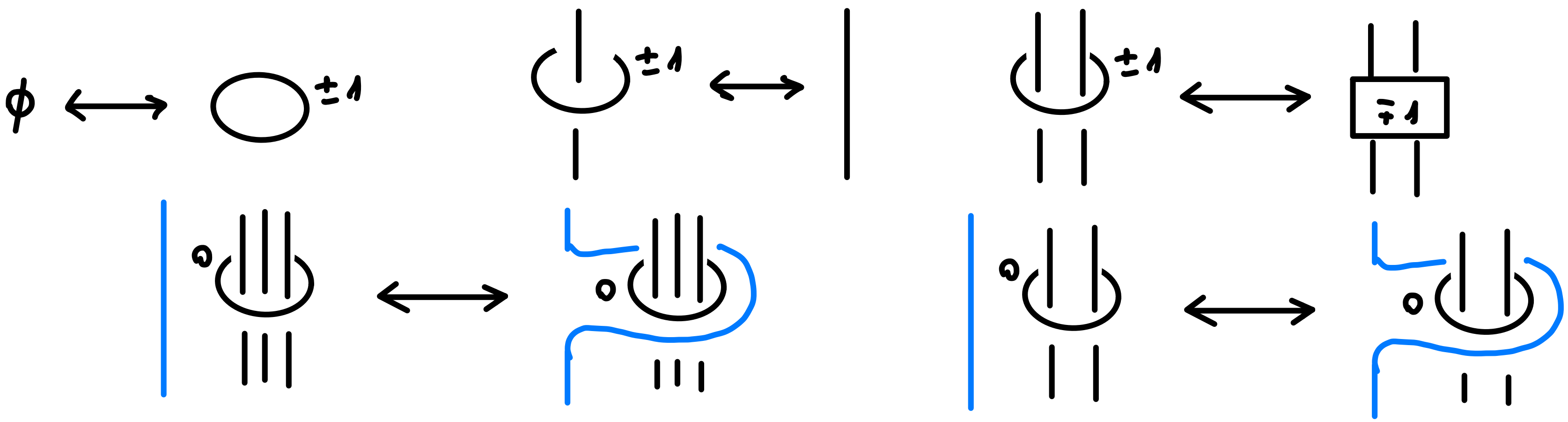


ARE ENOUGH.

STILL NOT FINITE

THM (MARTELLI '11): THE FOLLOWING MOVES SUFFICE:



APPLICATION OF KIRBY'S THEOREM

BASICALLY EVERYWHERE IN LOW DIMENSIONAL TOPOLOGY

(179 CITATIONS)

MOST IMPORTANT 3D APPLICATIONS

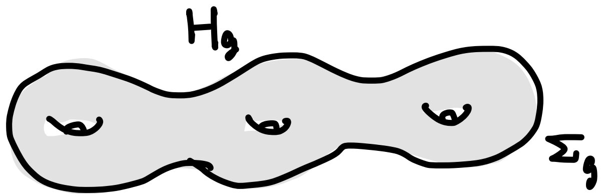
- CASSON '86: THE "CASSON INVARIANT" IS AN INTEGRAL LIFT
  OF THE ARF INVARIANT

- RESHETIKHIN - TURAEV '91: 3-MANIFOLD INVARIANT
  $\rightsquigarrow$ $\boxed{\text{TOPOLOGICAL QUANTUM FIELD THEORY}}$

THM (LICKORISH '62, WALLACE '60): ANY CLOSED, CONNECTED, ORIENTED, SMOOTH 3-MANIFOLD M^3 IS OBTAINED VIA SURGERY ALONG A FRAMED LINK $\mathbb{L} \subset S^3$

IDEA OF PROOF:

- **HEEGAARD SPLITTING:** $M = U \cup_{\Sigma} V$ HANDLEBODY



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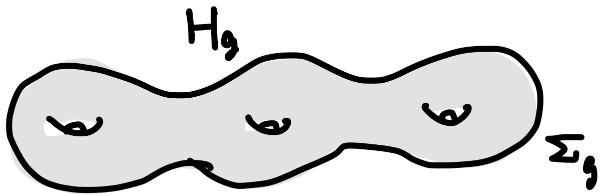

E.G.: $S^3 =$

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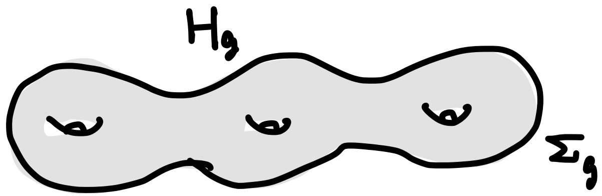

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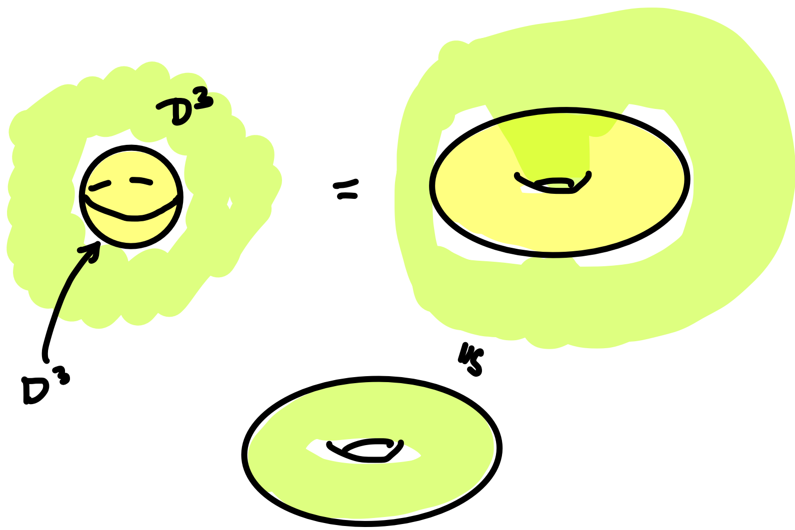

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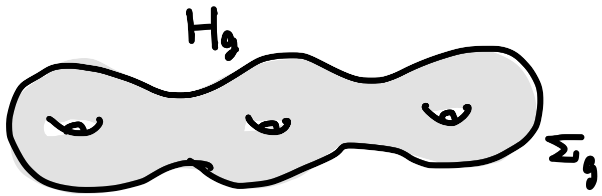

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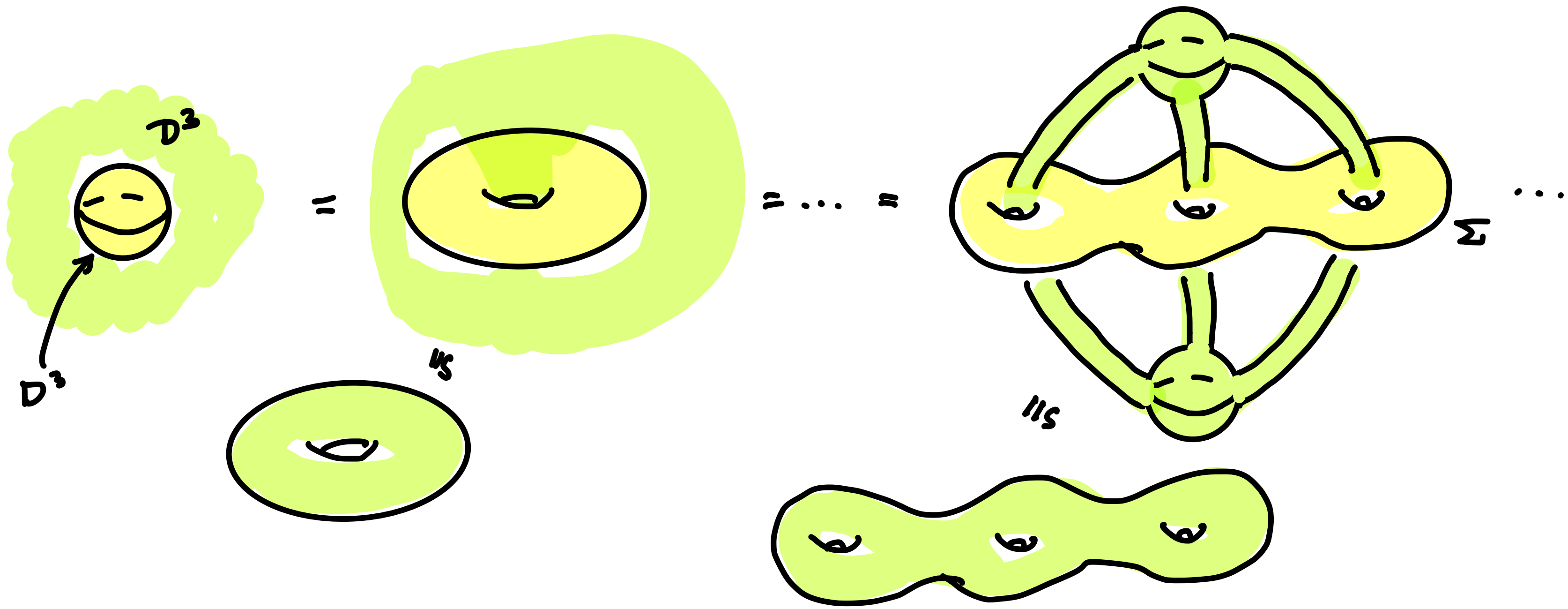

STANDARD HEEGAARD SPLITTING OF S^3 W/ GENUS g

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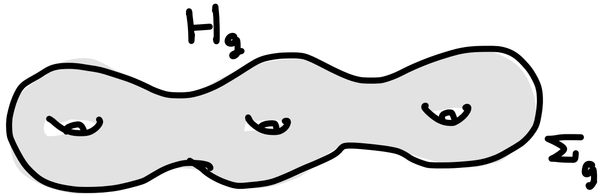

WE CAN THINK OF A HS AS $H_g \cup_\varphi H_g$ GLUED ALONG $\varphi: \Sigma_g \to \Sigma_g$

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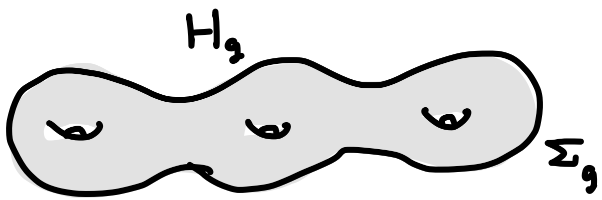
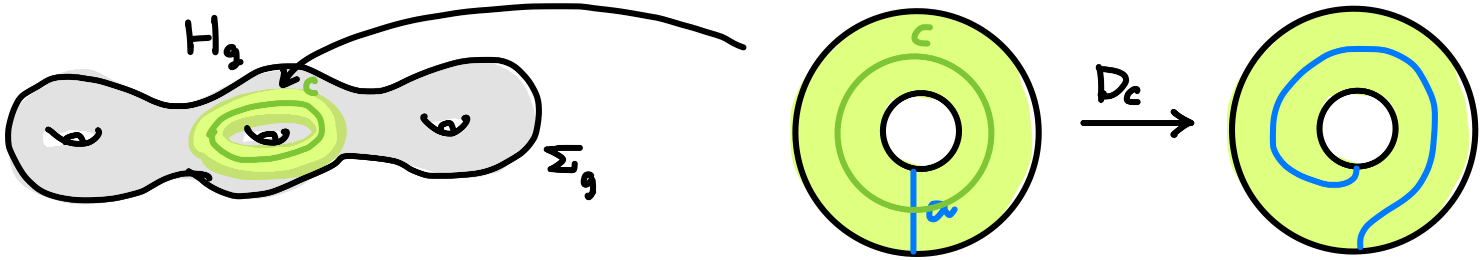


WE CAN THINK OF A HS AS $H_g \cup_{\varphi} H_g$ GLUED ALONG $\varphi : \Sigma_g \to \Sigma_g$

- **MAPPING CLASS GROUP:** $MCG(\Sigma_g) = \{\varphi : \Sigma_g \to \Sigma_g\} / \text{ISOTOPY}$

