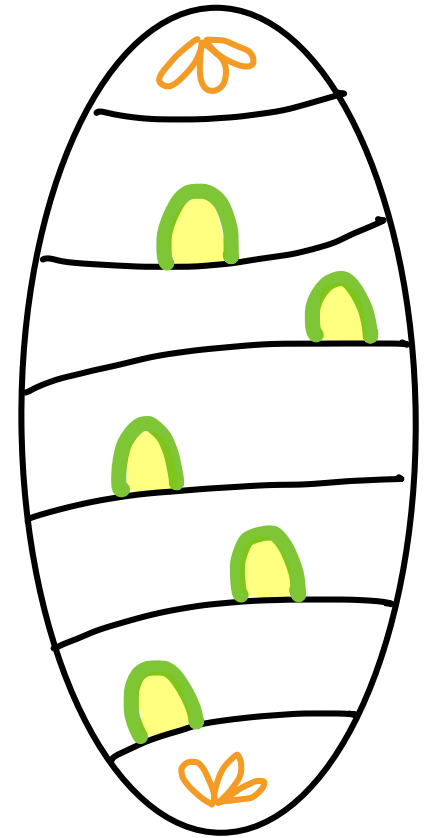


THE GIROUX CORRESPONDENCE VIA CONVEX SURFACES

VERA VÉRTESI

JOINT WORK WITH JOAN LICATA

UNIVERSITY OF VIENNA



INTRODUCTION

1973 WINKELNKEMPER: FIRST USED THE WORD
"OPEN BOOK DECOMPOSITION"

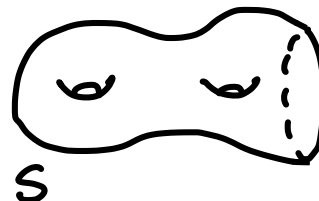
BUT IT WAS ALREADY KNOWN & STUDIED UNDER DIFFERENT NAMES:

- GLOBAL POINCARÉ-BIRKHOFF SECTION
- RELATIVE MAPPING TORUS
- LEFSHETZ/HILNOR FIBRATION
- FIBERED LINKS
- SPINNABLE STRUCTURES

DEF: OPEN BOOK: (S, ψ)

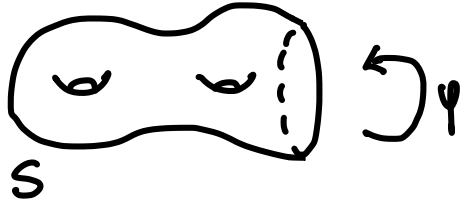
$\psi: S \rightarrow S$ HOMEOMORPHISM: MONODROMY

SURFACE WITH BOUNDARY



INTRODUCTION

OPEN BOOKS



THURSTON
WINKENKEMPER

CONTACT STRUCTURES

(M^3, ξ)

INTRODUCTION

OPEN BOOKS

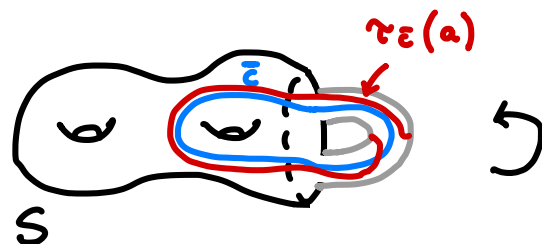
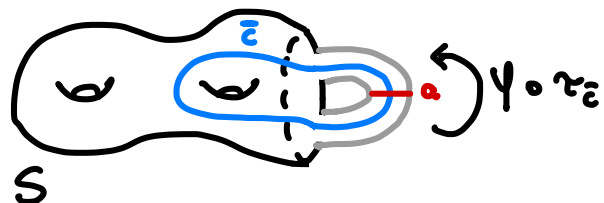


THURSTON
WINKENKEMPLER

CONTACT STRUCTURES

(M^3, ξ)

POSITIVE
STABILISATION



INTRODUCTION

OPEN BOOKS

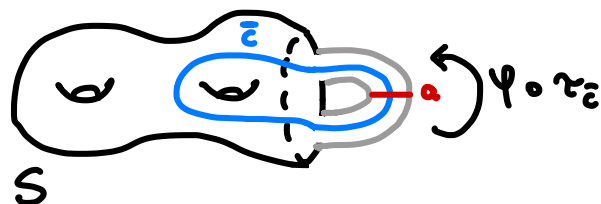


THURSTON
WINKENKEMPLER

CONTACT STRUCTURES

(M^3, ξ)

POSITIVE
STABILISATION



(S, h)

GIROUX

(M^3, ξ)

GIROUX CORRESPONDENCE

OPEN BOOKS

POSITIVE STABILISATION



CONTACT STRUCTURES

CONTACTOMORPHISM

INTRODUCTION

GIROUX CORRESPONDENCE

OPEN BOOKS

POSITIVE STABILISATION

$\leftrightarrow$ CONTACT STRUCTURES

CONTACTOMORPHISM

GC HAS BEEN EXTENSIVELY USED TO PROVE THMS ABOUT CTCT 3-MFDS

2000 THE ORIGINAL PROOF OF GIROUX WAS INCOMPLETE

(MASSOT WROTE DOWN A COMPLETE PROOF BUT DIDN'T
PUBLISH IT)

2023 BREEN - HONDA - HUANG: PROOF OF THE GIROUX CORRESPONDENCE
FOR CONTACT STRUCTURES IN ANY ODD DIMENSIONS

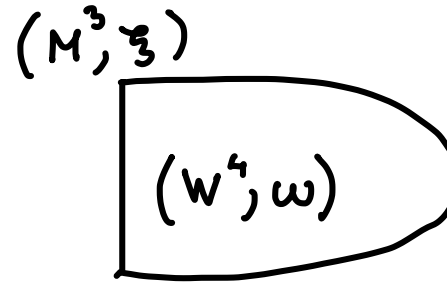
2023 LICATA - Y.: PROOF OF THE GIROUX CORRESPONDENCE
FOR TIGHT CONTACT 3-MANIFOLDS (INDEPENDENT)

2024 LICATA - Y.: EXTENDED OUR PROOF TO WORK FOR ANY
CONTACT 3-MANIFOLD

APPLICATIONS

IN CONTACT TOPOLOGY

→ FILLABILITY



GIROUX: TOPOLOGICAL DESCRIPTION OF STEIN-FILLABLE CONTACT 3-MANIFOLDS

ELIASBERG, ETNYRE: ANY WEAK SYMPLECTIC FILLING OF A CONTACT 3-MANIFOLD CAN BE EMBEDDED INTO A CLOSED SYMPLECTIC MANIFOLD

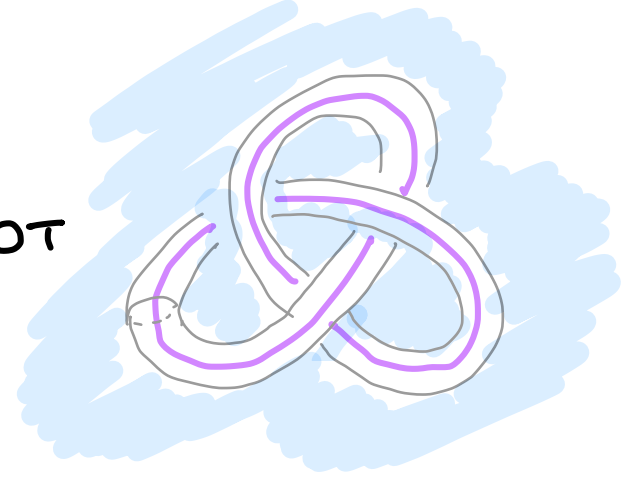
→ CONTACT SURGERY

WAND: CONTACT SURGERY PRESERVES TIGHTNESS

KEGEL-STENHENDE-V-ZUDDAS: CLASSIFICATION OF LEGENDRIAN SURGERY DIAGRAMS DESCRIBING THE SAME CONTACT MANIFOLD

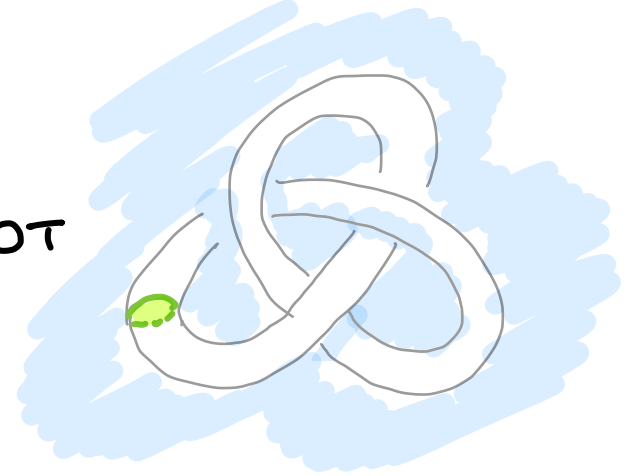
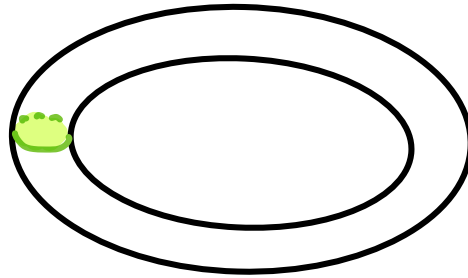
APPLICATIONS

SURGERY: REMOVE NEIGHBOURHOOD OF A KNOT



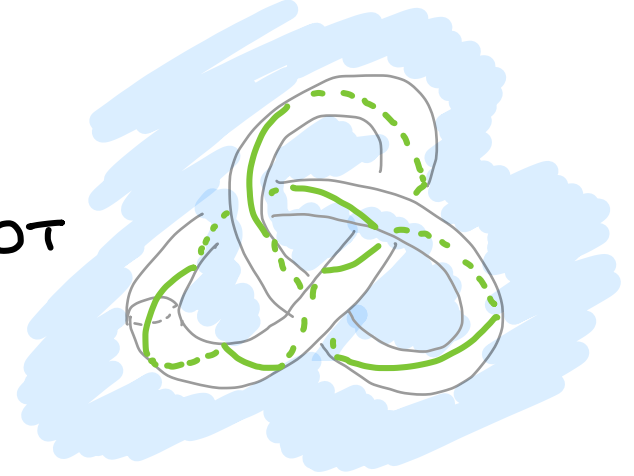
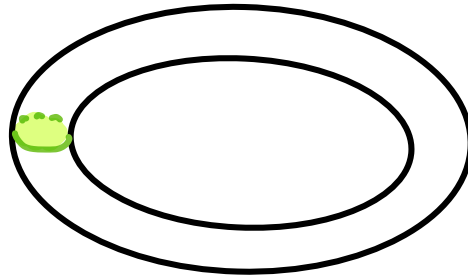
APPLICATIONS

SURGERY: REMOVE NEIGHBOURHOOD OF A KNOT
& GLUE BACK A $D^2 \times S^1$ DIFFERENTLY



APPLICATIONS

SURGERY: REMOVE NEIGHBOURHOOD OF A KNOT
& GLUE BACK A $D^2 \times S^1$ DIFFERENTLY



TOPOLOGY

KRONHEIMER - MROWKA: EVERY NONTRIVIAL KNOT HAS PROPERTY P

OZSVÁTH - SZABÓ: THE UNKNOT, TREFOIL & FIGURE-EIGHT KNOT
ARE CHARACTERISED BY THEIR SURGERIES

OZSVÁTH - SZABÓ: THE THURSTON NORM IS DETERMINED BY
HEEGAARD FLOER HOMOLOGY

GIROUX - GOODMAN: INDUCTIVE CONSTRUCTION OF FIBERED
KNOTS IN S^3

PLAN OF TALKS

- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES
- LECTURE 2: DESCRIBING CONTACT STRUCTURES
- LECTURE 3: PROOF OF GIROUX CORRESPONDENCE

PLAN OF TALKS

- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES

- CONTACT STRUCTURES

- (1D) LEGENDRIAN & TRANSVERSE KNOTS , LEGENDRIAN GRAPHS

- (2D) CONVEX SURFACES

- NEIGHBOURHOOD THEOREMS

- TIGHT & OVERTWISTED CONTACT STRUCTURES

- LECTURE 2: DESCRIBING CONTACT STRUCTURES

- LECTURE 3: PROOF OF GIROUX CORRESPONDENCE

PLAN OF TALKS

- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES
- LECTURE 2: DESCRIBING CONTACT STRUCTURES
 - CONTACT CELL DECOMPOSITIONS
 - CONVEX SURFACE THEORY - BYPASSES
 - CONTACT HEEGAARD SPLITTINGS (PROOF OF EXISTENCE)
 - OPEN BOOK DECOMPOSITIONS
 - OPEN BOOK DECOMPOSITIONS & CONTACT HEEGAARD SPLITTINGS
- LECTURE 3: PROOF OF GIROUX CORRESPONDENCE

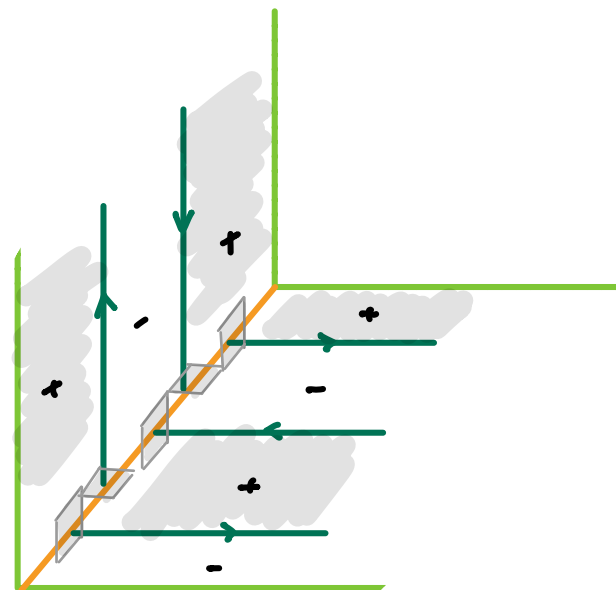
PLAN OF TALKS

- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES
- LECTURE 2: DESCRIBING CONTACT STRUCTURES
- LECTURE 3: PROOF OF GIROUX CORRESPONDENCE
 - STABILISATION
 - STATEMENT OF GIROUX CORRESPONDENCE
 - IDEA OF PROOF
 - FURTHER DIRECTIONS

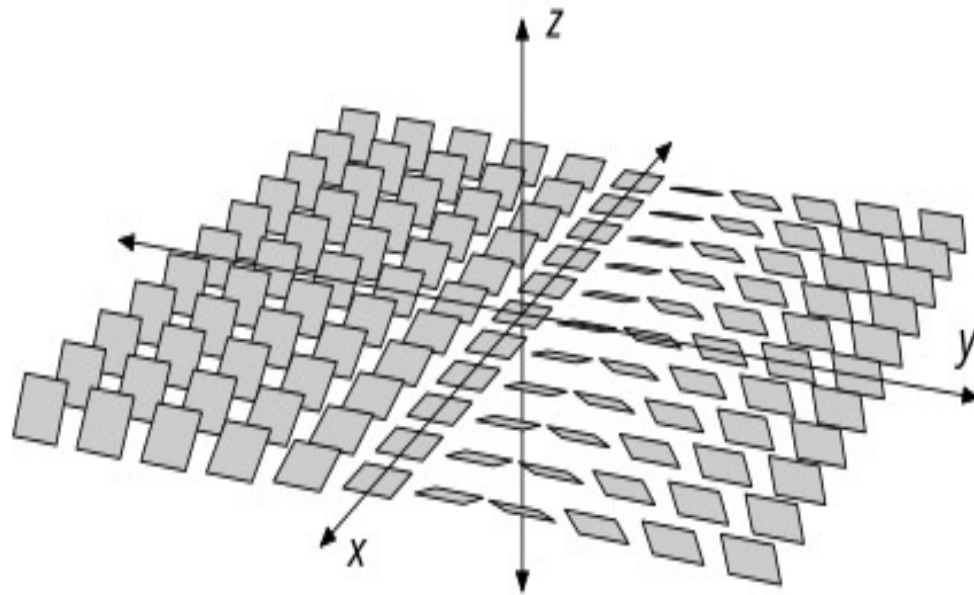
LECTURE 1

SUBMANIFOLDS IN

CONTACT STRUCTURES

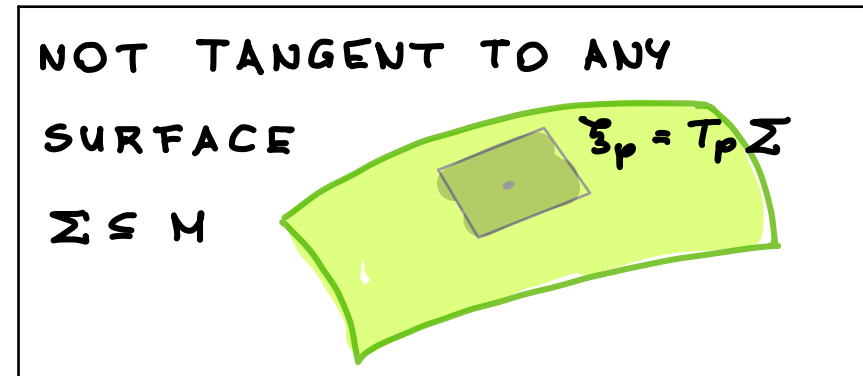
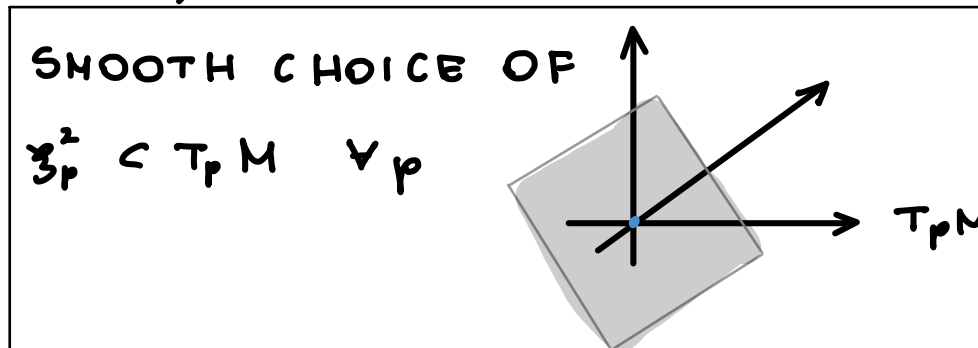


CONTACT STRUCTURES



CONTACT STRUCTURES

DEF: A CONTACT STRUCTURE ON A CLOSED, ORIENTED SMOOTH 3-MANIFOLD M^3 IS A TOTALLY NONINTEGRABLE 2-PLANE-DISTRIBUTION $\xi \subset TM$



$\updownarrow$ FROBENIUS

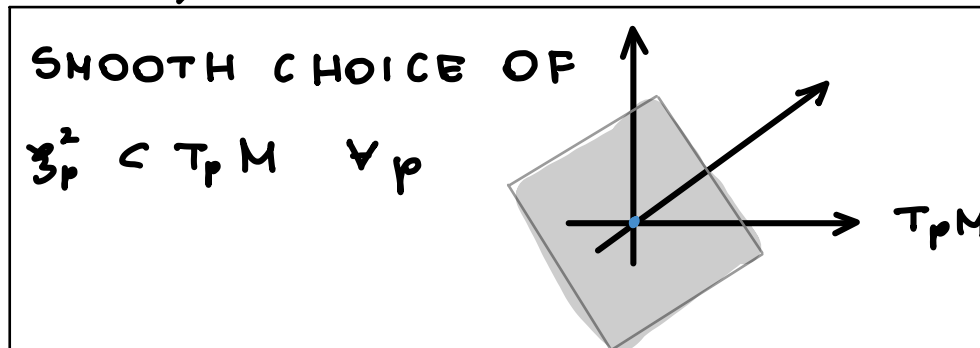
$$\alpha \wedge d\alpha > 0$$

LOCALLY: $\xi = \ker \alpha \quad \alpha \in \Omega^1(M)$

COORIENTED CONTACT STRUCTURE: GLOBAL α

CONTACT STRUCTURES

DEF: A CONTACT STRUCTURE ON A CLOSED, ORIENTED SMOOTH 3-MANIFOLD M^3 IS A TOTALLY NONINTEGRABLE 2-PLANE-DISTRIBUTION $\xi \subset TM$



NOT TANGENT TO ANY SURFACE $\Sigma \subseteq M$

$\xi_p = T_p \Sigma$

A light green curved surface Σ is shown. A small gray square represents a plane ξ_p tangent to the surface at a point p . The text 'NOT TANGENT TO ANY SURFACE $\Sigma \subseteq M$ ' is written to the left, and ' $\xi_p = T_p \Sigma$ ' is written next to the square.

$\updownarrow$ FROBENIUS

$$\alpha \wedge d\alpha > 0$$

LOCALLY: $\xi = \ker \alpha \quad \alpha \in \Omega^1(M)$

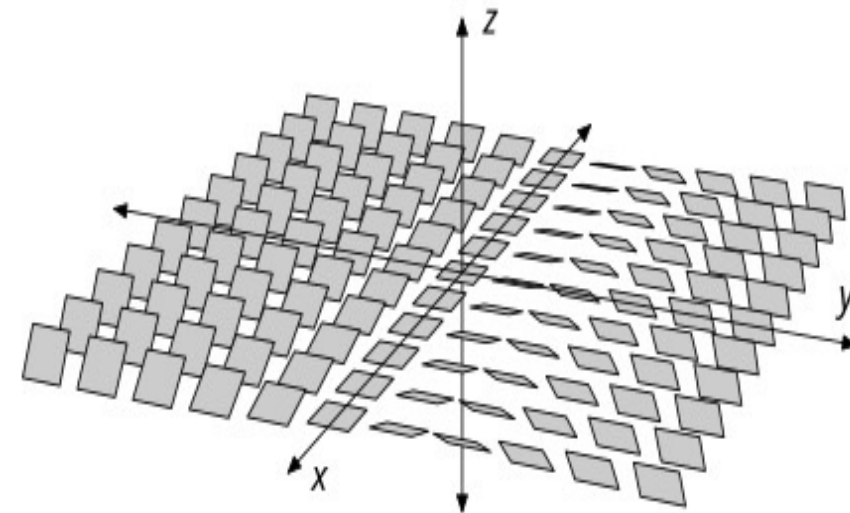
COORIENTED CONTACT STRUCTURE: GLOBAL α

DARBOUX THM: LOCALLY ANY CONTACT

STRUCTURE IS CONTACTOMORPHIC

TO $\mathbb{R}^3$, $\xi_{st} = \ker(dz - ydx)$

DIFFEOMORPHISM THAT CARRIES ξ TO ξ'



EQUIVALENCE OF CONTACT STRUCTURES

$(M, \xi) \& (M', \xi')$ CONTACT STRUCTURES

- CONTACTOMORPHISM: $(M, \xi) \cong (M', \xi')$ IF $\exists$ DIFFEOMORPHISM $\phi: M \rightarrow M'$ THAT CARRIES ξ TO ξ' : $\phi_* \xi = \xi'$

WHEN $M = M'$

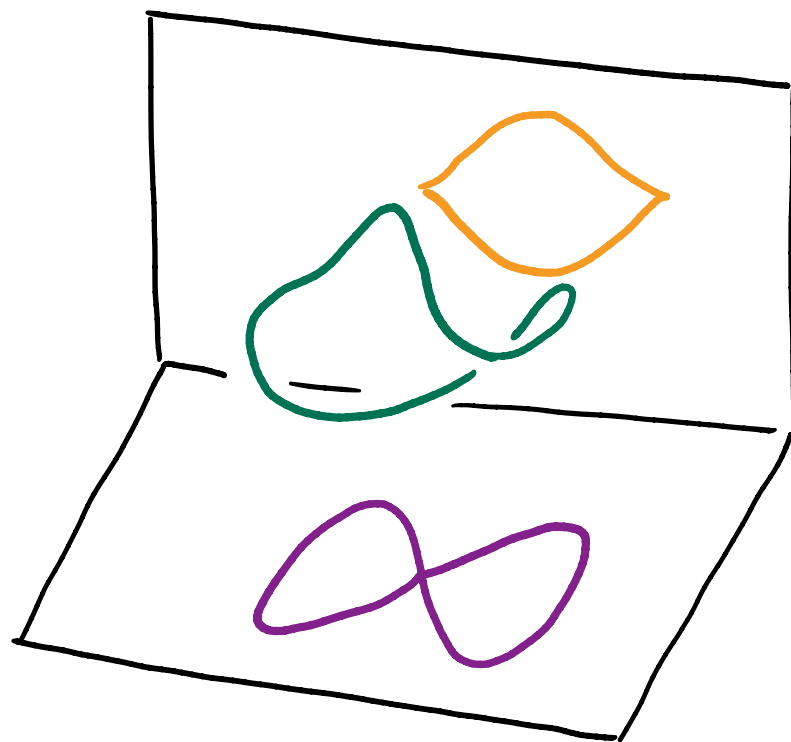
- HOMOTOPY: $\xi \approx \xi'$ IF $\exists$ 1-PARAMETER FAMILY OF CONTACT STRUCTURES $(\xi_t)_{t \in [0,1]}$ ON M WITH $\xi = \xi_0$ & $\xi' = \xi_1$
- ISOTOPY: $\xi \approx \xi'$ IF $\exists$ 1-PARAMETER FAMILY OF SELF-DIFFEOMORPHISM $(\phi_t)_{t \in [0,1]}$ OF M WITH
 - $\phi_0 = \text{Id}$ &
 - $\xi' = (\phi_1)_* \xi$

THM (GRAY STABILITY): "HOMOTOPY = ISOTOPY"

ANY HOMOTOPY $(\xi_t)_{t \in [0,1]}$ OF CONTACT STRUCTURES IS INDUCED BY AN ISOTOPY $(\phi_t)_{t \in [0,1]}$:

- $\phi_0 = \text{Id}$ &
- $\xi_t = (\phi_t)_* \xi_0$

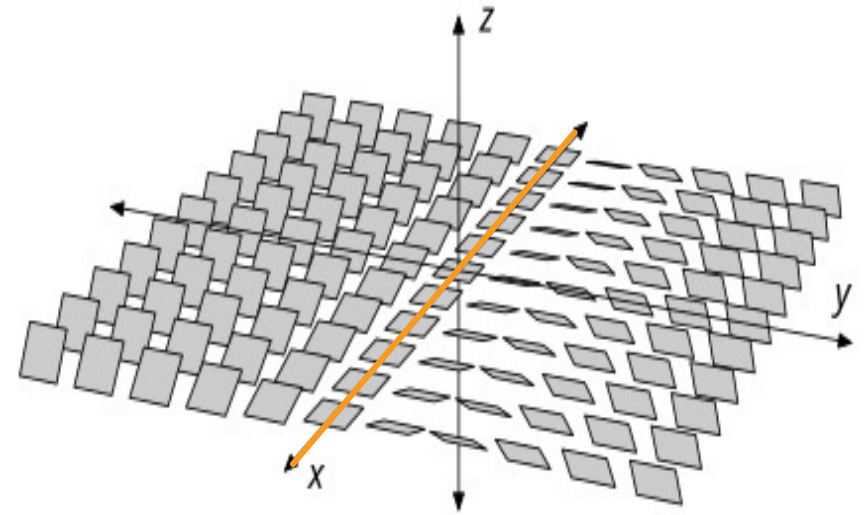
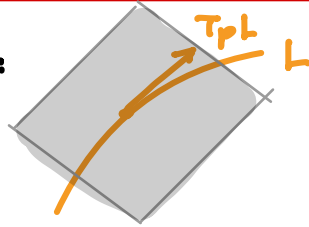
1-DIM: KNOTS