GENERATED BY **DEHN TWISTS** ALONG SIMPLE CLOSED CURVES



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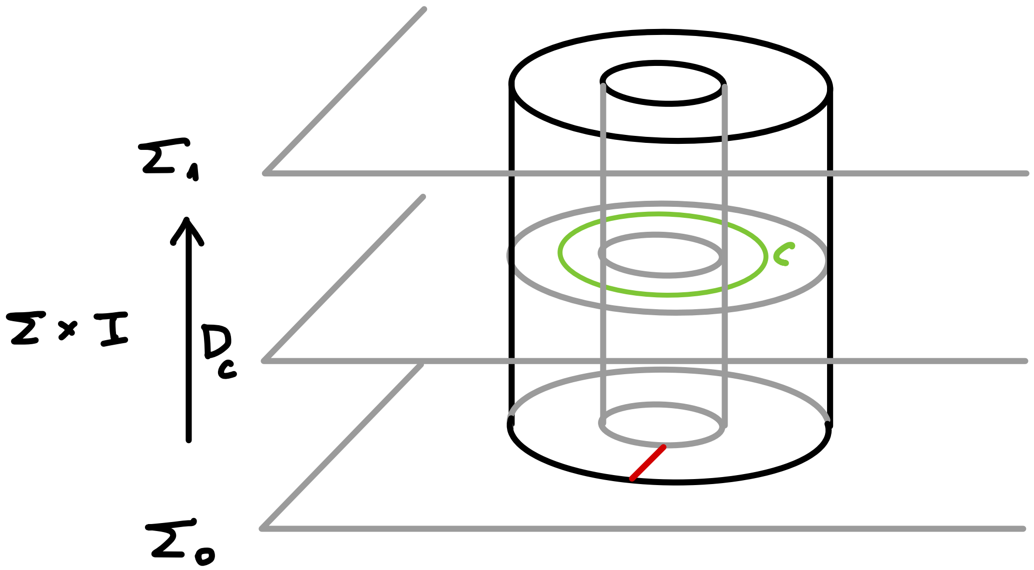
IDEA OF PROOF:

- NOTE: CONNECTION BTWN DEHN TWIST & SURGERY



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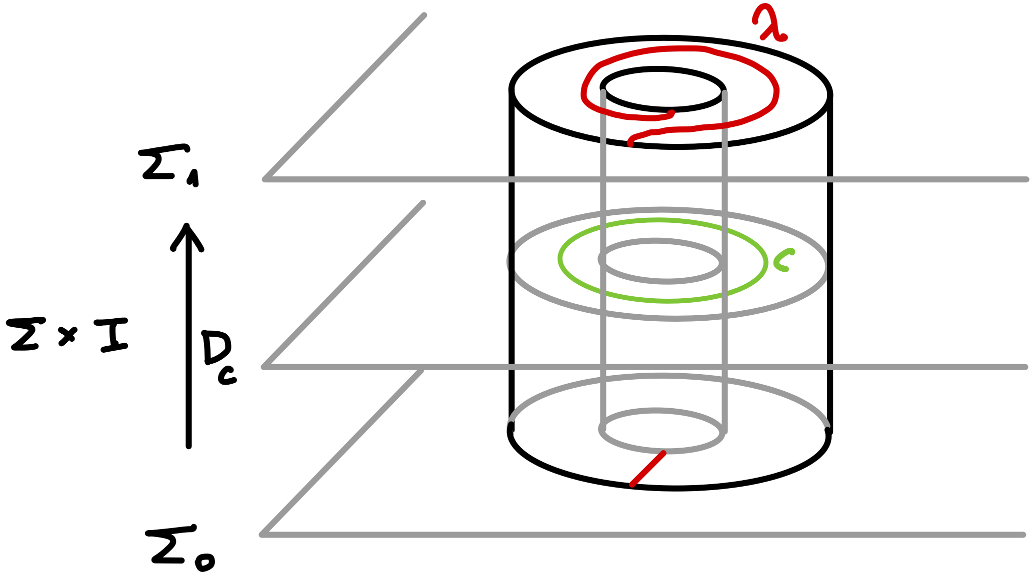

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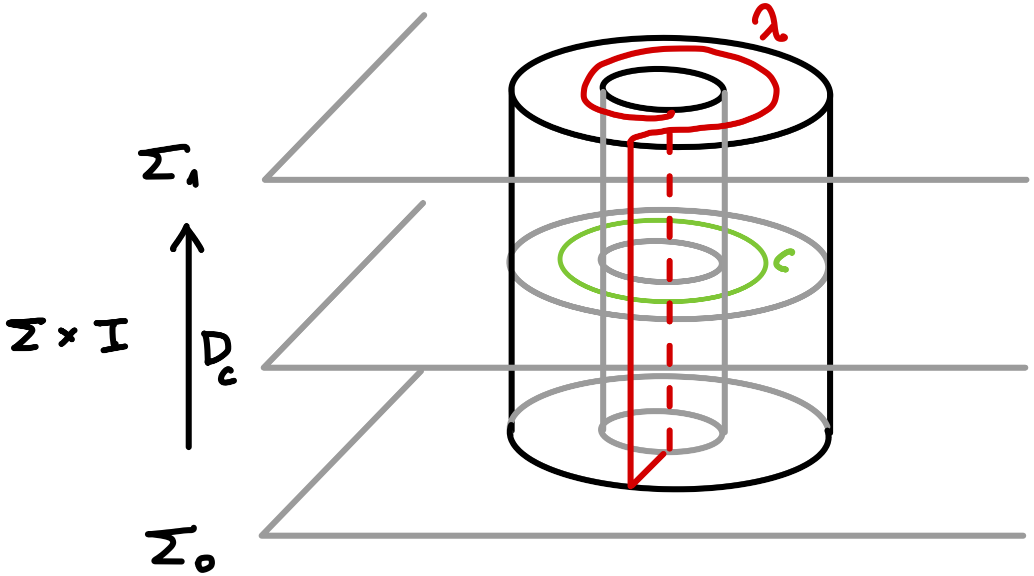

$\rightsquigarrow$ $D_c \longleftrightarrow$ SURGERY ALONG c W/ FRAMING λ

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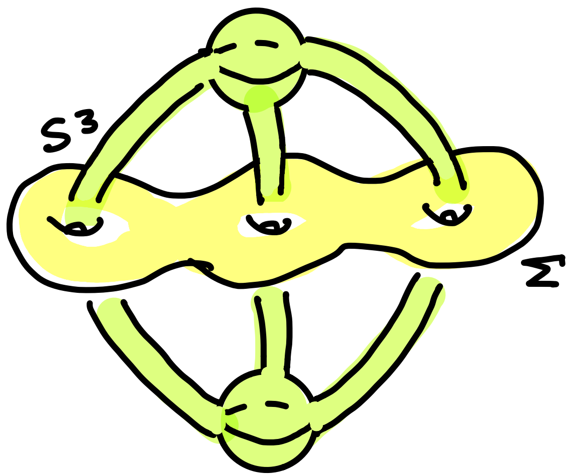
IDEA OF PROOF:

- TAKE A HS OF M: $M = H_g \cup_\varphi H_g$
- TAKE THE STANDARD GENUS g HS OF S^3



OR $S^3 = H_g \cup_{\varphi_0} H_g$

- DECOMPOSE $\varphi \circ \varphi_0^{-1} = D_{c_1} \circ \cdots \circ D_{c_n}$ AS PRODUCT OF DEHN TWISTS

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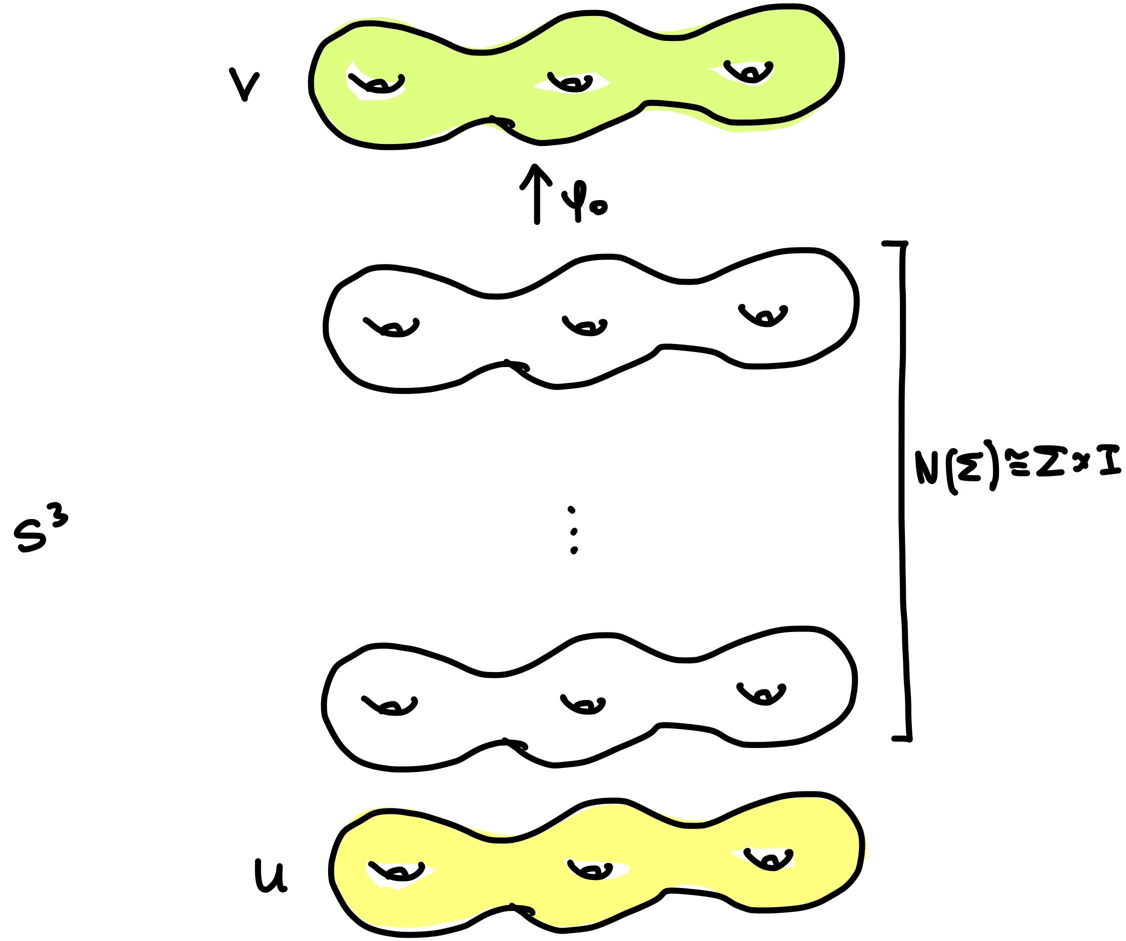
IDEA OF PROOF:

- $M = H_g \cup_\varphi H_g$

- $\varphi \circ \varphi_0^{-1} = D_{c_1} \circ \cdots \circ D_{c_n}$



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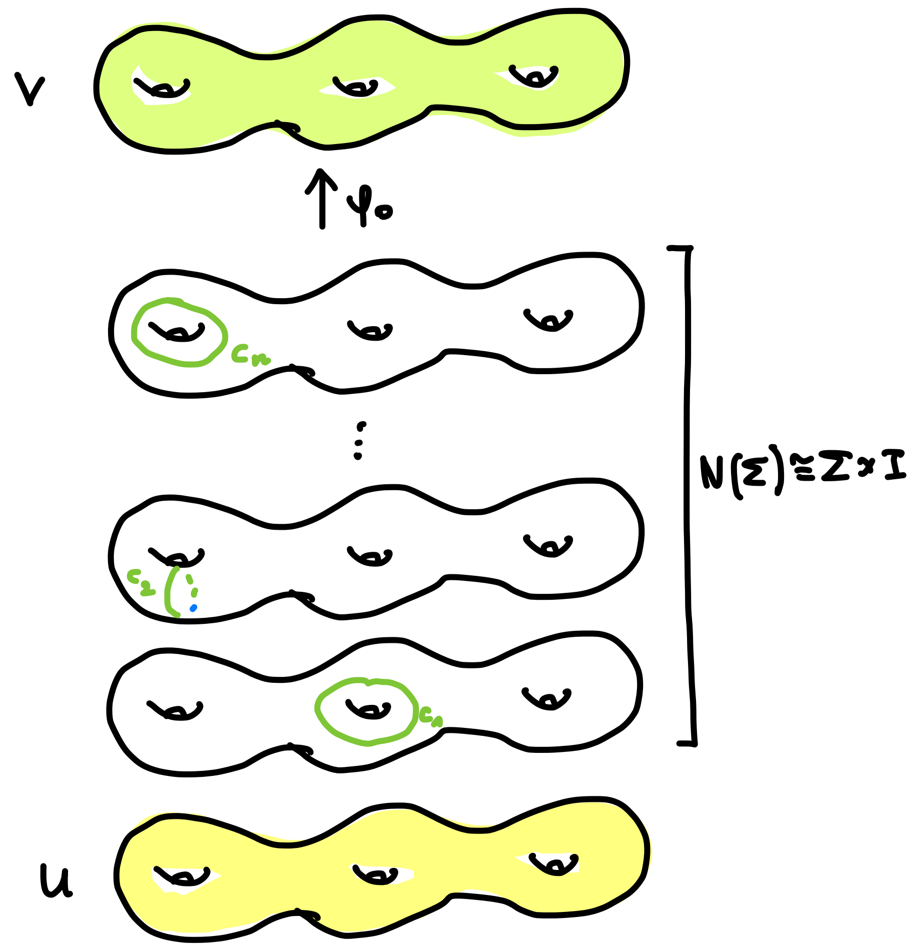
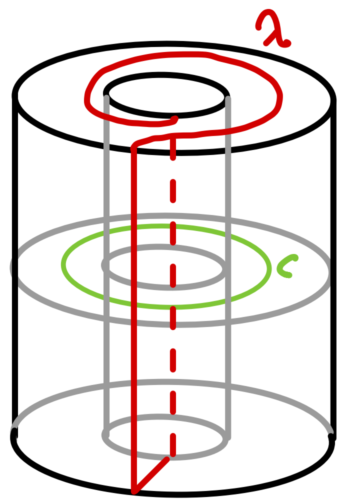


IDEA OF PROOF:

- $M = H_g \cup_\varphi H_g$
- $\varphi \circ \varphi_0^{-1} = D_{c_1} \circ \cdots \circ D_{c_n}$
- EMBED $c_i \hookrightarrow \Sigma \times \frac{i}{n} \subseteq N(\Sigma) \; : \; \mathbb{L}$

S^3

$\rightsquigarrow$ M IS OBTAINED VIA SURGERY ALONG THE FRAMED LINK $\mathbb{L}$



QED

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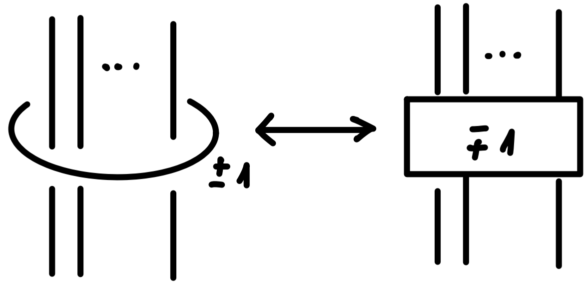
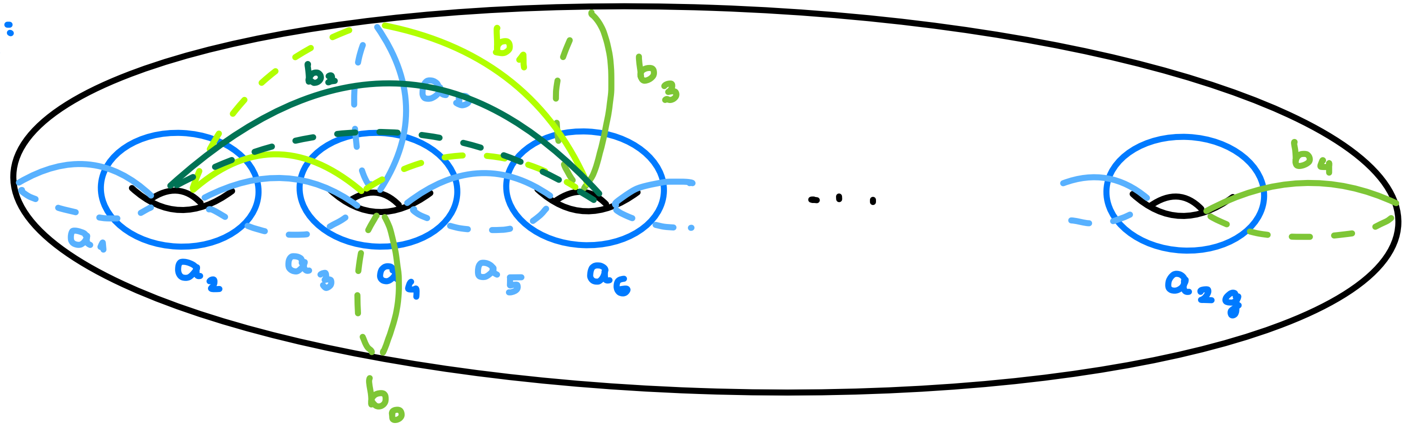


IDEA OF PROOF (N. LU '95)

- **PRESENTATION OF MCG(Σ)** (HATCHER-THURSTON-WAJNRYB '83)

GENERATORS:



RELATIONS:

- $a_i a_j = a_j a_i$ IF $a_i \cap a_j = \phi$
- $a_i a_j a_i = a_j a_i a_j$ IF $\# a_i \cap a_j = 1$
- $a_0 b_0 = (a_1 a_2 a_3)^4$
- $a_0 b_1 b_2 = a_1 a_3 a_5 b_3$
- $b_4 = w b_4 w^{-1}$
- $b_i = w' a_0 w'^{-1}$

w & w' SOME WORDS IN a_i

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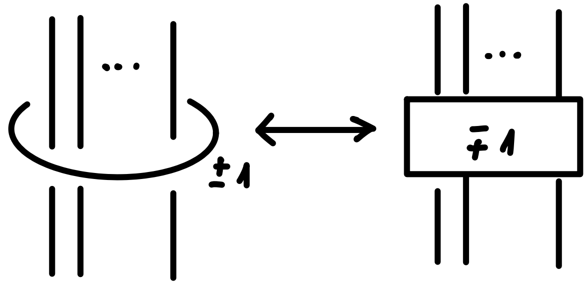
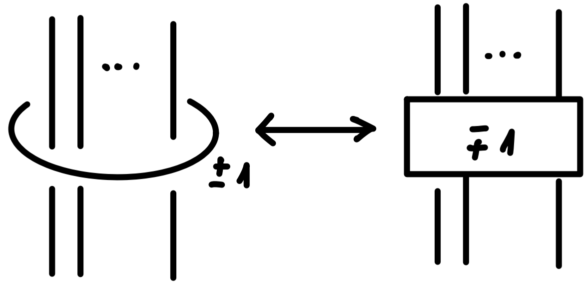


IDEA OF PROOF (N. LU '95)

- EMBED $\mathbb{L}$ & $\mathbb{L}'$ INTO A STANDARD HS OF S^3 AS IN THE PROOF OF LICKORISH-WALLACE (NEED TO BLOW-UP!)

THM (KIRBY '78): SURGERY ALONG THE FRAMED LINKS $\mathbb{L}$ & $\mathbb{L}' \subset S^3$ GIVE DIFFEOMORPHIC MANIFOLDS IFF $\mathbb{L}$ & $\mathbb{L}'$ ARE RELATED VIA



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- EMBED $\mathbb{L}$ & $\mathbb{L}'$ INTO A STANDARD HS OF S^3 AS IN THE PROOF OF LICKORISH-WALLACE (NEED TO BLOW-UP!)

- STABILISE THE HS'S TO HAVE THE SAME GENUS (BLOW-UP!)

$$\mathbb{L} \rightsquigarrow D_{c_1} \circ \cdots \circ D_{c_n}$$
$$\| \longleftarrow \text{IN } MCG(\Sigma)$$
$$\mathbb{L}' \rightsquigarrow D_{c_1'} \circ \cdots \circ D_{c_m'}$$

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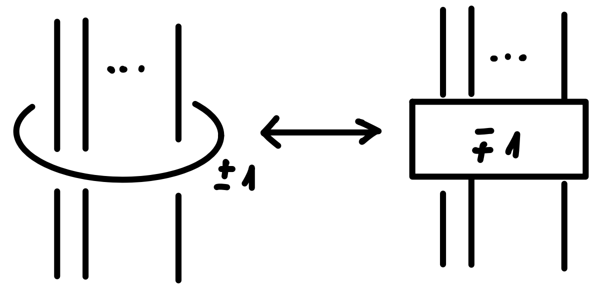

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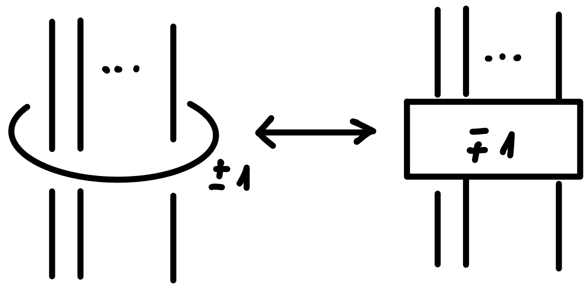
$$\mathbb{L}' \rightsquigarrow D_{c_1'} \circ \cdots \circ D_{c_m'}$$

- EXCHANGE THE DEHN TWISTS TO THE GENERATORS OF $MCG(\Sigma)$ (BLOW-UP!) $\Rightarrow$ RELATED BY THE RELATIONS

THM (KIRBY '78): SURGERY ALONG THE FRAMED LINKS $\mathbb{L}$ & $\mathbb{L}' \subset S^3$ GIVE DIFFEOMORPHIC MANIFOLDS IFF $\mathbb{L}$ & $\mathbb{L}'$ ARE RELATED VIA



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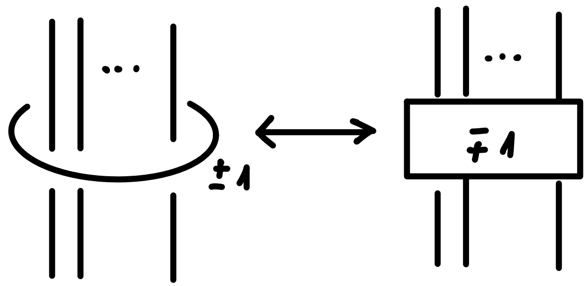
- STABILISE THE HS'S TO HAVE THE SAME GENUS (BLOW-UP!)

$$\begin{array}{l} \mathbb{L} \rightsquigarrow D_{c_1} \circ \cdots \circ D_{c_n} \\ \qquad\qquad \| \longleftarrow \text{IN } MCG(\Sigma) \\ \mathbb{L}' \rightsquigarrow D_{c_1'} \circ \cdots \circ D_{c_m'} \end{array}$$

- EXCHANGE THE DEHN TWISTS TO THE GENERATORS OF $MCG(\Sigma)$ (BLOW-UP!) $\Rightarrow$ RELATED BY THE RELATIONS
- EACH RELATION CORRESPONDS TO BLOW-UPS

THM (KIRBY '78): SURGERY ALONG THE FRAMED LINKS $\mathbb{L}$ & $\mathbb{L}' \subset S^3$ GIVE DIFFEOMORPHIC MANIFOLDS IFF $\mathbb{L}$ & $\mathbb{L}'$ ARE RELATED VIA