KNOTS IN CONTACT STRUCTURES

DEF: $L \subset M$ IS A LEGENDRIAN KNOT

IF $T_p L \subset \xi_p \quad \forall p:$

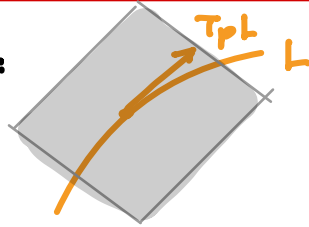


$$L = \{y = z = 0\}$$

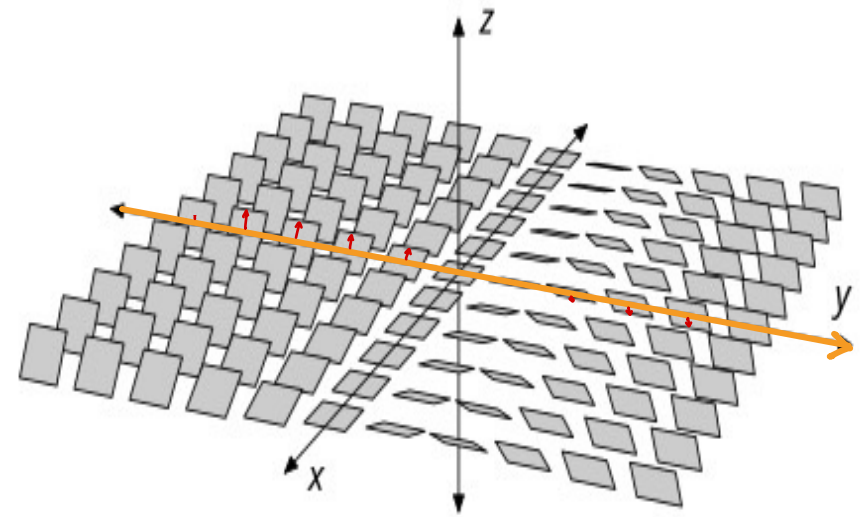
KNOTS IN CONTACT STRUCTURES

DEF: $L \subset M$ IS A LEGENDRIAN KNOT

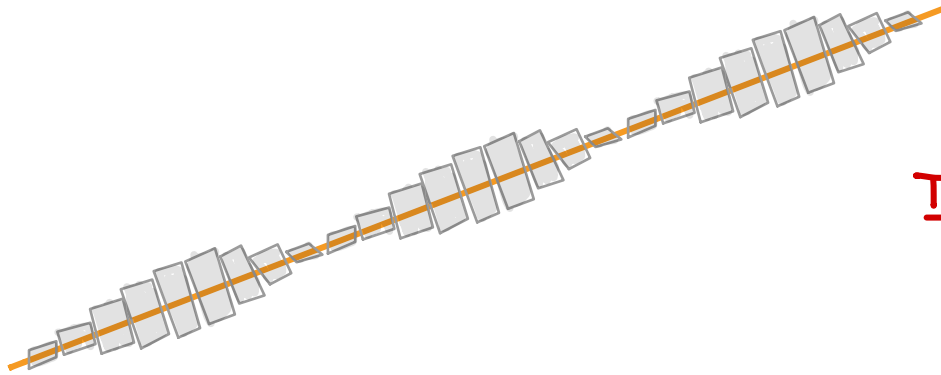
IF $T_p L \subset \xi_p \quad \forall p:$



MOTTO: THE CONTACT STRUCTURE
ALWAYS "ROTATES"
ALONG LEGENDRIANS



$$L = \{x = z = 0\}$$

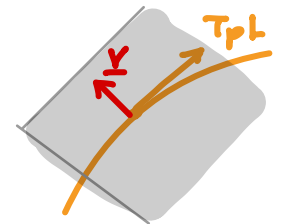


THURSTON-BENNEQUIN FRAMING:

PUSH L IN THE

DIRECTION OF $\underline{y}$

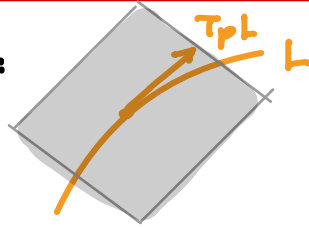
WHERE $\underline{y}_p \perp T_p L$ & $\underline{y}_p \in \xi_p$



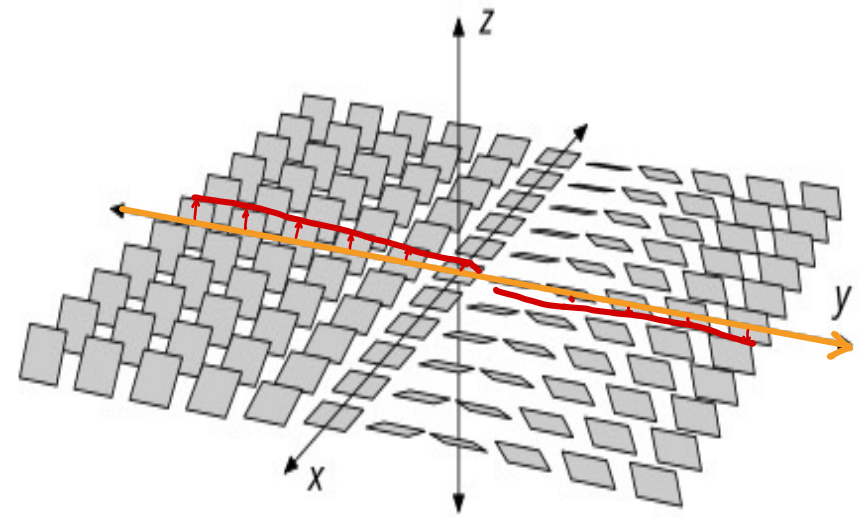
KNOTS IN CONTACT STRUCTURES

DEF: $L \subset M$ IS A LEGENDRIAN KNOT

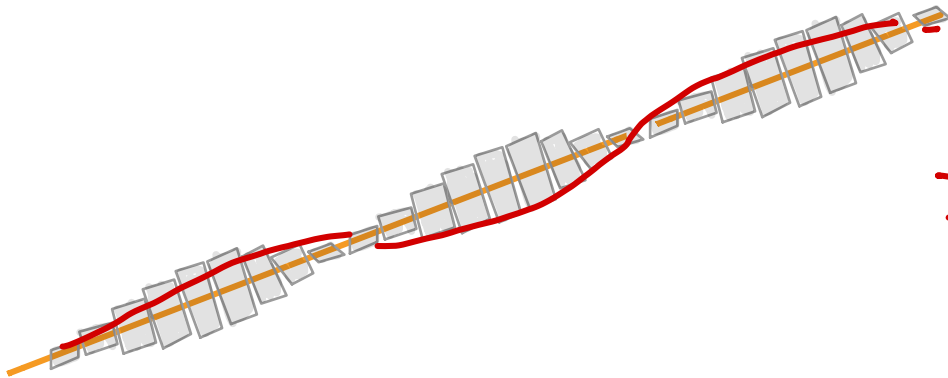
IF $T_p L \subset \xi_p \quad \forall p:$



MOTTO: THE CONTACT STRUCTURE
ALWAYS "ROTATES"
ALONG LEGENDRIANS



$$L = \{x = z = 0\}$$

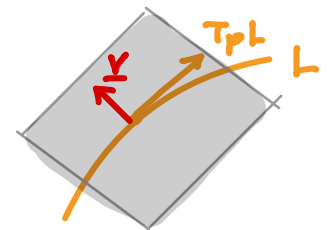


THURSTON-BENNEQUIN FRAMING:

PUSH L IN THE

DIRECTION OF $\underline{y}$

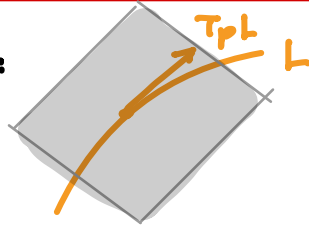
WHERE $\underline{y}_p \perp T_p L$ & $\underline{y}_p \in \xi_p$



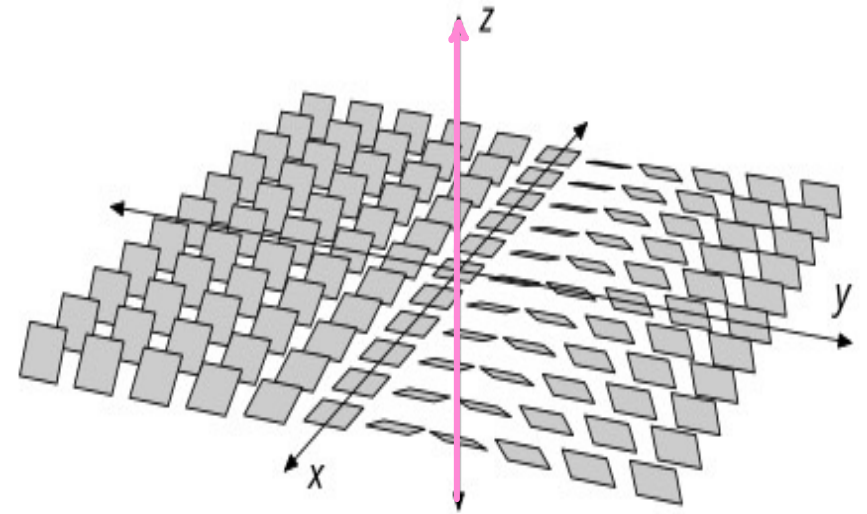
KNOTS IN CONTACT STRUCTURES

DEF: $L \subset M$ IS A LEGENDRIAN KNOT

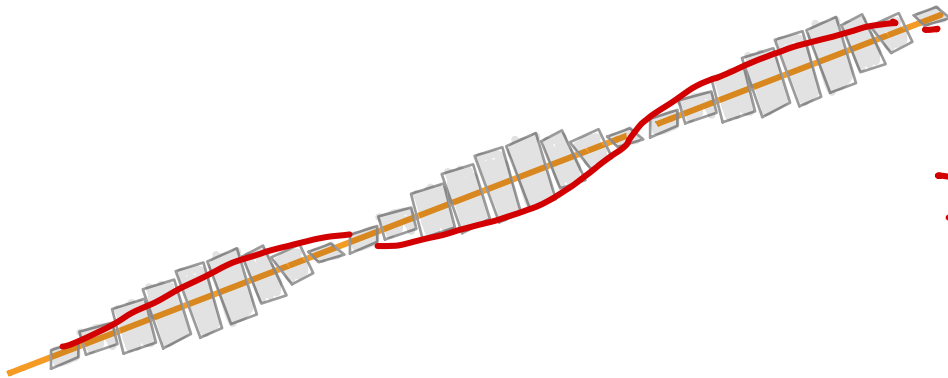
IF $T_p L \subset \xi_p \quad \forall p:$



MOTTO: THE CONTACT STRUCTURE
ALWAYS "ROTATES"
ALONG LEGENDRIANS



$$T = \{x=y=0\}$$

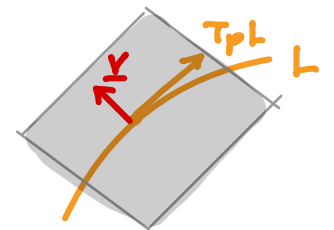


THURSTON-BENNEQUIN FRAMING:

PUSH L IN THE

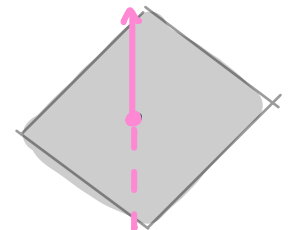
DIRECTION OF $\underline{y}$

WHERE $\underline{y}_p \perp T_p L$ & $\underline{y}_p \in \xi_p$



DEF: $T \subset M$ IS A TRANSVERSE KNOT IF $T_p T \nsubseteq \xi_p \quad \forall p:$

$$\alpha(T_p T) > 0$$

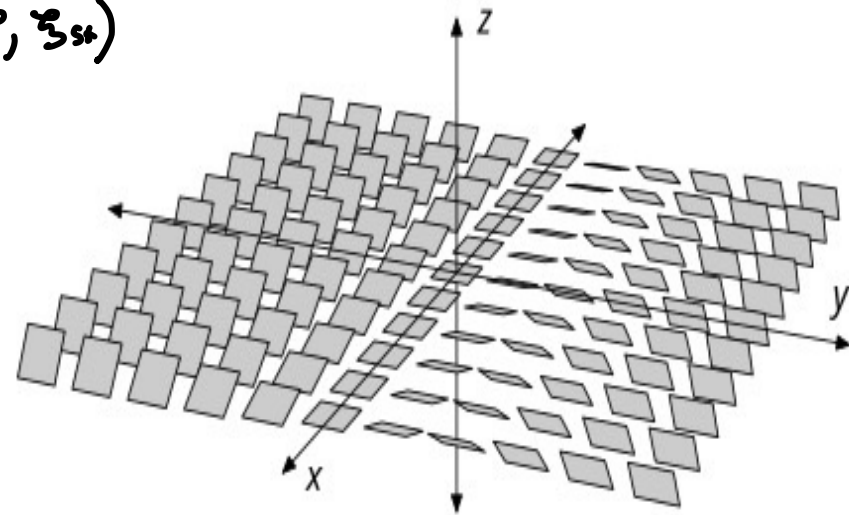


LEGENDRIAN APPROXIMATION

THM: ANY KNOT $K \hookrightarrow (M, \xi)$ CAN BE C^0 -APPROXIMATED BY A LEGENDRIAN KNOT

IDEA OF PROOF: ENOUGH TO APPROXIMATE LOCALLY & BY DARBOUX THM WE CAN WORK IN $(\mathbb{R}^3, \xi_{st})$

$$\xi_{st} = \ker(dz - ydx)$$



LEGENDRIAN APPROXIMATION

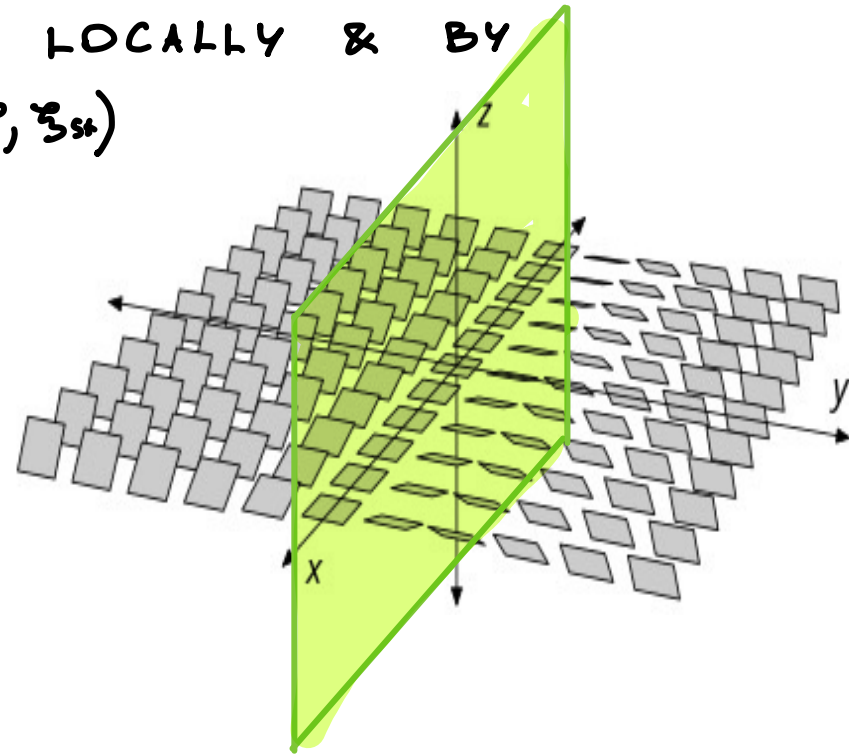
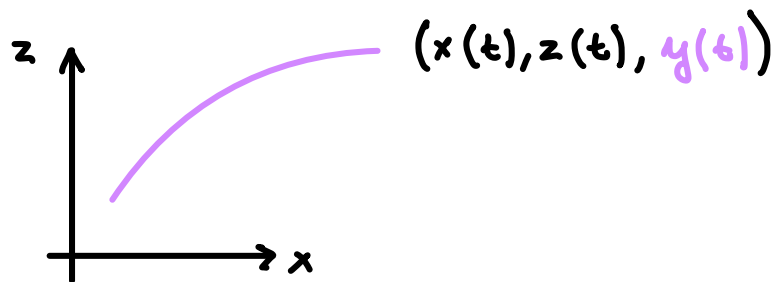
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IDEA OF PROOF: ENOUGH TO APPROXIMATE LOCALLY & BY DARBOUX THM WE CAN WORK IN $(\mathbb{R}^3, \xi_{st})$

$$\xi_{st} = \ker(dz - y dx) \iff "y = \frac{dz}{dx}"$$

WE CAN READ OFF y -COORDINATE FROM THE PROJECTION TO (x, z) -PLANE

- PROJECT K TO THE (x, z) -PLANE



LEGENDRIAN APPROXIMATION

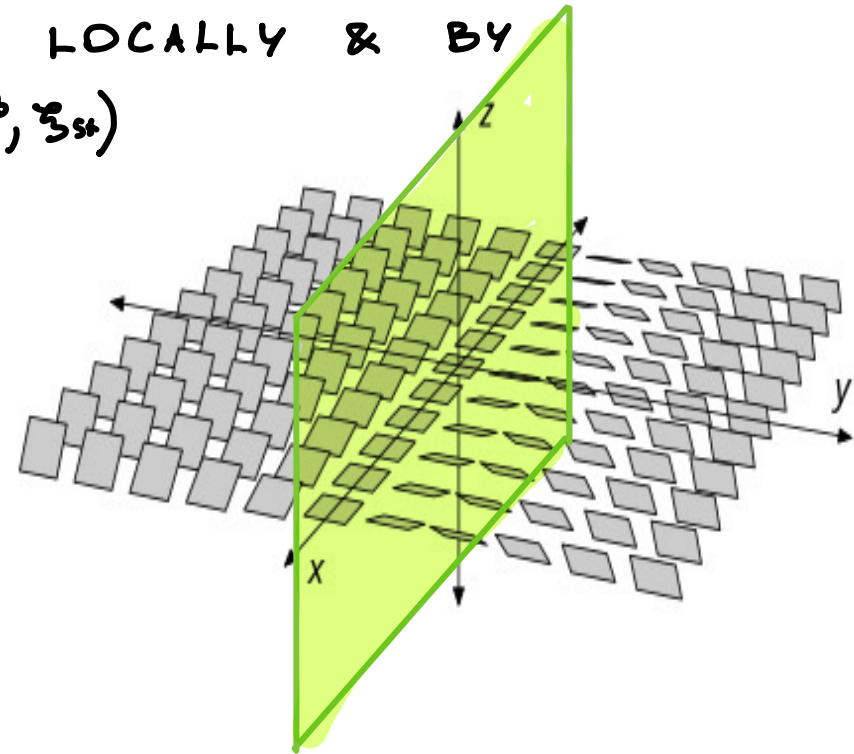
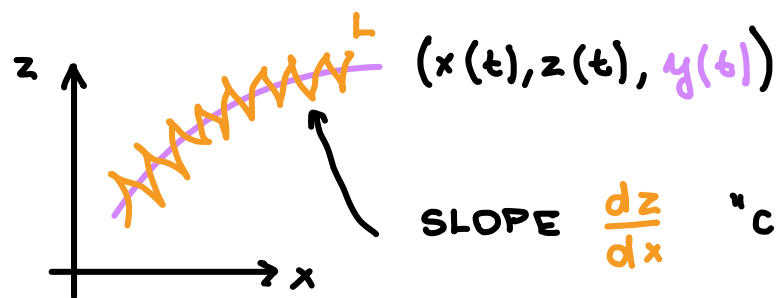
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IDEA OF PROOF: ENOUGH TO APPROXIMATE LOCALLY & BY DARBOUX THM WE CAN WORK IN $(\mathbb{R}^3, \xi_{st})$