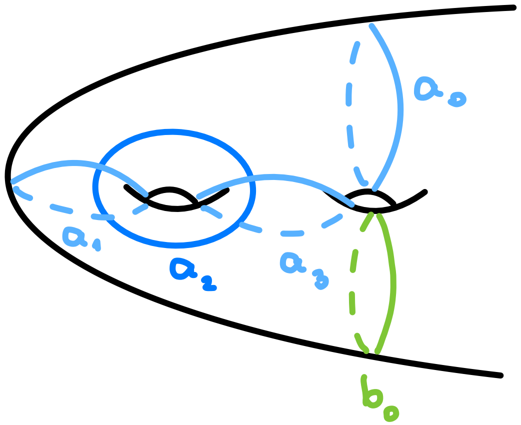


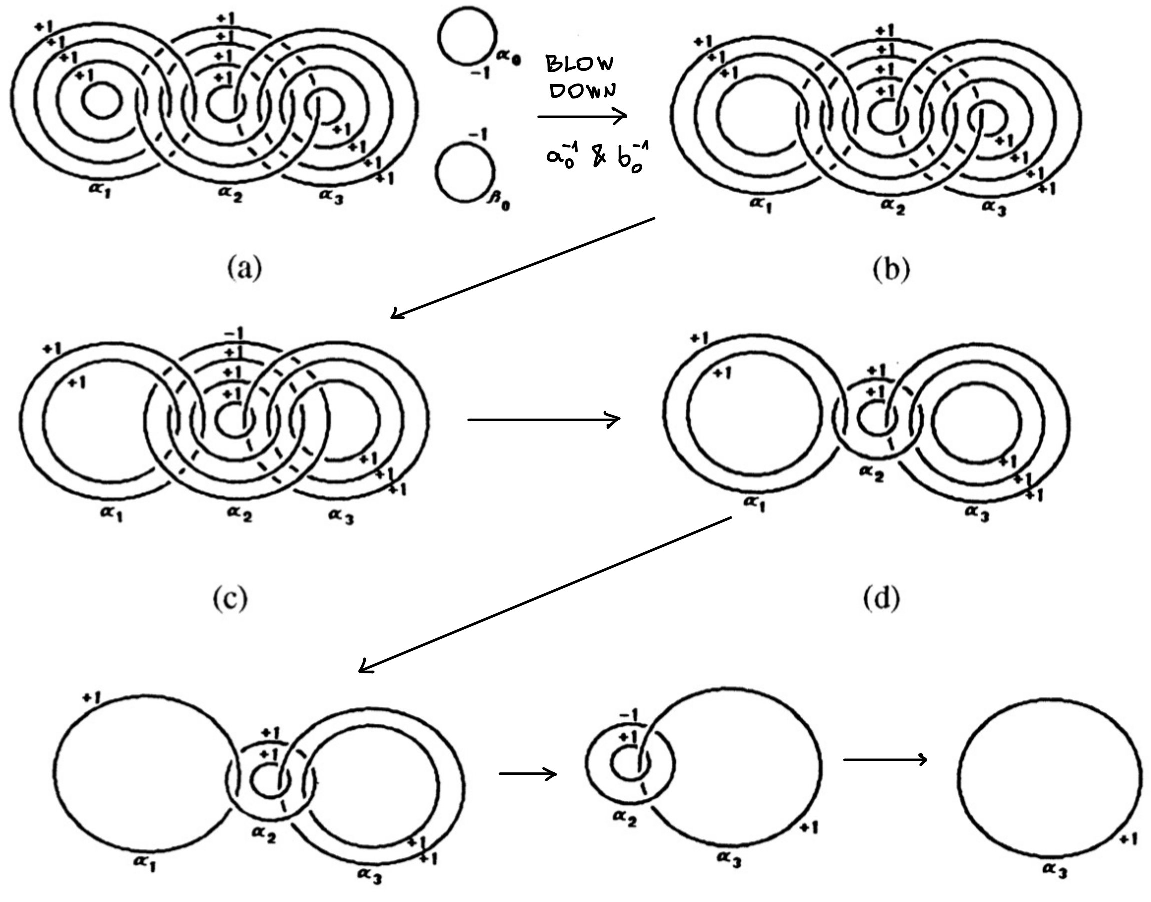
IDEA OF PROOF (N. LU '95)

EACH RELATION CORRESPONDS TO BLOW-UPS

E.G.: $\boxed{a_0 b_0 = (a_1 a_2 a_3)^4}$

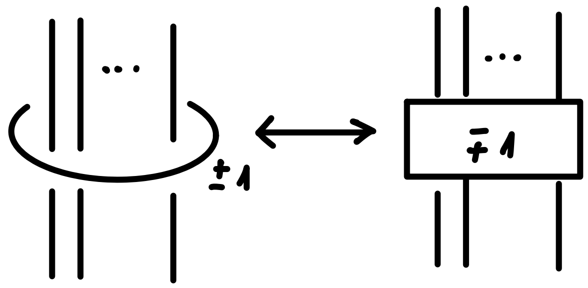
WHERE





(a) (b) (c) (d) (e) (f) (g)

THM (KIRBY '78): SURGERY ALONG THE FRAMED LINKS $\mathbb{L}$ & $\mathbb{L}' \subset S^3$ GIVE DIFFEOMORPHIC MANIFOLDS IFF $\mathbb{L}$ & $\mathbb{L}'$ ARE RELATED VIA



IDEA OF PROOF (N. LU '95)

- EMBED $\mathbb{L}$ & $\mathbb{L}'$ INTO A STANDARD HS OF S^3 AS IN THE PROOF OF LICKORISH-WALLACE (NEED TO BLOW-UP!)

- STABILISE THE HS'S TO HAVE THE SAME GENUS (BLOW-UP!)

$$\begin{array}{l} \mathbb{L} \rightsquigarrow D_{c_1} \circ \cdots \circ D_{c_n} \\ \qquad\qquad \| \longleftarrow \text{IN } MCG(\Sigma) \\ \mathbb{L}' \rightsquigarrow D_{c_1'} \circ \cdots \circ D_{c_m'} \end{array}$$

- EXCHANGE THE DEHN TWISTS TO THE GENERATORS OF $MCG(\Sigma)$ (BLOW-UP!) $\Rightarrow$ RELATED BY THE RELATIONS

- EACH RELATION CORRESPONDS TO BLOW-UPS ✓

QED

GOAL: GENERALISE THESE FOR CONTACT STRUCTURES

CONTACT STRUCTURES

DEF: A **CONTACT STRUCTURE** ON A CLOSED, ORIENTED SMOOTH 3-MANIFOLD M^3 IS A TOTALLY NONINTEGRABLE 2-PLANE-DISTRIBUTION $\xi \subset TM$

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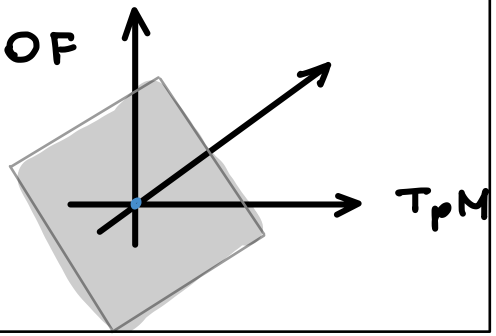
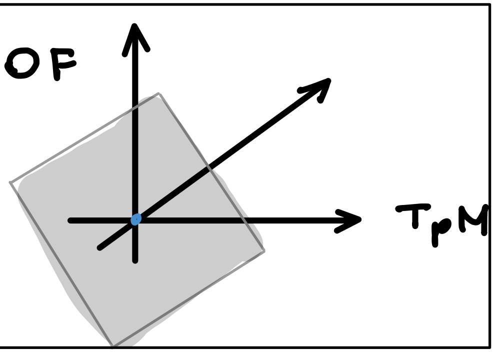
SMOOTH CHOICE OF $\xi_p^2 \subset T_pM \quad \forall p$



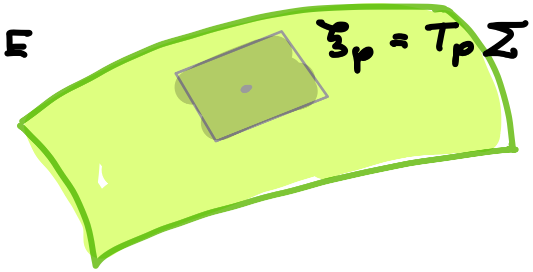
Contact Structures

DEF: A **contact structure** on a closed, oriented smooth 3-manifold M^3 is a totally nonintegrable 2-plane-distribution $\xi \subset TM$

Smooth choice of $\xi_p^2 \subset T_pM$ $\forall p$



Not tangent to any surface $\Sigma \subseteq M$



$\Updownarrow$ Frobenius

$\alpha \wedge d\alpha > 0$

Contact Structures

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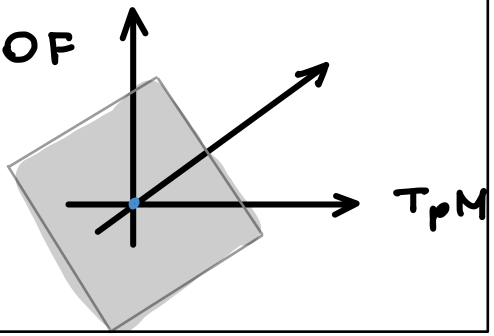
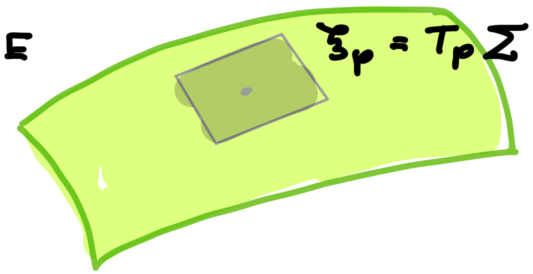


NOT TANGENT TO ANY SURFACE $\Sigma \subseteq M$



$\Updownarrow$ FROBENIUS

$\alpha \wedge d\alpha > 0$

LOCALLY: $\xi = \ker \alpha \qquad \alpha \in \Omega^1(M)$

COORIENTED CONTACT STRUCTURE: GLOBAL α

Contact Structures

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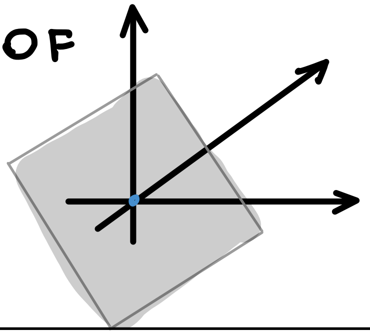
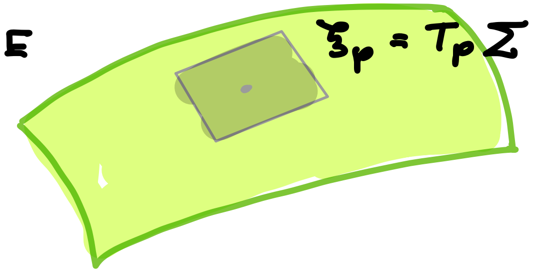


NOT TANGENT TO ANY SURFACE $\Sigma \subseteq M$



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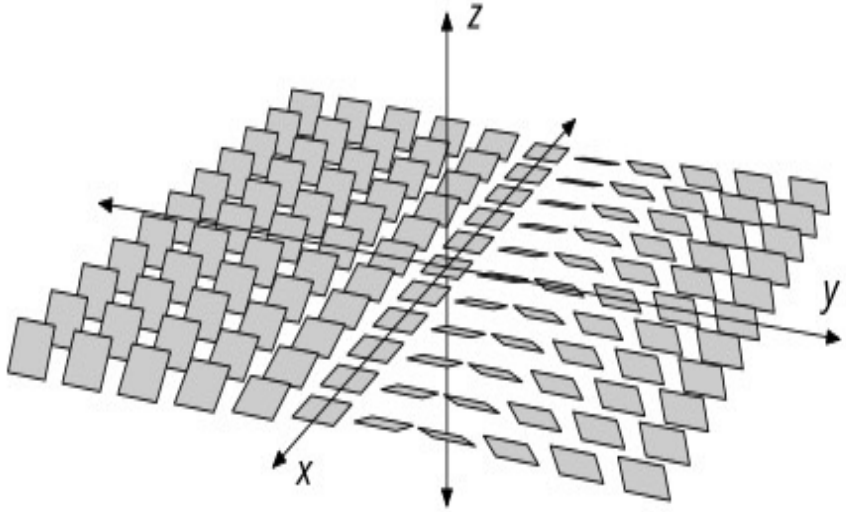
$\alpha \wedge d\alpha > 0$