$$\xi_{st} = \ker(dz - y dx) \iff "y = \frac{dz}{dx}"$$

WE CAN READ OFF y -COORDINATE FROM THE PROJECTION TO (x, z) -PLANE

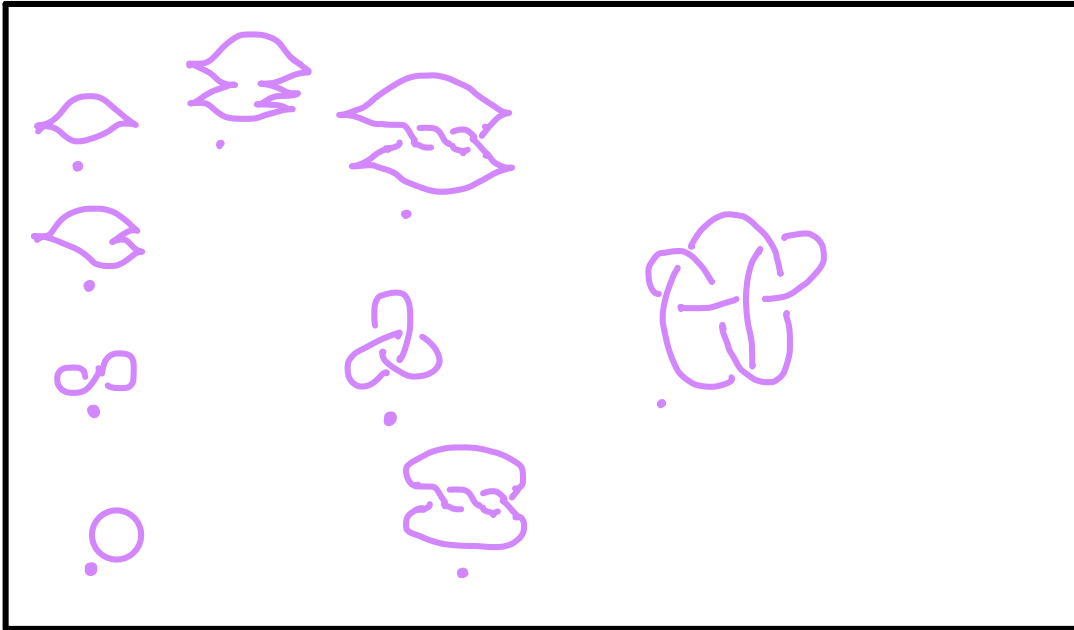
• PROJECT K TO THE (x, z) -PLANE



COR: ANY SMOOTH KNOT CAN BE REPRESENTED BY A LEGENDRIAN KNOT

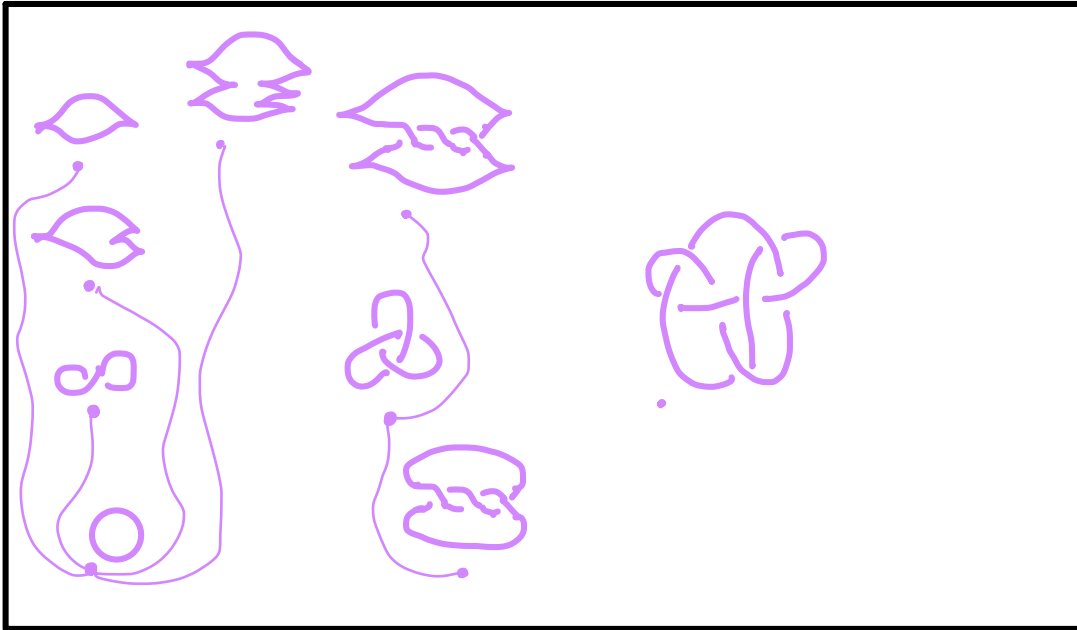
ISOTOPIES OF KNOTS

SMOOTH KNOTS



ISOTOPIES OF KNOTS

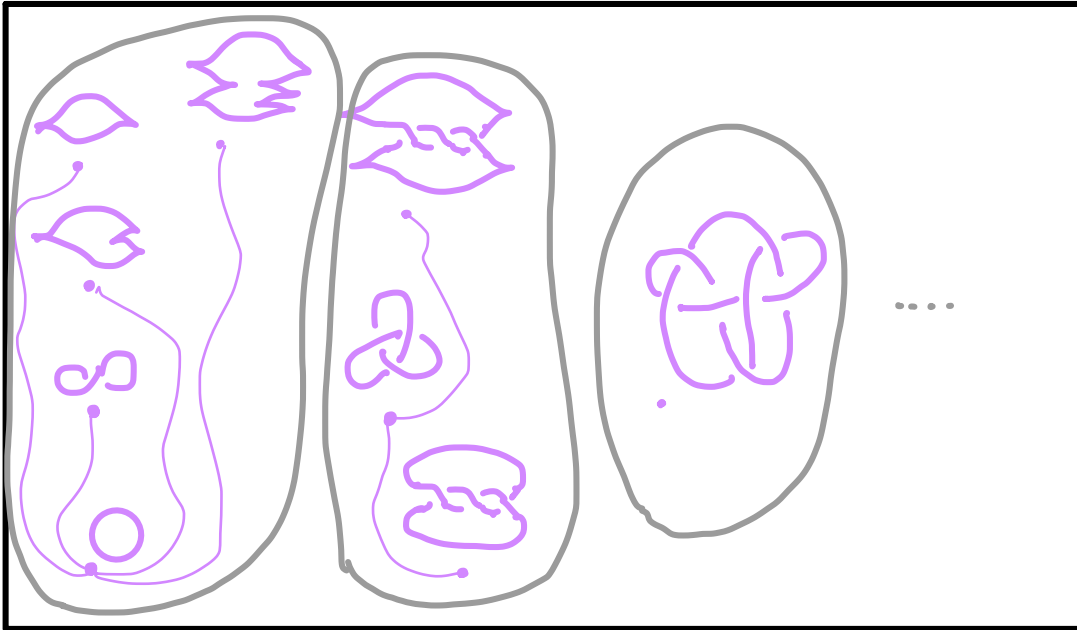
SMOOTH KNOTS



ISOTOPY: PATH IN THE
SPACE OF KNOTS

ISOTOPIES OF KNOTS

SMOOTH KNOTS

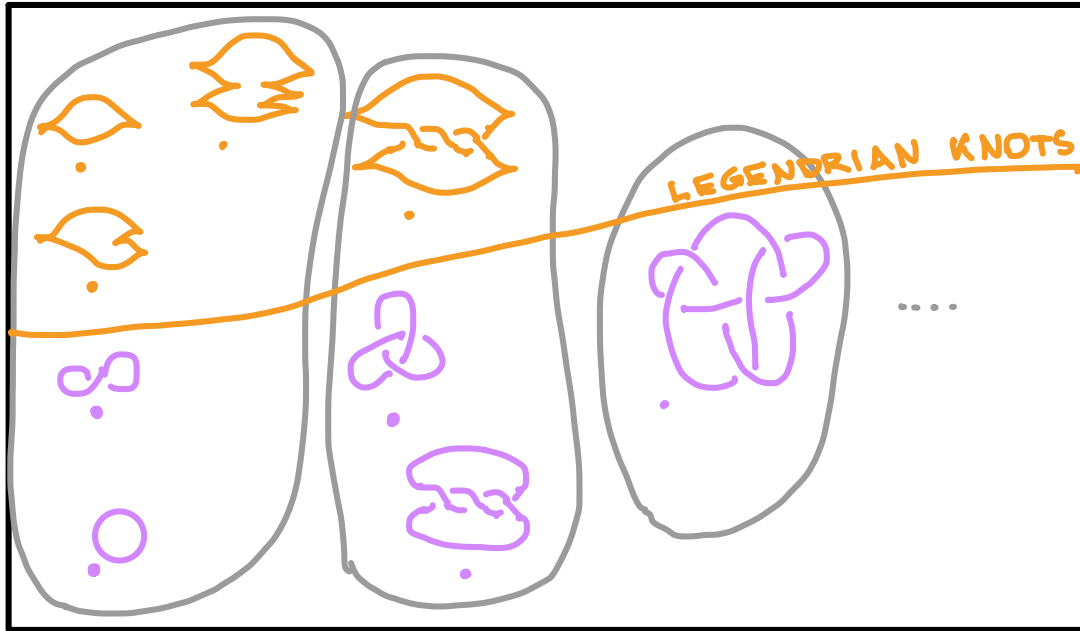


ISOTOPY: PATH IN THE
SPACE OF KNOTS

ISOTOPY CLASS: CONNECTED
COMPONENT

ISOTOPIES OF KNOTS

SMOOTH KNOTS

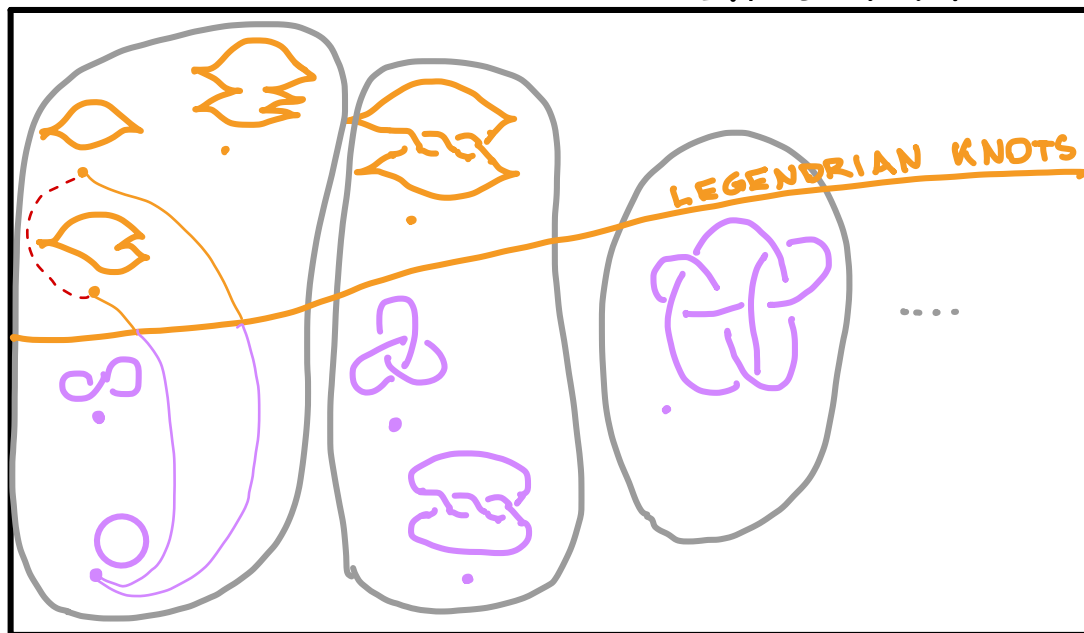


ISOTOPY: PATH IN THE
SPACE OF KNOTS

ISOTOPY CLASS: CONNECTED
COMPONENT

ISOTOPIES OF KNOTS

SMOOTH KNOTS



ISOTOPY: PATH IN THE
SPACE OF KNOTS

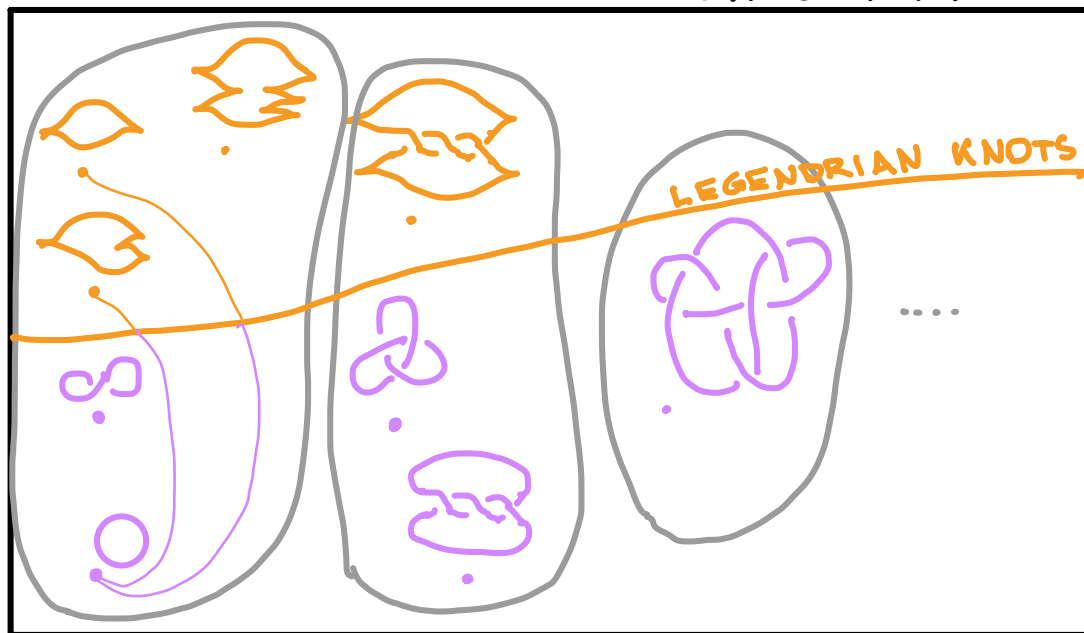
ISOTOPY CLASS: CONNECTED
COMPONENT

LEGENDRIAN ISOTOPY: PATH
IN THE SPACE OF LEGENDRIAN
KNOTS

IF L ISOTOPIC TO L' IMPLIES L LEGEDRIAN ISOTOPIC TO L' ?

ISOTOPIES OF KNOTS

SMOOTH KNOTS



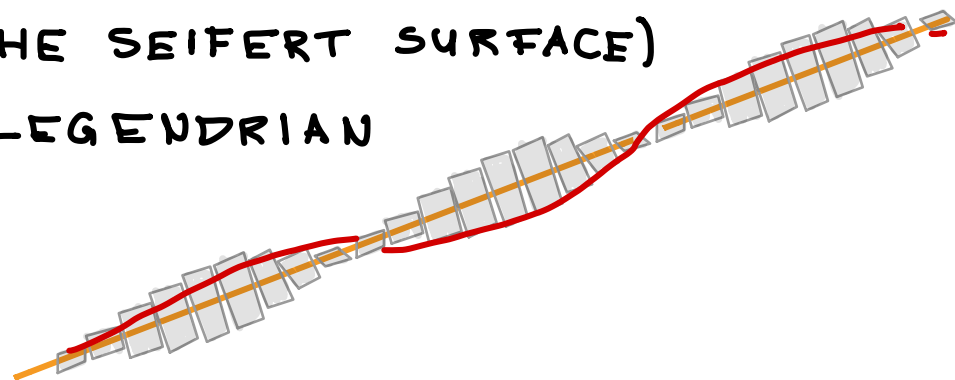
ISOTOPY: PATH IN THE SPACE OF KNOTS

ISOTOPY CLASS: CONNECTED COMPONENT

LEGENDRIAN ISOTOPY: PATH IN THE SPACE OF LEGENDRIAN KNOTS

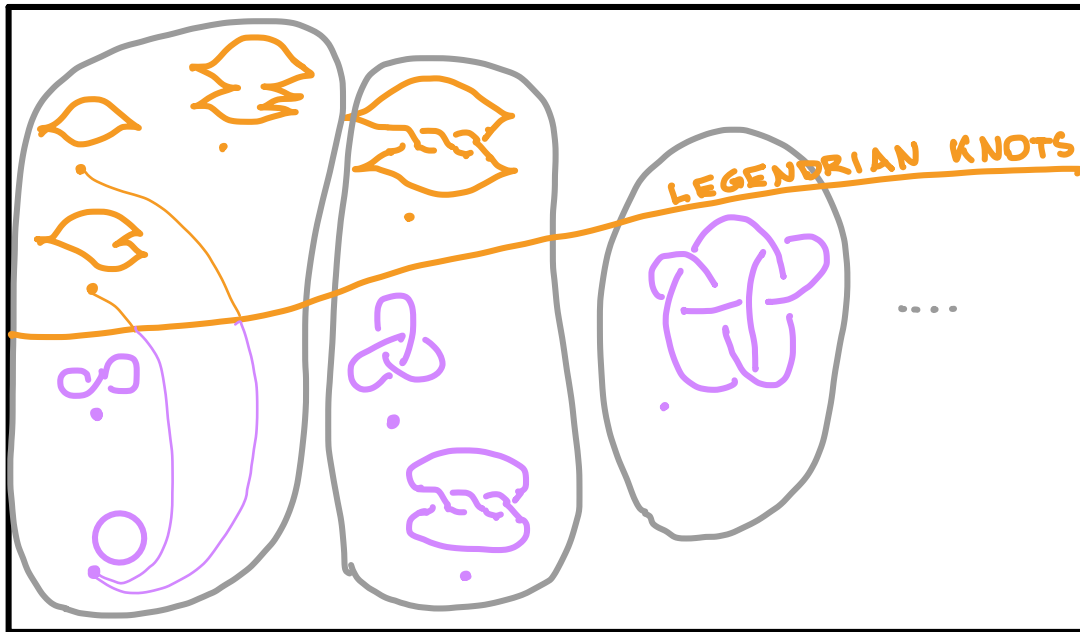
IF L ISOTOPIC TO L' IMPLIES L LEGENDRIAN ISOTOPIC TO L' ?

NO! - THE TWISTING OF $\mathfrak{z}$ (W.R.T THE SEIFERT SURFACE) DOESN'T CHANGE DURING LEGENDRIAN ISOTOPY



ISOTOPIES OF KNOTS

SMOOTH KNOTS



ISOTOPY: PATH IN THE SPACE OF KNOTS

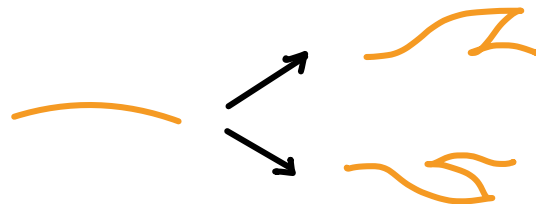
ISOTOPY CLASS: CONNECTED COMPONENT

LEGENDRIAN ISOTOPY: PATH IN THE SPACE OF LEGENDRIAN KNOTS

IF L ISOTOPIC TO L' IMPLIES L LEGENDRIAN ISOTOPIC TO L' ?

NO! - THE TWISTING OF $\mathfrak{z}$ (W.R.T THE SEIFERT SURFACE) DOESN'T CHANGE DURING LEGENDRIAN ISOTOPY

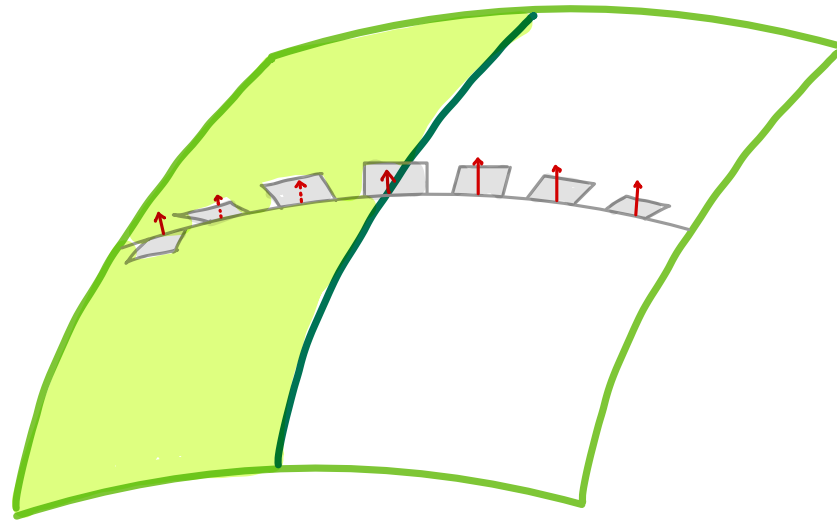
STABILISATION:



CHANGES TWISTING

THM (FUCHS-TABACHNIKOV): L IS ISOTOPIC TO $L' \iff$ AFTER SOME STABILISATIONS $St(L)$ IS LEGENDRIAN ISOTOPIC TO $St(L')$

2-DIM: SURFACES

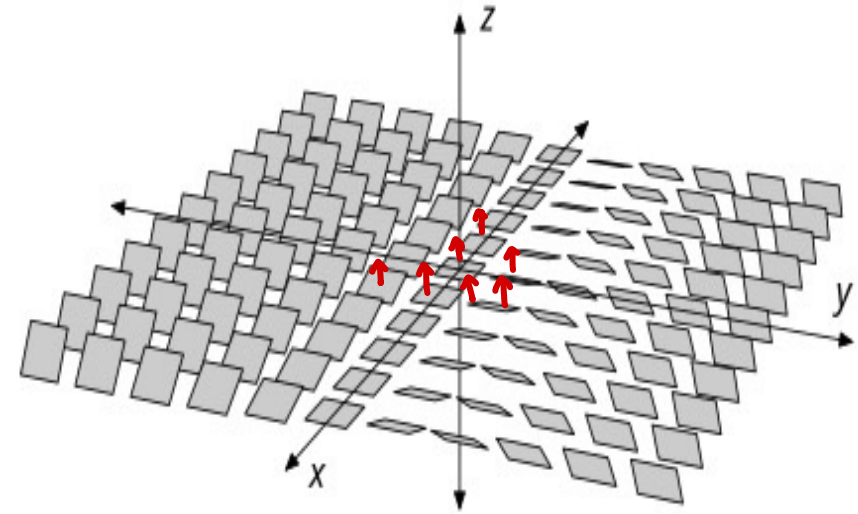


SURFACES IN CONTACT STRUCTURES

DEF: A CONTACT VECTORFIELD $X \in \mathfrak{X}(M)$
IS A VECTORFIELD WHOSE FLOW
PRESERVES ξ



$$\mathcal{L}_X \alpha = g\alpha \quad \text{FOR SOME } g: M \rightarrow \mathbb{R}$$



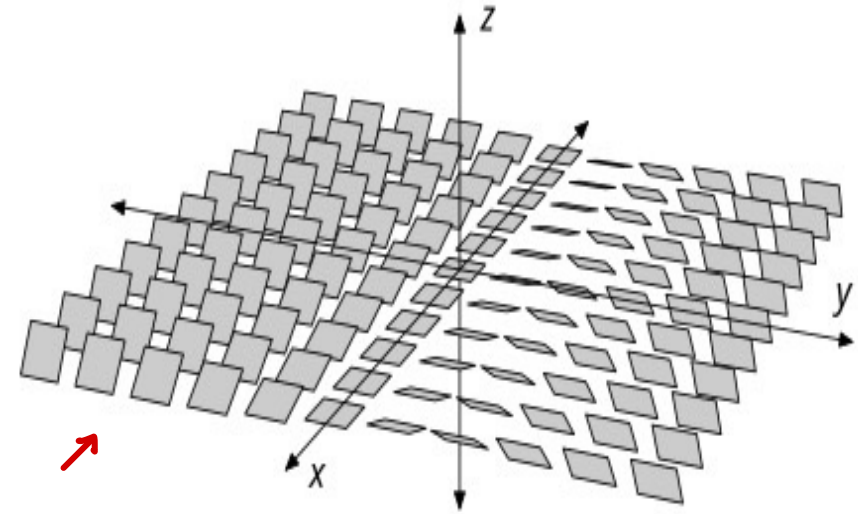
$$X = \frac{\partial}{\partial z}$$

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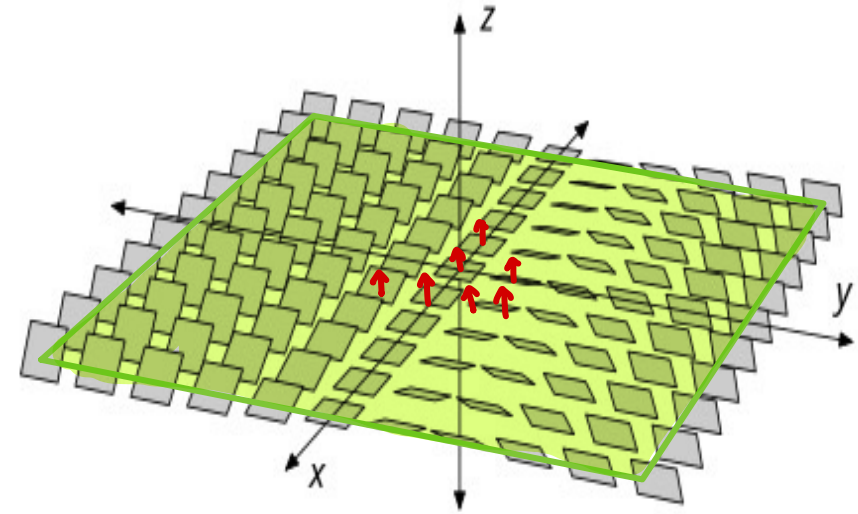
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DEF: $\Sigma \hookrightarrow M$ IS CONVEX IF $\exists$ X CONTACT VECTORFIELD $X \nmid \Sigma$