LOCALLY: $\xi = \ker \alpha \qquad \alpha \in \Omega^1(M)$

COORIENTED CONTACT STRUCTURE: GLOBAL α

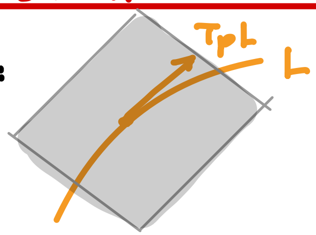
DARBOUX THM: LOCALLY ANY CONTACT STRUCTURE IS **CONTACTOMORPHIC** TO $\mathbb{R}^3$, $\xi_{st} = \ker(dz - y\,dx)$

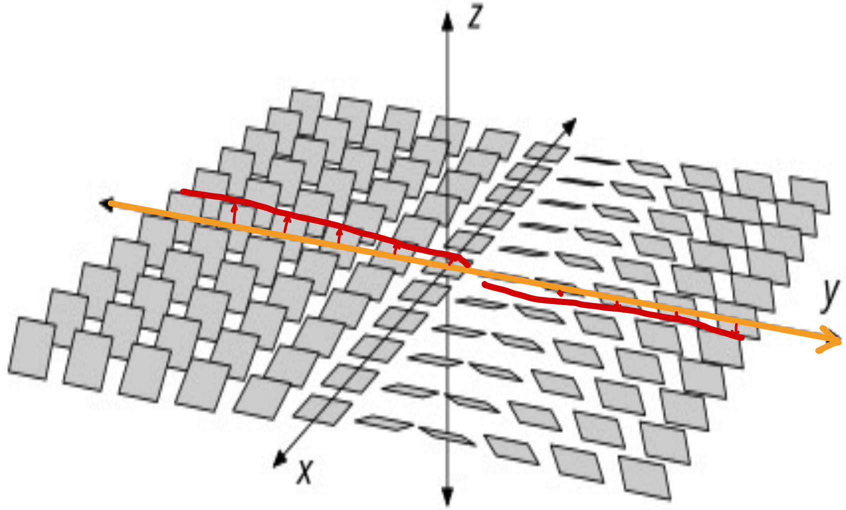
DIFFEOMORPHISM THAT CARRIES ξ TO ξ'



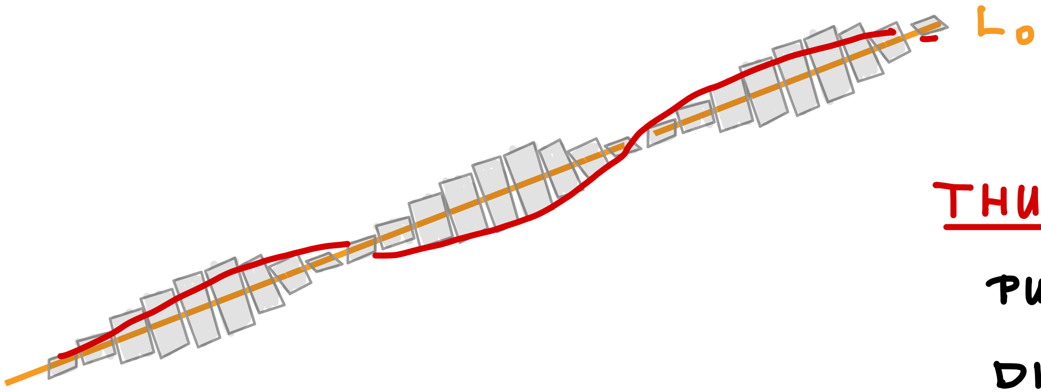
Knots in Contact Structures

DEF: $L \hookrightarrow M$ is a **Legendrian knot** if $T_pL \subset \xi_p \ \forall p$:



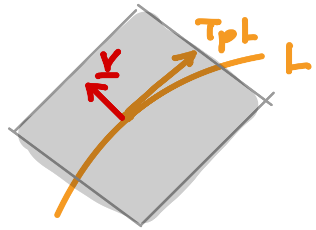


MOTTO: The contact structure always "rotates" along Legendrians



THURSTON-BENNEQUIN FRANING:

Push L in the direction of $\underline{v}$ where $\underline{v}_p \perp T_pL$ & $\underline{v}_p \in \xi_p$

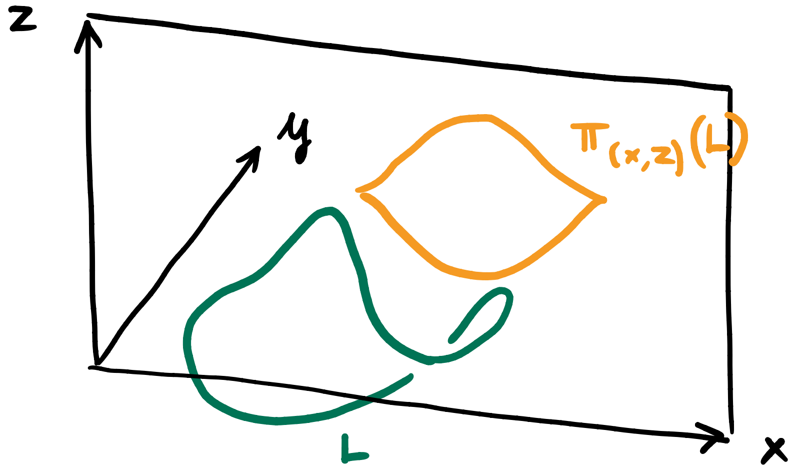


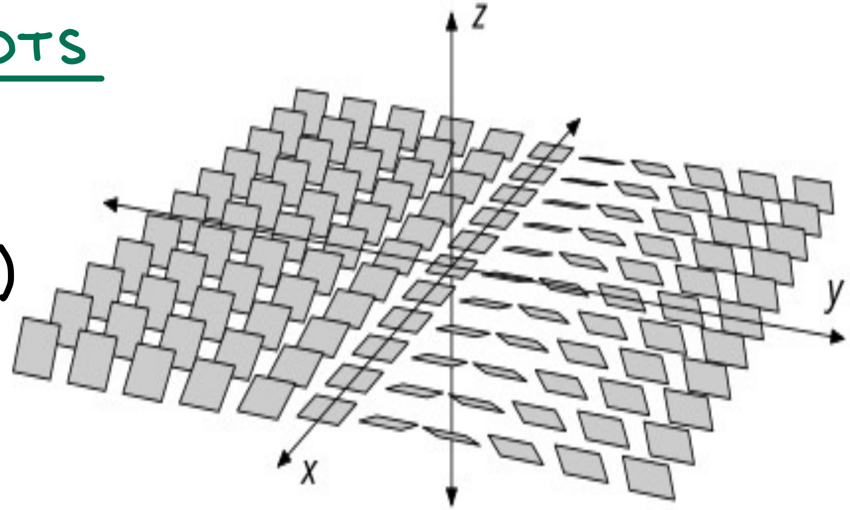
THM: Any Legendrian knot $L \hookrightarrow (M,\xi)$ has a neighbourhood $N(L)$ contactomorphic to $N(L_0)$

Front Projection of Legendrian Knots

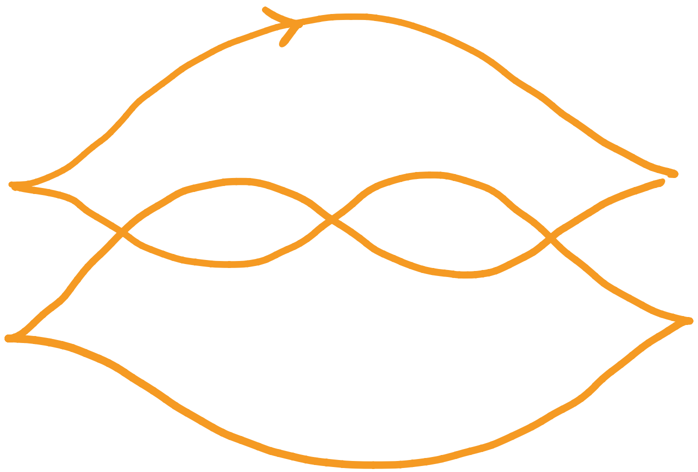
L Legendrian in $(\mathbb{R}^3, \xi_{st}) \Leftrightarrow y = \frac{dz}{dx}$

SO: y can be recovered from $\pi_{(x,z)}(L)$





$\xi_{st} = \ker(dz - y\,dx)$

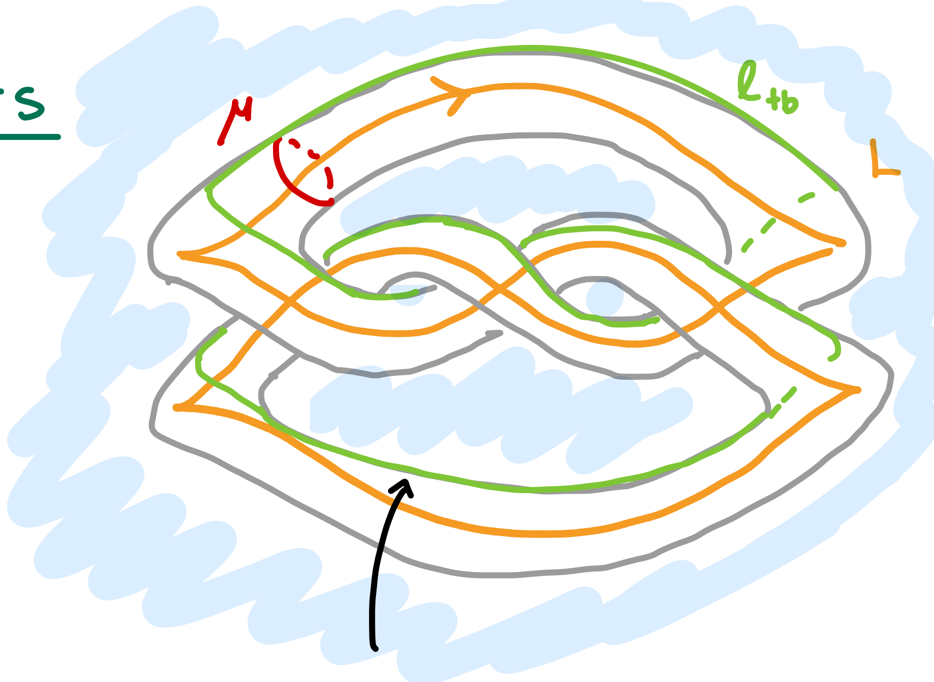


SURGERY ALONG LEGENDRIAN KNOTS

- GIVEN $L \hookrightarrow (S^3, \xi_{st})$ LEG KNOT
- $X := S^3 - \nu(K)$ ← STANDARD NBHD

$\rightsquigarrow$ $S^3_\pm(L) = X \cup_\varphi (D^2 \times S^1)$

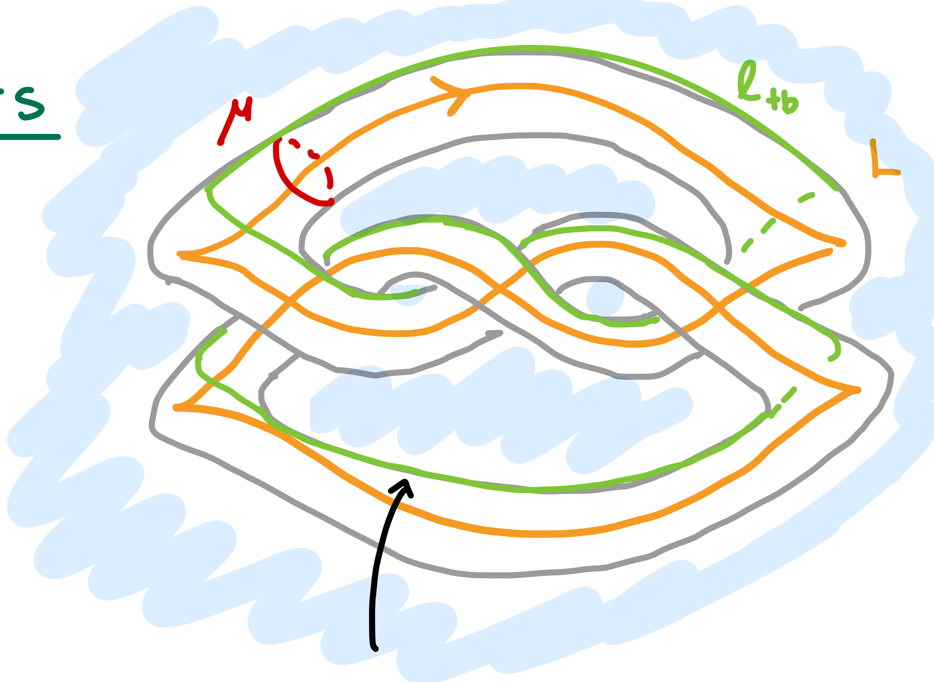
WHERE $\varphi(m) = \ell_{tb} \pm \mu$



THURSTON - BENNEQUIN FR

Surgery along Legendrian Knots

- Given $L \hookrightarrow (S^3, \xi_{st})$ leg knot
- $X := S^3 - \nu(K)$ ← standard nbhd



Thurston-Bennequin fr

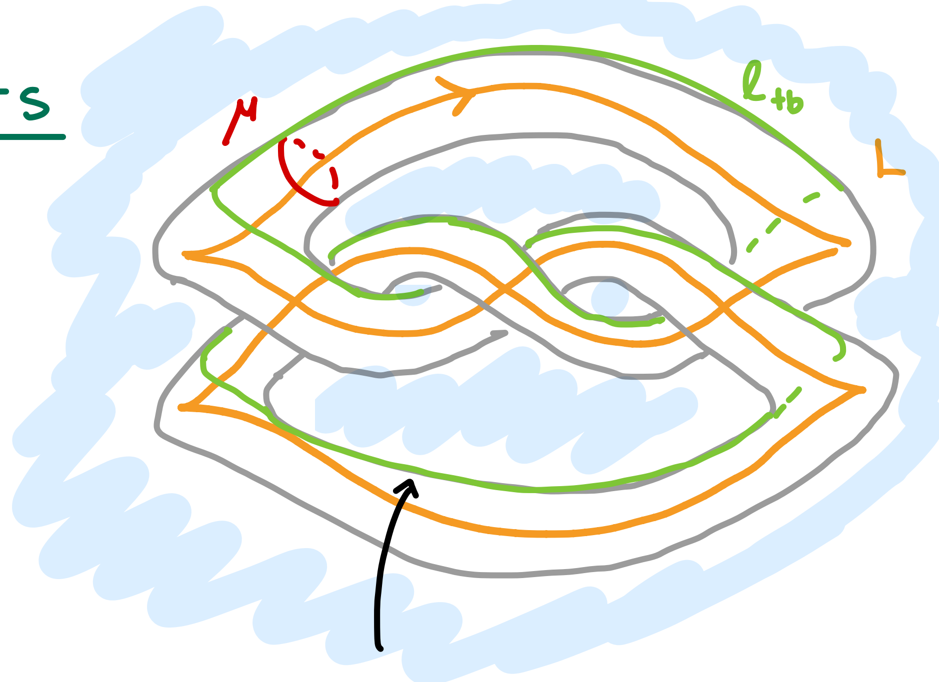
$\rightsquigarrow \quad S^3_\pm(L) = X \cup_\varphi (D^2 \times S^1)$

where $\varphi(m) = \ell_{tb} \pm \mu$

Fact: There is a unique (minimal) way to extend $\xi_{st}|_X$ to $S^3_\pm(L)$

(minimal = tight)

SURGERY ALONG LEGENDRIAN KNOTS



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FACT: THERE IS A UNIQUE (MINIMAL) WAY TO EXTEND $\xi_{st}|_X$ TO $S^3_{\pm}(L)$

$\shortparallel$

TIGHT

THM (DING - GEIGES '04) ANY CONTACT 3-MANIFOLD (M, ξ) CAN BE OBTAINED FROM (S^3, ξ_{st}) VIA ± 1 SURGERY ALONG A LEGENDRIAN LINK $\mathcal{L} \hookrightarrow (S^3, \xi_{st})$

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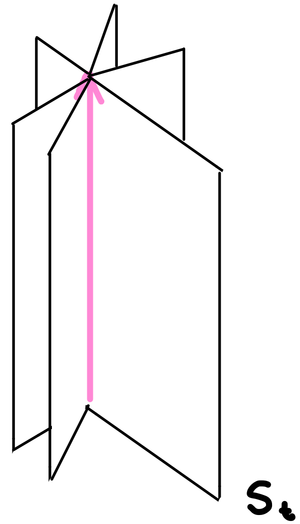
IDEA OF PROOF:

- **OPEN BOOK DECOMPOSITION**

PAIR (B, π), WHERE

- $B^1 \hookrightarrow M$ EMBEDDED 1-MANIFOLD: **BINDING**
- $\pi : M - B \longrightarrow S^1$ FIBRATION SUCH THAT
  - $\forall\, t \in S^1$ $\quad S_t := \overline{\pi^{-1}(t)}$ IS A SEIFERT SURFACE FOR B
  - & ON $N(B) \cong B \times D^2$ $\quad \pi =$ ANGLE



$S_t := \pi^{-1}(t)$ ARE THE **PAGES** OF (B, π)

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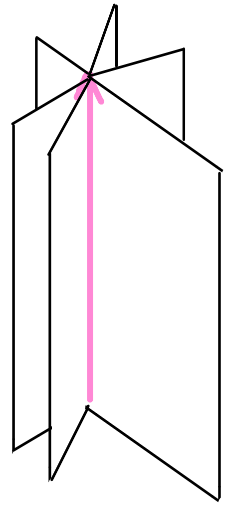

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  $S_t := \pi^{-1}(t)$ ARE THE **PAGES** OF (B,π)

- (B,π) COMPATIBLE WITH $(M, \xi = \ker(\alpha))$
  - $\alpha(B) > 0$
  - $d\alpha|_{S_t} > 0$

THM (DING – GEIGES '04) ANY CONTACT 3-MANIFOLD (M,ξ) CAN BE OBTAINED FROM (S^3,ξ_{st}) VIA ± 1 SURGERY ALONG A LEGENDRIAN LINK $\mathcal{L} \hookrightarrow (S^3,\xi_{st})$

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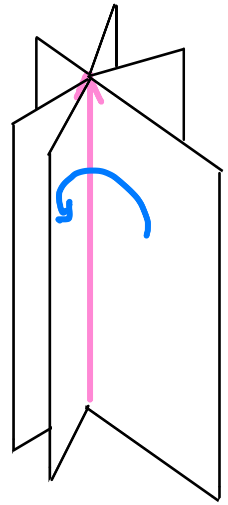
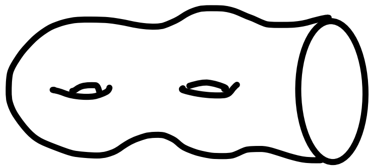


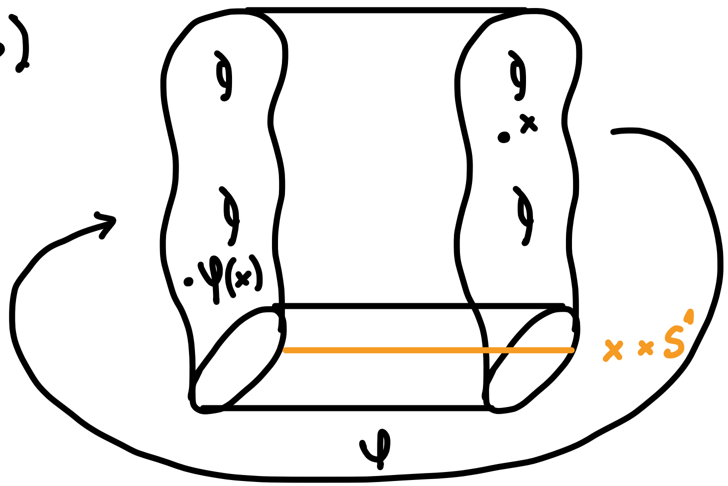
WE CAN THINK OF AN OB VIA ITS FIRST RETURN MAP (S,φ)

SURFACE W/ ∂



MONODROMY $\in MCG(S)$

$$M \cong {S \times I} \Big/ {\substack{(x,1) \sim (\varphi(x),0) \\ (x,t)\sim(x,t') \quad x \in \partial S}}$$



- **MCG(S)** IS STILL GENERATED BY DEHN TWISTS ALONG S.C.CS

THM (DING - GEIGES '04) ANY CONTACT 3-MANIFOLD (M,ξ) CAN BE OBTAINED FROM (S^3,ξ_{st}) VIA ± 1 SURGERY ALONG A LEGENDRIAN LINK $\mathcal{L} \hookrightarrow (S^3,\xi_{st})$

IDEA OF PROOF: SIMILAR TO SMOOTH CASE

- $M \rightsquigarrow (S,\varphi)$ OB
- TAKE AN OB W/ PAGE S FOR S^3: (S,φ_0)
- WRITE $\varphi \circ \varphi_0^{-1} = D_{c_1} \circ \cdots \circ D_{c_n}$, WHERE c_i IS NONSEPARATING ON S

$\Rightarrow$ CAN EMBED c_i ON $S_{\frac{i}{n}} \subseteq S^3$ AS A LEGENDRIAN : $\mathcal{L}$

(S.T. $S_{\frac{i}{n}}$ GIVES THE THURSTON - BENNEQUIN FRAMING FOR c_i)

$\rightsquigarrow$ (M,ξ) IS OBTAINED VIA CONTACT SURGERY ALONG $\mathcal{L}$

QED

EARLIER RESULTS

MOVES THAT DON'T CHANGE (M,ξ)

- DING - GEIGES '01: CANCELLATION
- DING - GEIGES '09: CONTACT HANDLE SLIDE

  CONTACT ANNULUS TWIST
- LISCA - STIPSICZ '11: LANTERN DESTABILISATION
- CASALS - ETNYRE - KEGEL: CONTACT ROLFSEN TWIST
- AVDEK '13: RIBBON MOVES

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THM (AVDEK '13) ± 1 - SURGERY ALONG THE LEGENDRIAN LINKS $\mathcal{L}$ & $\mathcal{L}' \hookrightarrow (S^3, \xi_{st})$ GIVE CONTACTOMORPHIC CONTACT MANIFOLDS IFF $\mathcal{L}$ & $\mathcal{L}'$ ARE RELATED VIA RIBBON MOVES

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$\boxed{\text{THM (AVDEK '13) } \pm 1\text{-SURGERY ALONG THE LEGENDRIAN LINKS } \mathcal{L} \,\&\, \mathcal{L}' \subset (S^3, \xi_{st}) \text{ GIVE CONTACTOMORPHIC CONTACT MANIFOLDS IFF } \mathcal{L} \,\&\, \mathcal{L}' \text{ ARE RELATED VIA RIBBON MOVES}}$

BUT! RIBBON MOVES CAN BE ARBITRARILY COMPLICATED

& CANNOT BE RECOGNISED

Towards a Contact Kirby's Theorem

THE STRATEGY FROM THE SMOOTH CASE DOESN'T EASILY GENERALISE:

PROBLEMS:

1) THERE IS NO CONTACT BLOW UP

(THIS WAS KEY IN ALL THE SMOOTH STEPS)

2) THERE IS NO UNIQE OB. OF (S^3, ξ_{st}) FOR A GIVEN PAGE-TYPE

3) ONLY NONSEPARATING CURVES ON S GIVE LEGENDRIANS

Towards a Contact Kirby's Theorem

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3) ONLY NONSEPARATING CURVES ON S GIVE LEGENDRIANS

WE STRUGGLED THE MOST W/ 3, EVEN THOUGH THE SOLUTION ALREADY EXISTED HIDDEN IN A '96 PAPER

TOWARDS A CONTACT KIRBY'S THEOREM

- PRESENTATION OF MCG(S) (GERVAIS '96)

  GENERATORS: DEHN TWISTS ALONG NONSEPARATING CURVES

TOWARDS A CONTACT KIRBY'S THEOREM

- **PRESENTATION OF MCG(S)** (GERVAIS '96)