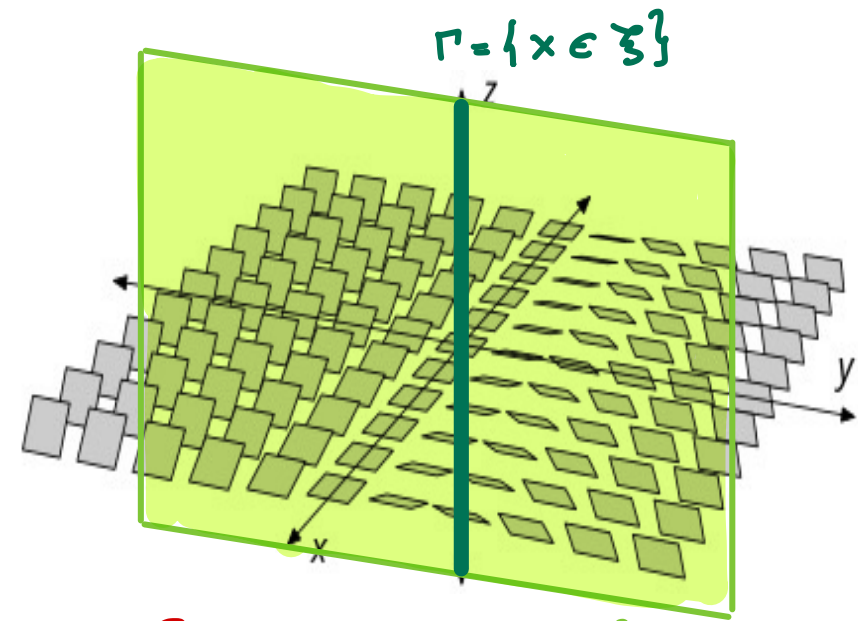
$$X = \frac{\partial}{\partial z} \quad \Sigma = \{z = 0\}$$

SURFACES IN CONTACT STRUCTURES

DEF: A CONTACT VECTORFIELD $X \in \mathfrak{X}(M)$
IS A VECTORFIELD WHOSE FLOW
PRESERVES ξ



$$\mathcal{L}_X \alpha = g \alpha \quad \text{FOR SOME } g: M \rightarrow \mathbb{R}$$



$$X = \frac{\partial}{\partial x} \quad \Sigma = \{x = 0\}$$

DEF: $\Sigma \hookrightarrow M$ IS CONVEX IF $\exists$ X CONTACT VECTORFIELD $X \pitchfork \Sigma$

EQUIVALENTLY: Σ HAS A NEIGHBOURHOOD $N(\Sigma) \cong \Sigma \times I$ WITH
I-INVARIANT CONTACT STRUCTURE ξ



$$\alpha = \beta + g dt \quad \text{WHERE } \beta \in \Omega^1(\Sigma) \text{ \& } g: M \rightarrow \mathbb{R}$$

FACT (GIROUX): TO UNDERSTAND ξ ON $N(\Sigma)$ ONE ONLY NEEDS
TO KNOW $\Gamma = \{g = 0\} = \{\frac{\partial}{\partial t} \in \xi\} = \{X \in \xi\}$

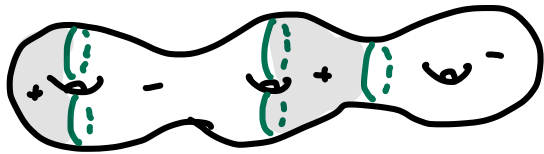
CONVEX SURFACES (GIROUX)

DEF: $\Sigma \hookrightarrow M$ IS CONVEX IF $\exists$ CONTACT VECTORFIELD $X : \Sigma \nrightarrow X$

DEF: $\Gamma = \{X \in \xi\} = \{\alpha(X) = 0\} \subset \Sigma$ IS THE DIVIDING CURVE

PROP' - THE ISOTOPY CLASS OF Γ IS INDEPENDENT OF THE CHOICE OF X

- Γ DIVIDES Σ INTO TWO PIECES: $\Sigma_+ = \{\alpha(X) > 0\}$
 $\Sigma_- = \{\alpha(X) < 0\}$



THM (THE DIVIDING CURVE DETERMINES ξ NEAR Σ)

Σ, Σ' CONVEX SURFACES W/ ISOTOPIC DIVIDING CURVES
 $\implies \exists N(\Sigma), N(\Sigma')$ NEIGHBOURHOODS THAT ARE
CONTACTOMORPHIC

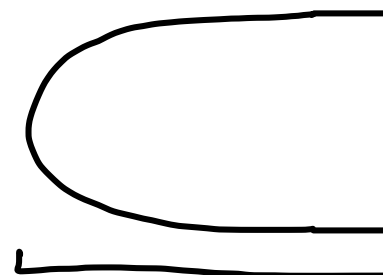
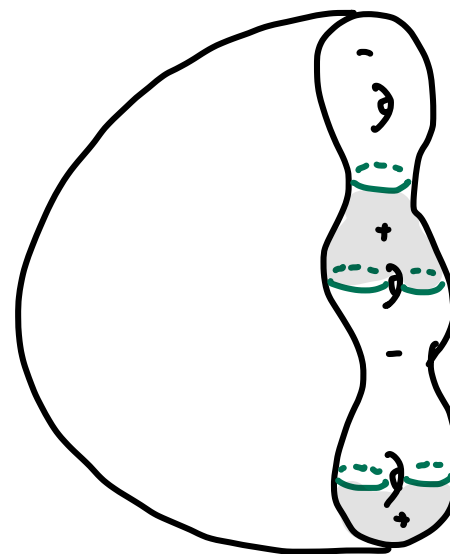
THM (CONVEX SURFACES ARE C^∞ -GENERIC): ANY SURFACE Σ CAN
BE C^∞ -SMALL ISOTOPED TO BE CONVEX

CONTACT MANIFOLDS WITH BOUNDARY

DEF: $(\Sigma_t)_{t \in [0,1]}$ IS A CONVEX ISOTOPY IF Σ_t IS CONVEX $(\forall t \in [0,1])$

WE WILL WORK WITH

M^3 3-MANIFOLD WITH BOUNDARY,
 ξ CONTACT STRUCTURE ON M , S.T
 ∂M IS CONVEX



(M, ξ)

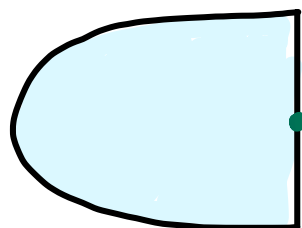
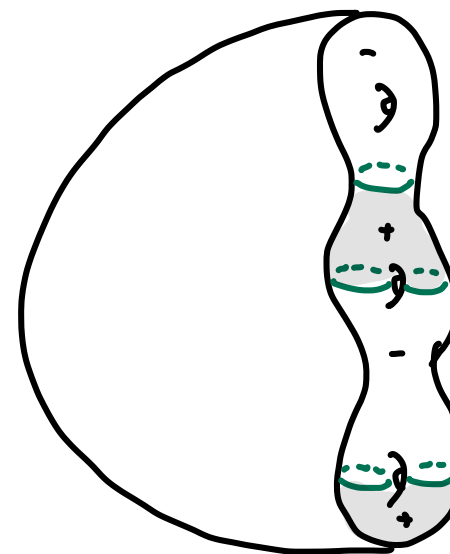
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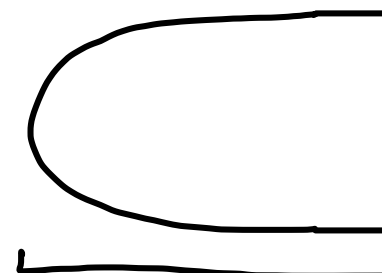
WE WILL WORK WITH

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 ξ CONTACT STRUCTURE ON M , S.T
 ∂M IS CONVEX

• SAME FOR (M', ξ')



(M', ξ')



(M, ξ)

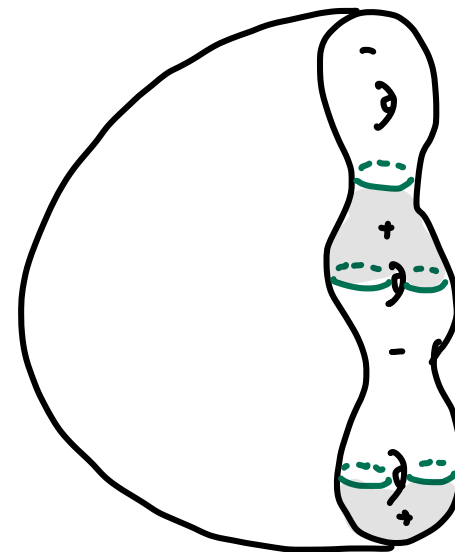
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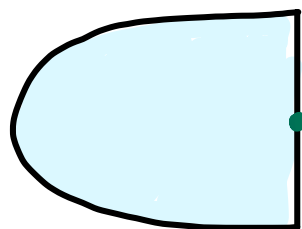
WE WILL WORK WITH

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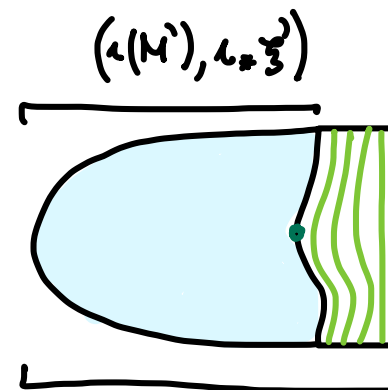
• SAME FOR (M', ξ')



DEF: WEAKLY CONTACTOMORPHIC: $\exists$ EMBEDDING $(M', \xi') \xhookrightarrow{\iota} (M, \xi)$
 SUCH THAT $\iota(\partial M')$ IS CONVEX ISOTOPIC TO ∂M



(M', ξ')

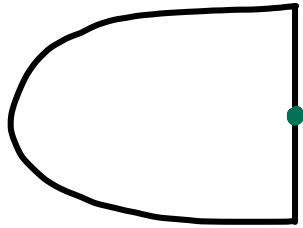


(M, ξ)

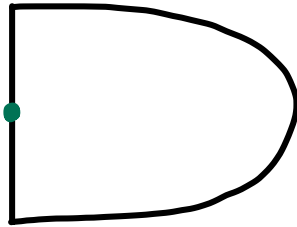
GLUING CONTACT STRUCTURES

WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH
MATCHING DIVIDING CURVES

IDEA



(M^L, ξ^L)

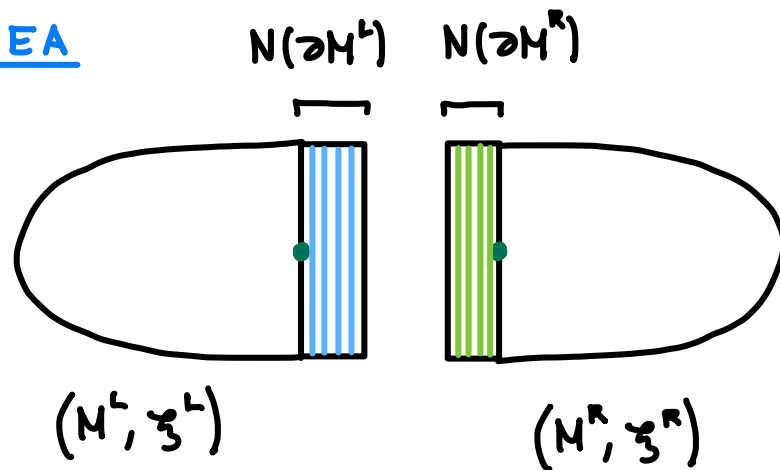


(M^R, ξ^R)

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IDEA

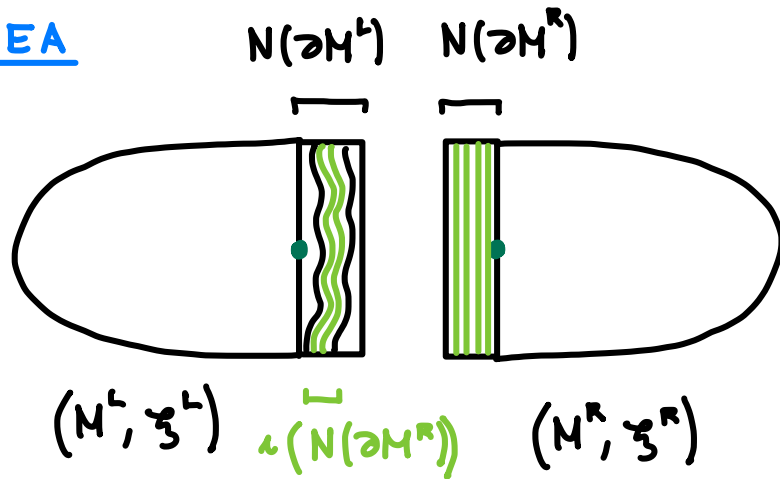


STEP 1: ADD I-INVARIANT PART TO EACH

GLUING CONTACT STRUCTURES

WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH
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IDEA