  GENERATORS: DEHN TWISTS ALONG NONSEPARATING CURVES

  RELATIONS: - COMMUTATIVITY: $ab = ba$ IF $a \cap b = 0$

TOWARDS A CONTACT KIRBY'S THEOREM

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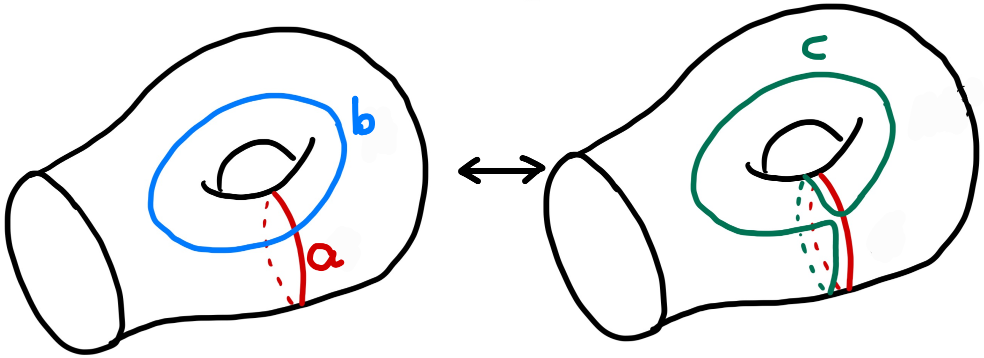
  RELATIONS: - COMMUTATIVITY: $ab = ba$ IF $a \cap b = 0$

  - BRAID RELATIONS:



$ab = ca$

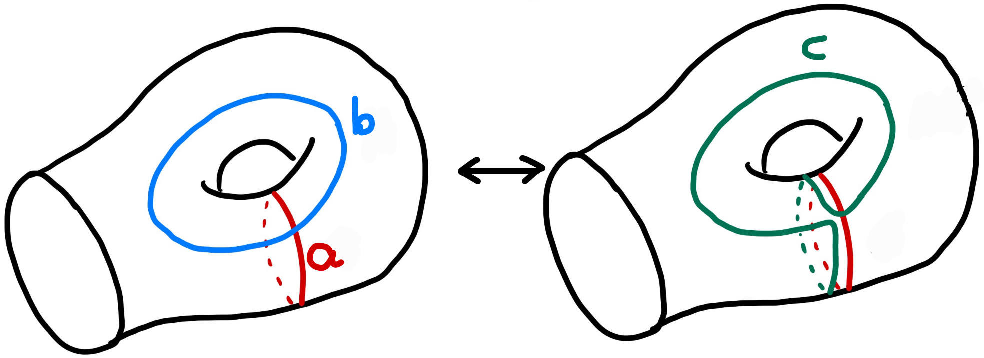
TOWARDS A CONTACT KIRBY'S THEOREM

- PRESENTATION OF MCG(S) (GERVAIS '96)

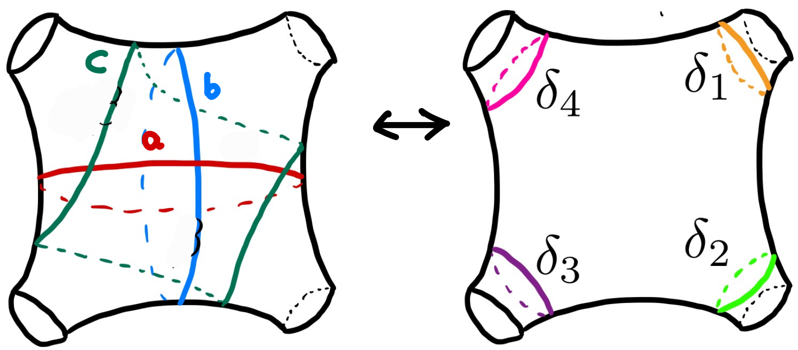
GENERATORS: DEHN TWISTS ALONG NONSEPARATING CURVES

RELATIONS: - COMMUTATIVITY: $ab = ba$ IF $a \cap b = 0$

- BRAID RELATIONS: - 1 LANTERN RELATION



$ab = ca$



$abc = d_1 d_2 d_3 d_4$

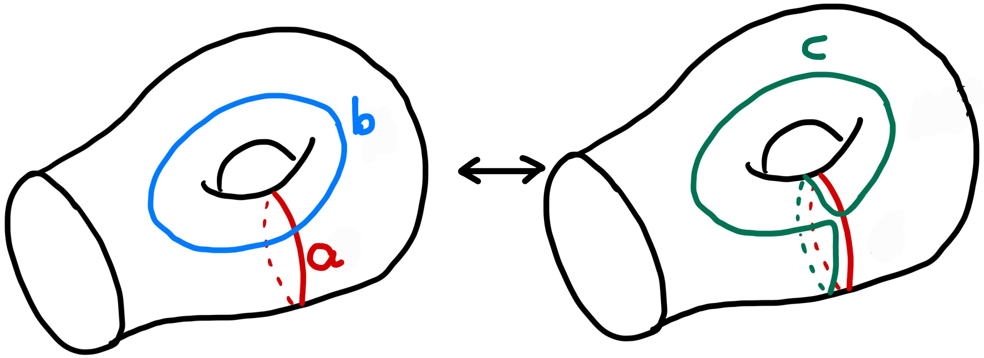
Towards a Contact Kirby's Theorem

• Presentation of MCG(S) (Gervais '96)

Generators: Dehn twists along nonseparating curves

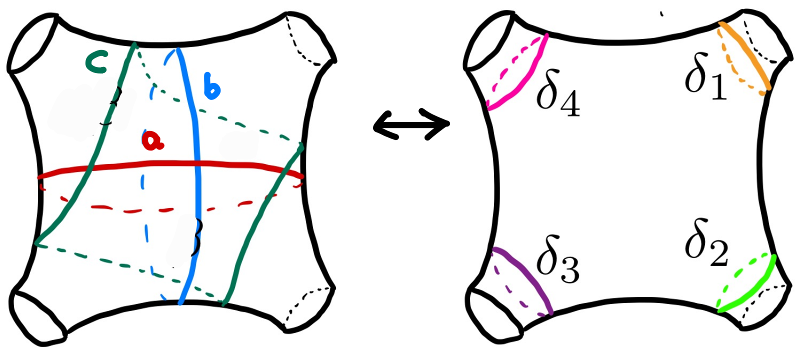
Relations: - Commutativity: $ab = ba$ if $a \cap b = 0$

- Braid relations:



$ab = ca$

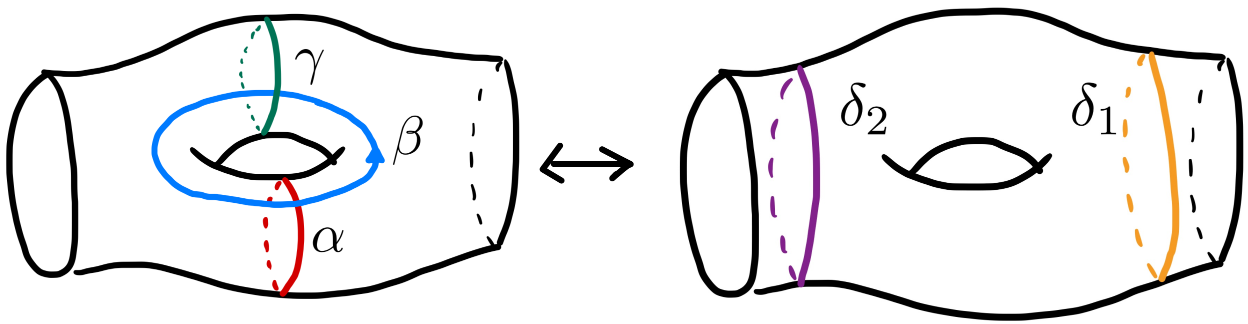
- 1 Lantern relation



$abc = d_1 d_2 d_3 d_4$

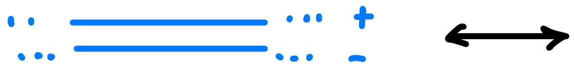
- 1 Chain relation

$(abc)^4 = d_1 d_2$

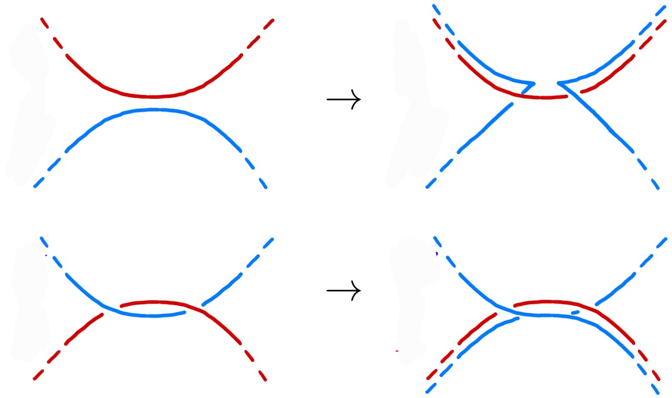


THM (KEGEL, STENHEDE, V, ZUDDAS) ± 1-SURGERY ALONG THE LEGENDRIAN LINKS $\mathcal{L}$ & $\mathcal{L}' \subset (S^3, \xi_{st})$ GIVE CONTACTOMORPHIC CONTACT MANIFOLDS IFF $\mathcal{L}$ & $\mathcal{L}'$ ARE RELATED VIA

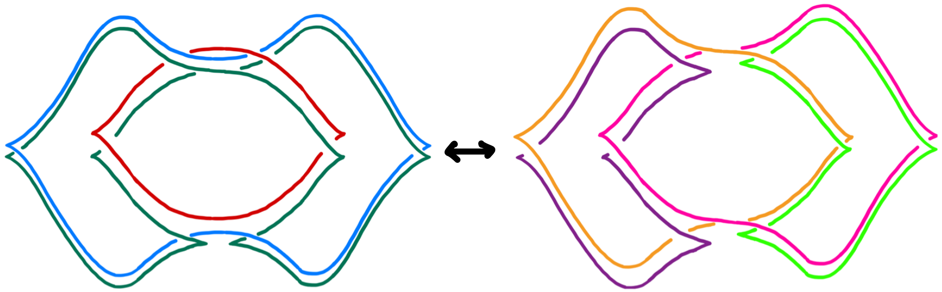
- CANCELLATION :



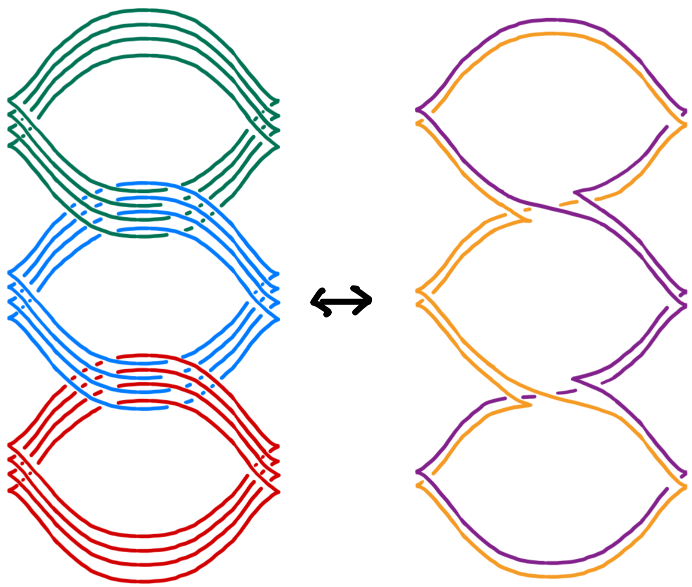
- LEGENDRIAN HANDLESLIDE :



- LANTERN MOVE :



- CHAIN MOVE



Idea of Proof

$\boxed{\Leftarrow}$ RIBBON MOVES $\checkmark$

IDEA OF PROOF

$\boxed{\Rightarrow}$

- EMBED $\mathcal{L}$ AND $\mathcal{L}'$ INTO PAGES OF OB'S (S,φ_0) OF (S^3,ξ):

$\mathcal{L}$

$\mathcal{L}'$

ASSUME THEY ARE "FAR" & EXTEND $\mathcal{L} \sqcup \mathcal{L}'$ INTO A "COMPLICATED ENOUGH" LEGENDRIAN GRAPH

Idea of Proof

$\boxed{\Rightarrow}$

- Embed $\mathcal{L}$ and $\mathcal{L}'$ into pages of OB's (S, φ_0) of (S^3, ξ):

$\mathcal{L}$

$\mathcal{L}'$

Assume they are "far" & extend $\mathcal{L} \amalg \mathcal{L}'$ into a "complicated enough" Legendrian graph

$\rightsquigarrow$ $c_1, \dots, c_k \subset S$ $\quad 0 \neq [c_i^{(\prime)}] \in H_1(S)$

$c_1', \dots, c_k' \subset S$

IDEA OF PROOF

$\boxed{\Rightarrow}$

- EMBED $\mathcal{L}$ AND $\mathcal{L}'$ INTO PAGES OF OB'S (S, φ_0) OF (S^3, ξ):

$\mathcal{L}$

$\mathcal{L}'$

ASSUME THEY ARE "FAR"
& EXTEND $\mathcal{L} \sqcup \mathcal{L}'$ INTO
A "COMPLICATED ENOUGH"
LEGENDRIAN GRAPH

$\rightsquigarrow$ $c_1, \dots, c_k$ / $c_1', \dots, c_k'$ $\subset S$

$0 \neq [c_i^{(\prime)}] \in H_1(S)$
& WE STABILISE (S, φ_0) S.T
$c_i^{(\prime)}$ IS NONSEPARATING

(CANCELLATION
&
LEG. HSLIDE)

Idea of Proof

$\boxed{\Rightarrow}$

- EMBED $\mathcal{L}$ AND $\mathcal{L}'$ INTO PAGES OF OB'S (S, φ_0) OF (S^3, ξ):

$\mathcal{L}$

$\mathcal{L}'$

ASSUME THEY ARE "FAR"
& EXTEND $\mathcal{L} \sqcup \mathcal{L}'$ INTO
A "COMPLICATED ENOUGH"
LEGENDRIAN GRAPH

$\rightsquigarrow$ $c_1, \dots, c_k \subset S$ $\quad 0 \neq [c_i^{(\prime)}] \in H_1(S)$
$c_1', \dots, c_k' \subset S$ & WE STABILISE (S, φ_0) S.T
$c_i^{(\prime)}$ IS NONSEPARATING

$\rightsquigarrow$ TWO OB'S FOR (M, ξ): $(S, D_{c_1}^{\pm} \circ \dots \circ D_{c_k}^{\pm} \circ \varphi_0)$
$(S, D_{c_1'}^{\pm} \circ \dots \circ D_{c_k'}^{\pm} \circ \varphi_0)$

(CANCELLATION
&
LEG. HSLIDE)

IDEA OF PROOF

$\boxed{\Rightarrow}$

- EMBED $\mathcal{L}$ AND $\mathcal{L}'$ INTO PAGES OF OB'S (S,φ_0) OF (S^3,ξ):

$\mathcal{L}$

$\mathcal{L}'$

ASSUME THEY ARE "FAR" & EXTEND $\mathcal{L} \sqcup \mathcal{L}'$ INTO A "COMPLICATED ENOUGH" LEGENDRIAN GRAPH

$\rightsquigarrow$ $\left.\begin{array}{l} c_1, \dots, c_k \\ c_1', \dots, c_{k'}' \end{array}\right\} \subset S$ $\quad 0 \neq [c_i^{(\prime)}] \in H_1(S)$

& WE STABILISE (S,φ_0) S.T $c_i^{(\prime)}$ IS NONSEPARATING

$\rightsquigarrow$ TWO OB'S FOR (M,ξ): $\to (S,\ D^{\pm}_{c_1} \circ \dots \circ D^{\pm}_{c_k} \circ \varphi_0)$

$\to (S,\ D^{\pm}_{c_1'} \circ \dots \circ D^{\pm}_{c_{k'}'} \circ \varphi_0)$

(CANCELLATION & LEG. HSLIDE)

$\xRightarrow[\text{CORRES}]{\text{GIROUX}}$ CAN BE STABILISED TO CONJUGATE OB'S

$\Updownarrow$

(CANCELLATION & LEG. HSLIDE)

IDEA OF PROOF

$\boxed{\Rightarrow}$ SO AFTER SOME CANCELLATIONS & LEG. HSLIDES

WE CAN ASSUME - $c_1, \dots, c_k$
- $c'_1, \dots, c'_{k'}$ $\subset S$ ARE NONSEPARATING,

(S, φ_0) OB FOR (S^3, ξ) & $\left.\begin{array}{l}(S, D^{\pm}_{c_1} \circ \dots \circ D^{\pm}_{c_k} \circ \varphi_0) \\ (S, D^{\pm}_{c'_1} \circ \dots \circ D^{\pm}_{c'_{k'}} \circ \varphi_0)\end{array}\right\}$ ARE CONF OB'S FOR (M, ξ)

IDEA OF PROOF

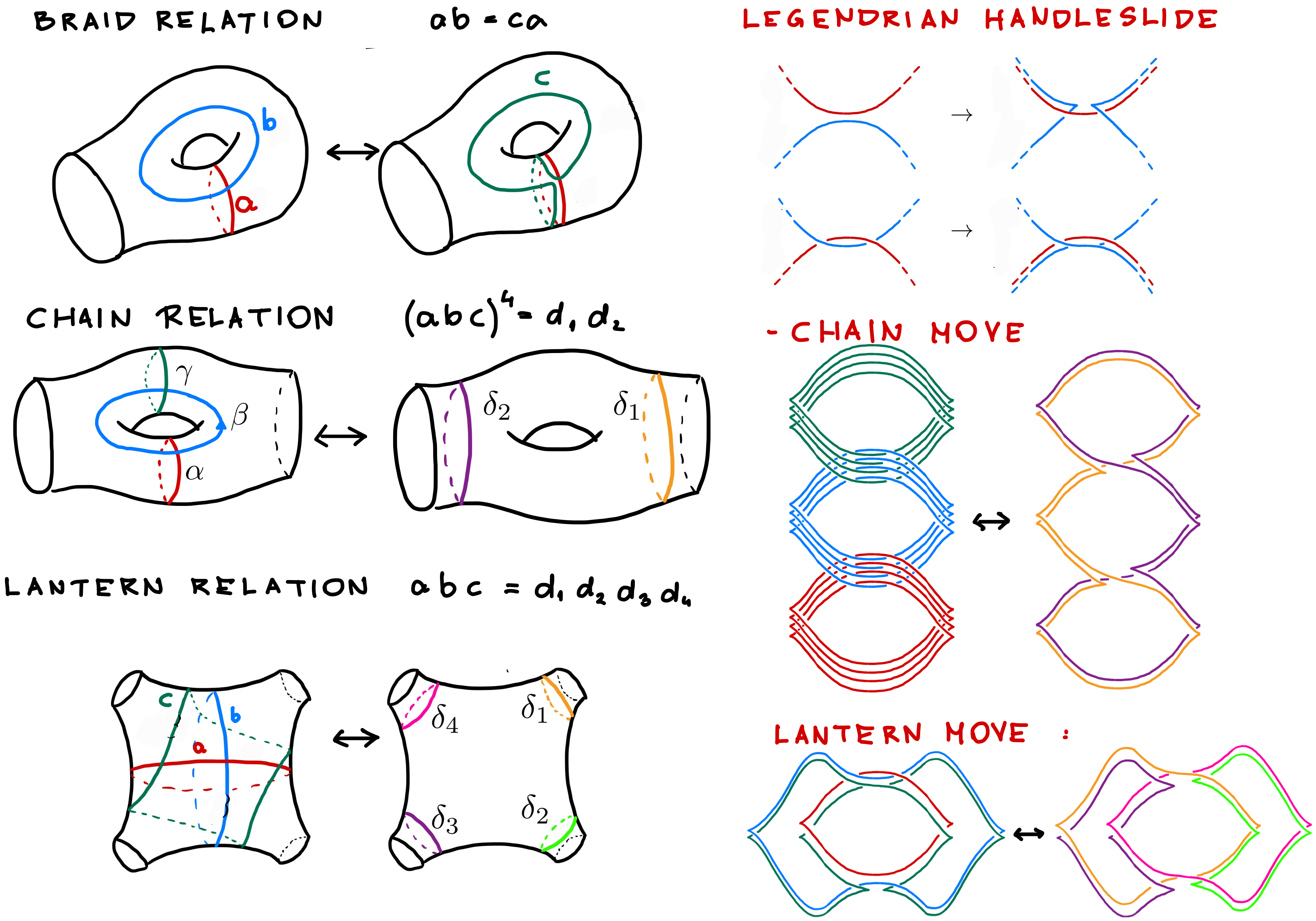
$\boxed{\Rightarrow}$ SO AFTER SOME CANCELLATIONS & LEG. HSLIDES

WE CAN ASSUME - $c_1, \dots, c_k \subset$
- $c_1', \dots, c_{k'}' \subset$ S ARE NONSEPARATING,

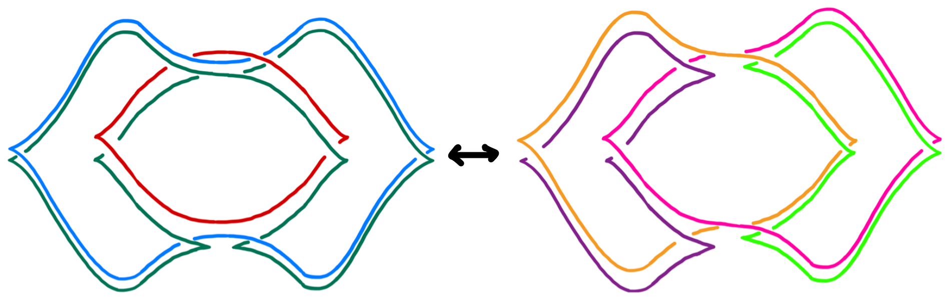
$$\left.\begin{array}{l} (S,\varphi_0) \text{ OB FOR } (S^3,\xi) \;\&\; (S, D^{\pm}_{c_1}\circ\cdots\circ D^{\pm}_{c_k}\circ\varphi_0) \\ (S, D^{\pm}_{c_1'}\circ\cdots\circ D^{\pm}_{c_{k'}'}\circ\varphi_0) \end{array}\right\} \text{ ARE CONJ OB'S FOR } (M,\xi)$$

$\xRightarrow{\text{GERVAIS}}$ WE CAN RELATE THE WORDS $D^{\pm}_{c_1}\circ\cdots\circ D^{\pm}_{c_k}\circ\varphi_0$ & $D^{\pm}_{c_1'}\circ\cdots\circ D^{\pm}_{c_{k'}'}\circ\varphi_0$ BY

- COMMUTATION $\longleftrightarrow$
- BRAID RELATION $\longleftrightarrow$ LEGENDRIAN HANDLESLIDE
- LANTERN RELATION $\longleftrightarrow$ LANTERN MOVE
- CHAIN RELATION $\longleftrightarrow$ CHAIN MOVE

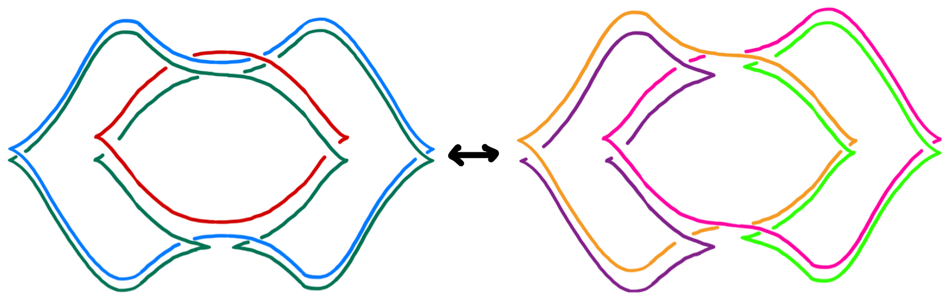

BRAID RELATION ab = ca
b
a
c
CHAIN RELATION (abc)^4 = d1 d2
γ
β
α
δ2
δ1
LANTERN RELATION abc = d1 d2 d3 d4
c
b
a
δ4
δ1
δ3
δ2
LEGENDRIAN HANDLESLIDE
- CHAIN MOVE
LANTERN MOVE :

THANKS FOR

YOUR 

ATTENTION!

THANKS FOR

THE 

NICE CONFERENCE!