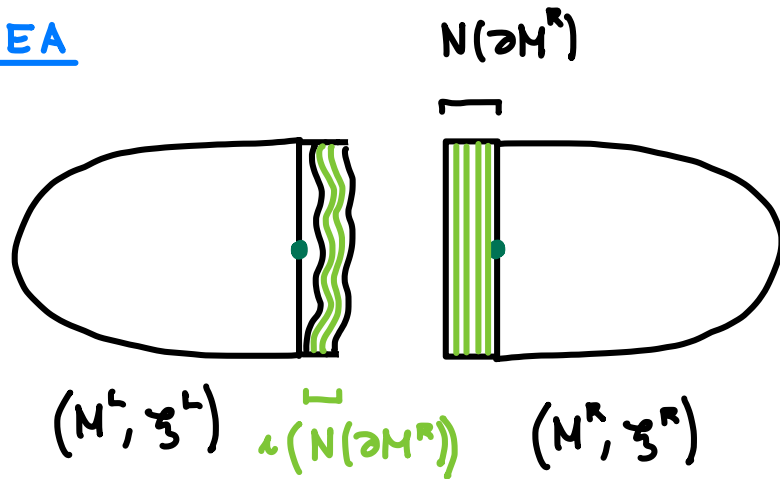
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STEP 2: FIND $N(\partial M^R)$ IN $N(\partial M^L)$

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IDEA



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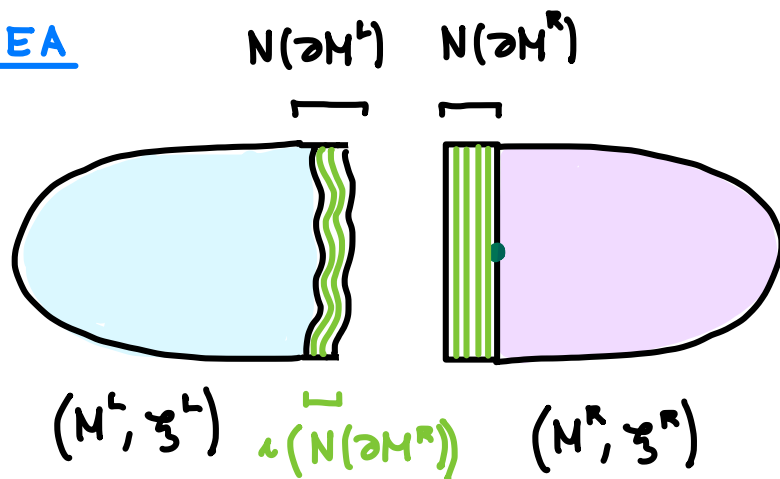
STEP 2: FIND $N(\partial M^R)$ IN $N(\partial M^L)$

STEP 3: TRUNCATE M^L AT $\iota(\partial M^R \times \{1\})$

GLUING CONTACT STRUCTURES

WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH MATCHING DIVIDING CURVES

IDEA

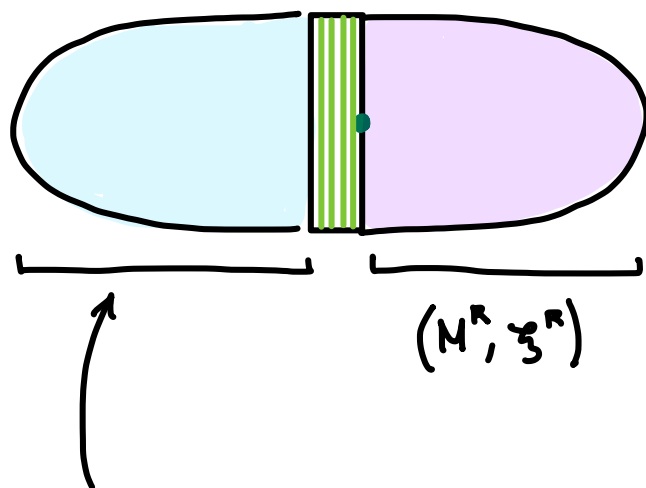


STEP 1: ADD I-INVARIANT PART TO EACH

STEP 2: FIND $N(\partial M^R)$ IN $N(\partial M^L)$

STEP 3: TRUNCATE M^L AT $\iota(\partial M^R \times \{1\})$

STEP 4: OVERLAP $\iota(N(\partial M^R))$ WITH $N(\partial M^R)$



THE OBTAINED CONTACT MANIFOLD IS $(M^L, \xi^L) \cup (M^R, \xi^R)$

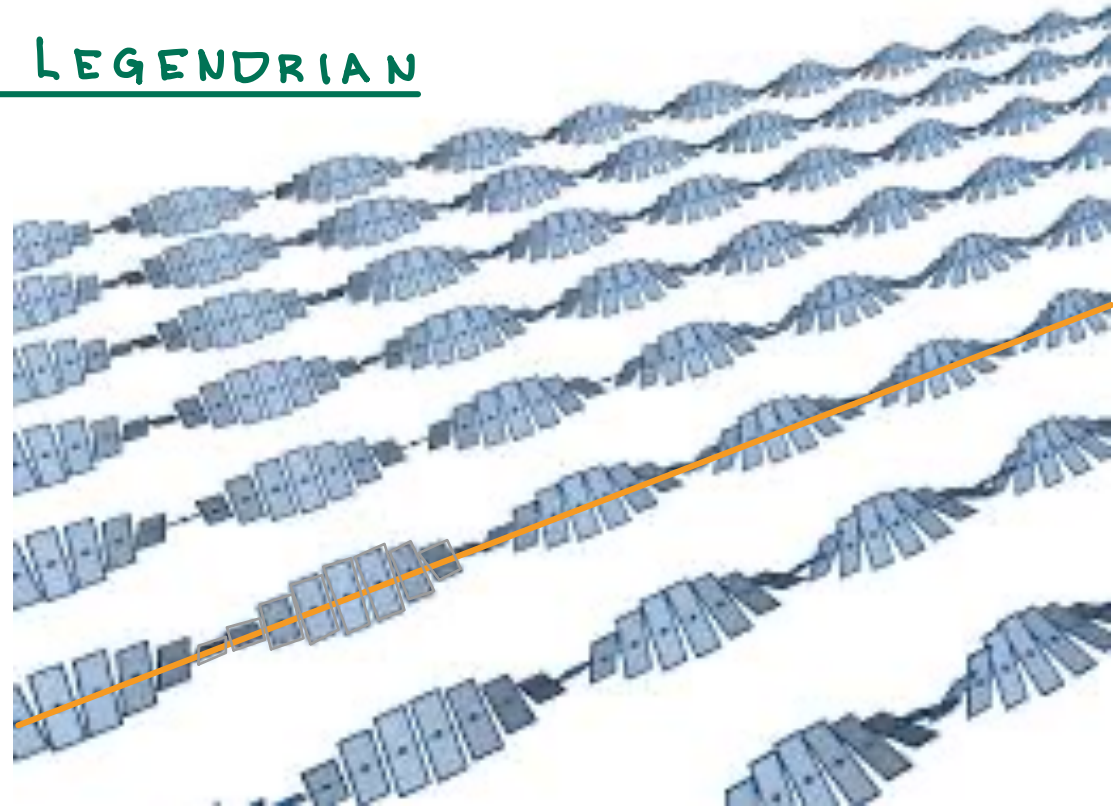
& IT IS WELL DEFINED UP TO CONTACTOMORPHISM

WEAKLY CONTACT ISOTOPIC TO (M^L, ξ^L)

STANDARD NEIGHBOURHOOD OF A LEGENDRIAN

E.G.: $\xi = \ker (\cos(z) dx - \sin(z) dy)$

(ISOTOPIC TO ξ_{st})



STANDARD NEIGHBOURHOOD OF A LEGENDRIAN

E.G.: $\mathfrak{z} = \ker (\cos(z) dx - \sin(z) dy)$

(ISOTOPIC TO $\mathfrak{z}_{st}$)

IDENTIFY $(x, y, z) \sim (x, y, z + 2\pi n)$

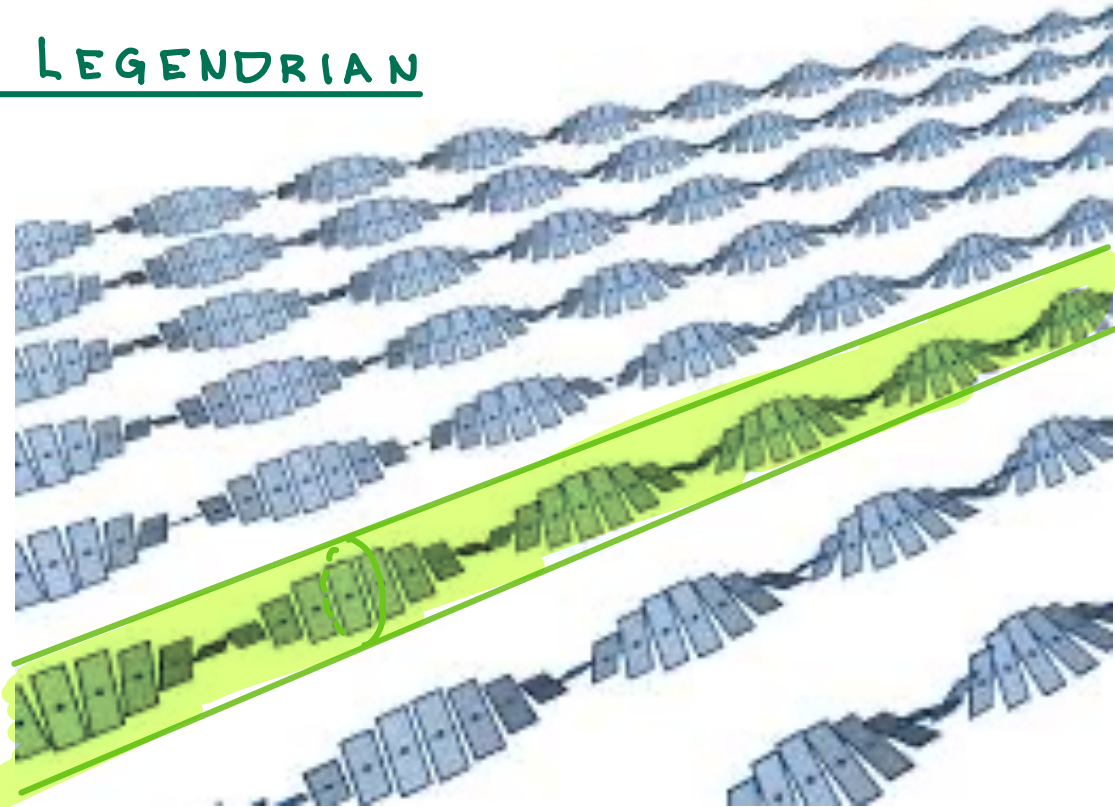
$\rightsquigarrow$ CONTACT STRUCTURE

ON $\mathbb{R}^2 \times S^1$

WITH LEGENDRIAN KNOT

$$L_0 = (0, 0) \times S^1 \subset \mathbb{R}^2 \times S^1$$

NEIGHBOURHOOD $N(L) \cong D^2 \times S^1$



STANDARD NEIGHBOURHOOD OF A LEGENDRIAN

E.G.: $\mathfrak{g} = kw (\cos(z) dx - \sin(z) dy)$

(ISOTOPIC TO $\mathfrak{g}_{st}$)

IDENTIFY $(x, y, z) \sim (x, y, z + 2\pi n)$

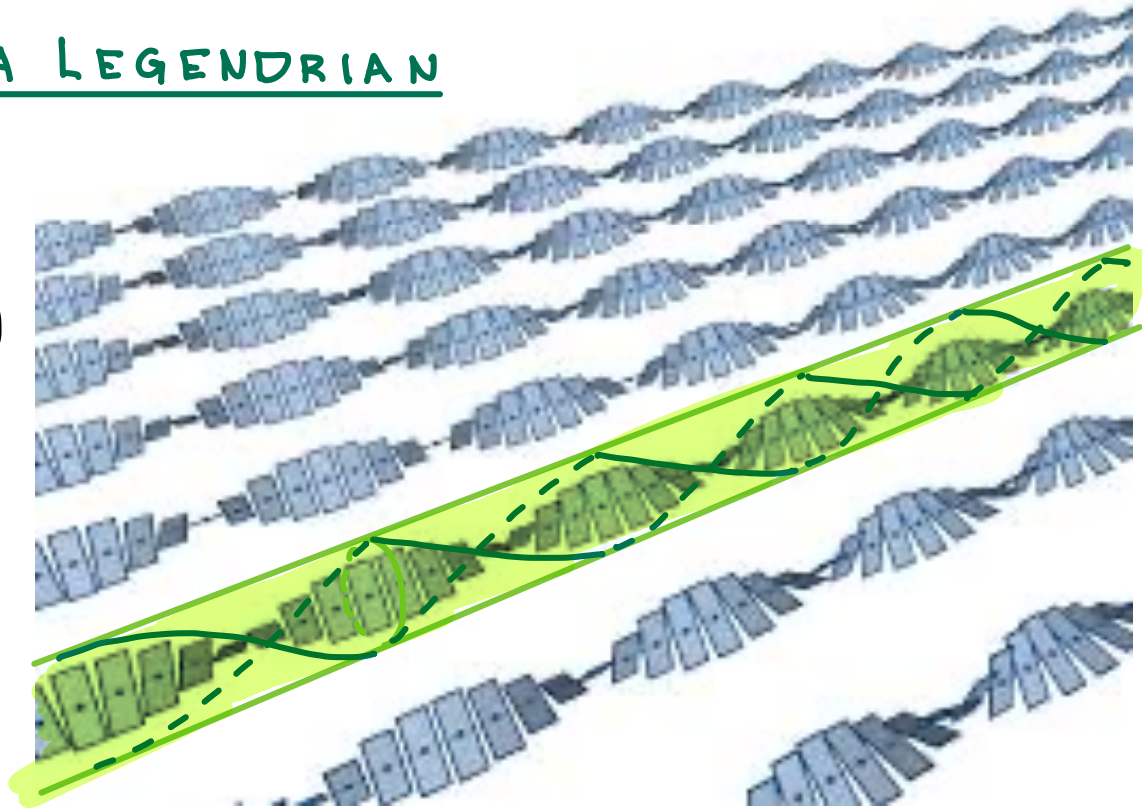
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RNK: $\partial N(L_0)$ IS NOT CONVEX, BUT BY A C^∞ -SMALL ISOTOPY
IT CAN BE MADE CONVEX WITH A TWO COMPONENT
DIVIDING CURVE PARALLEL TO $(\sin(z), \cos(z), z) / \sim$
 $\uparrow$
THIS GIVES THE THURSTON-BENNEQUIN FRAMING

STANDARD NEIGHBOURHOOD OF A LEGENDRIAN

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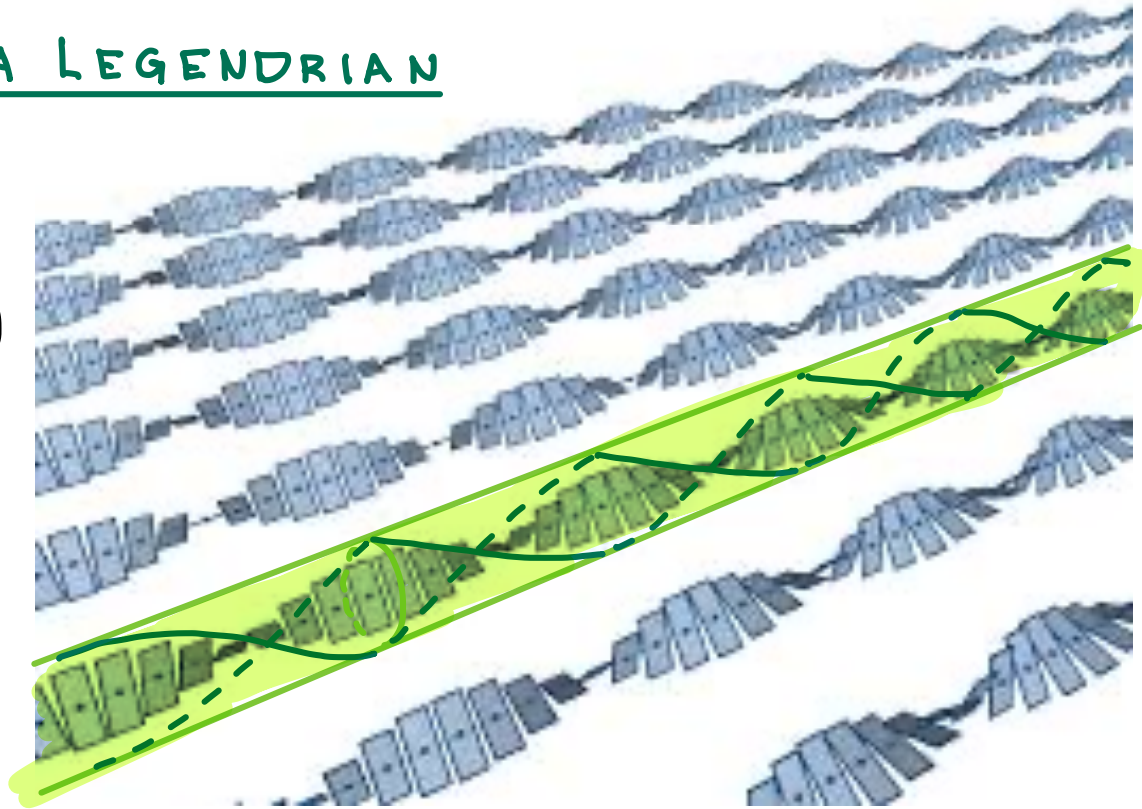
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DIVIDING CURVE PARALLEL TO $(\sin(z), \cos(z), z) / \sim$

THIS GIVES THE THURSTON-BENNEQUIN FRAMING

THM: ANY LEGENDRIAN KNOT $L \subset (M, \xi)$ HAS A NEIGHBOURHOOD $N(L)$
CONTACTOMORPHIC TO $N(L_0)$

LEGENDRIANS ON CONVEX SURFACES

DEF: $C \subset (\Sigma, \Gamma)$ IS AN ISOLATING CURVE, IF SOME COMPONENT OF $\Sigma \setminus C$ IS DISJOINT FROM Γ :



THM (LEGENDRIAN REALISATION PRINCIPLE)

(Σ, Γ) CONVEX SURFACE, $C \subset \Sigma$ NON-ISOLATING CURVE

$\implies \Sigma$ CAN BE ISOTOPED THROUGH CONVEX SURFACES $\Psi_t(\Sigma)$
S.T. AFTER THE ISOTOPY $\Psi_1(C) \subset \Psi_1(\Sigma)$ IS LEGENDRIAN

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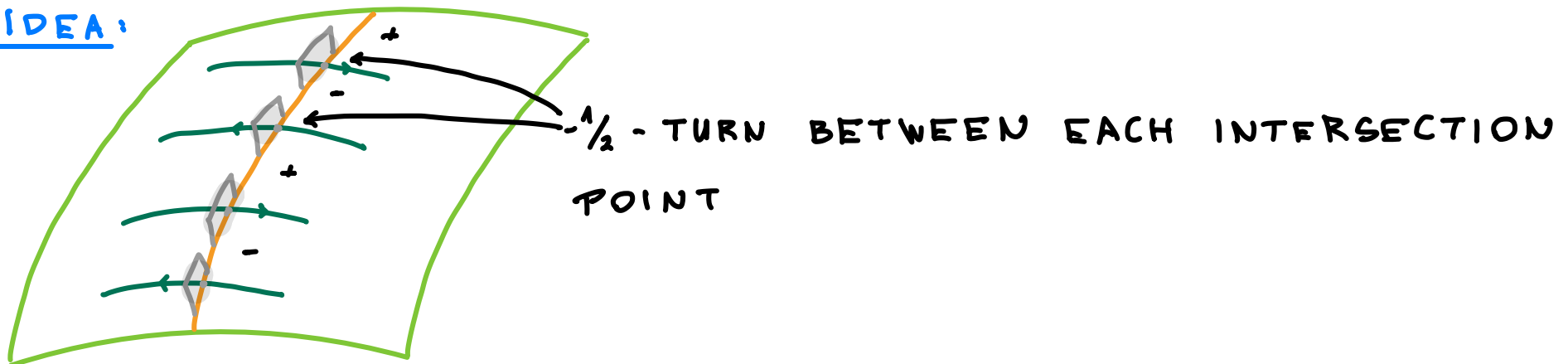
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RMK: THE TWISTING OF γ W.R.T. $T\Sigma$ ALONG $C = -\frac{1}{2} |C \cap \Gamma|$

IDEA:



CONVEX SURFACES WITH LEGENDRIAN BOUNDARY

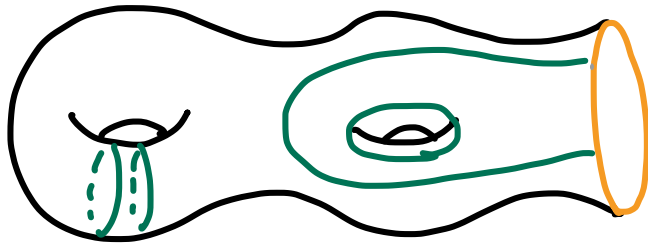
THM (KANDA): Σ SURFACE WITH LEGENDRIAN BOUNDARY L CAN
BE ISOTOPED REL ∂ TO BE CONVEX



TWISTING OF ξ W.R.T. Σ ALONG L IS ≤ 0

- THE ISOTOPY CAN BE ASSUMED TO BE C^0 -SMALL
(C^∞ -SMALL IF Σ IS ALREADY CONVEX NEAR $\partial\Sigma$)

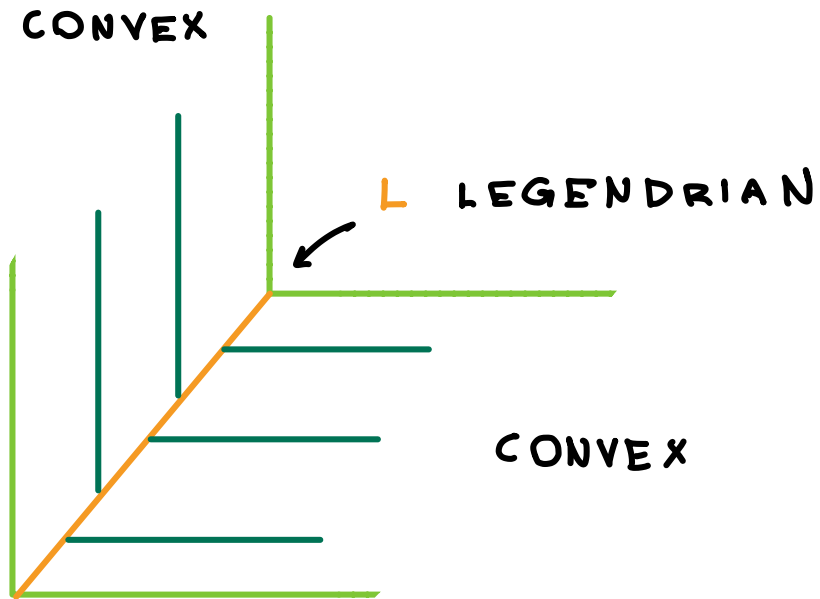
RMK: AFTER THE ISOTOPY



TWISTING OF ξ W.R.T. Σ ALONG $L = -\frac{1}{2} |\pi \cap L|$

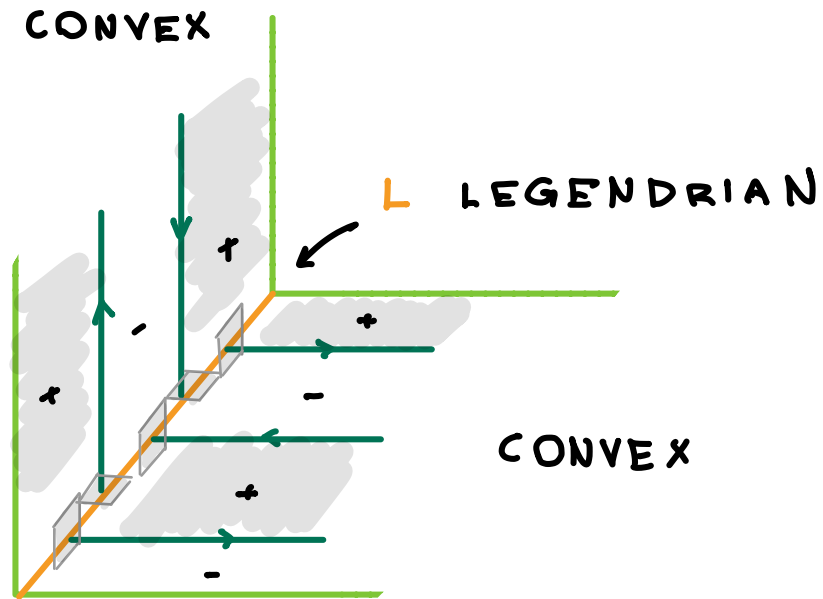
ROUNDING EDGES

Σ_1 & Σ_2 CONVEX SURFACES WITH COMMON LEGENDRIAN BOUNDARY L



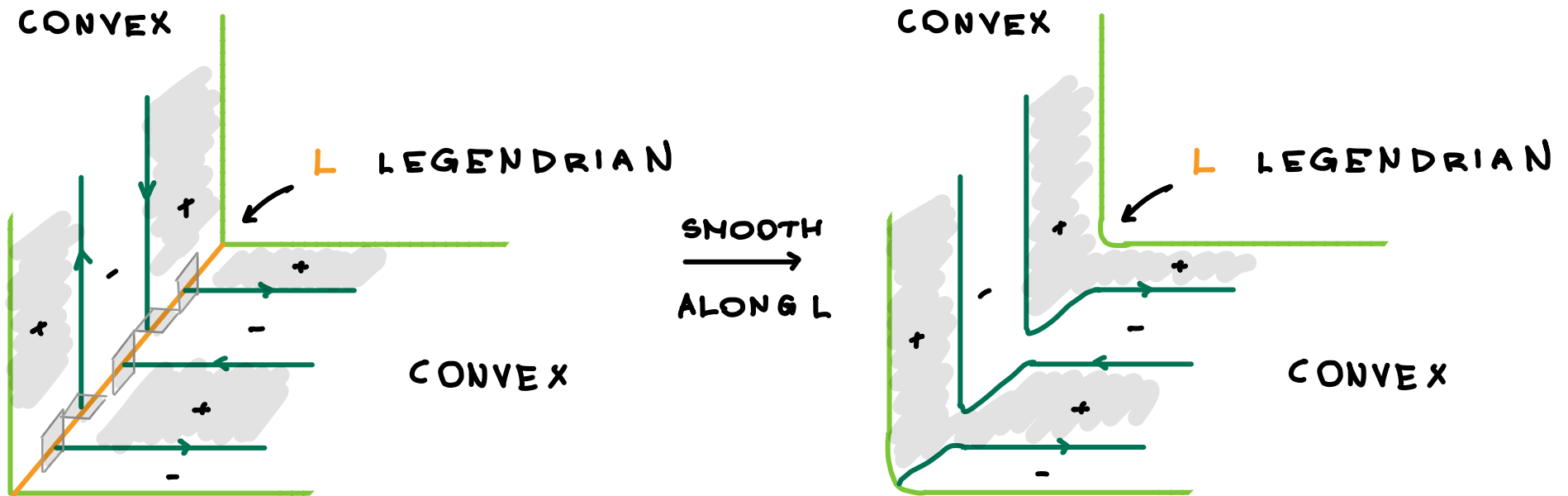
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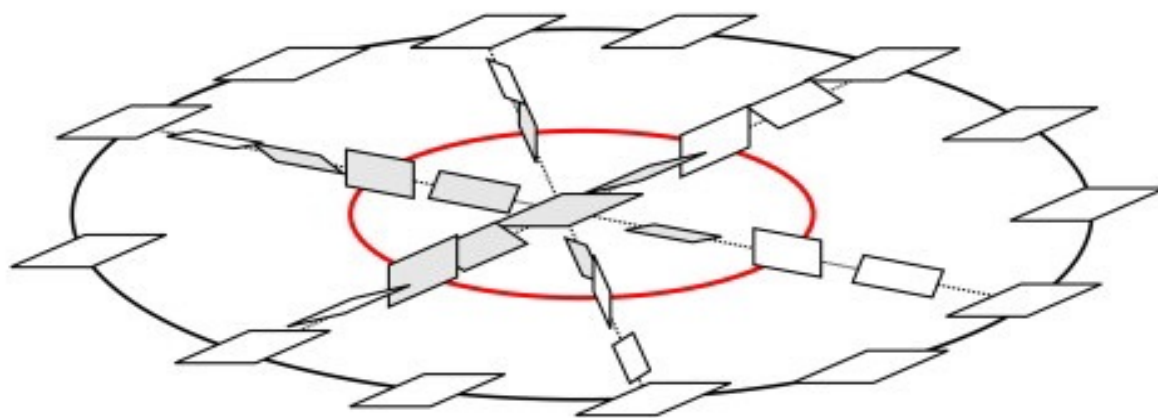
ROUNDING EDGES

Σ_1 & Σ_2 CONVEX SURFACES WITH COMMON LEGENDRIAN BOUNDARY L
THEN THE EDGE L CAN BE ROUNDED & WE GET A NEW
SMOOTH CONVEX SURFACE Z WITH DIVIDING CURVE AS BELOW



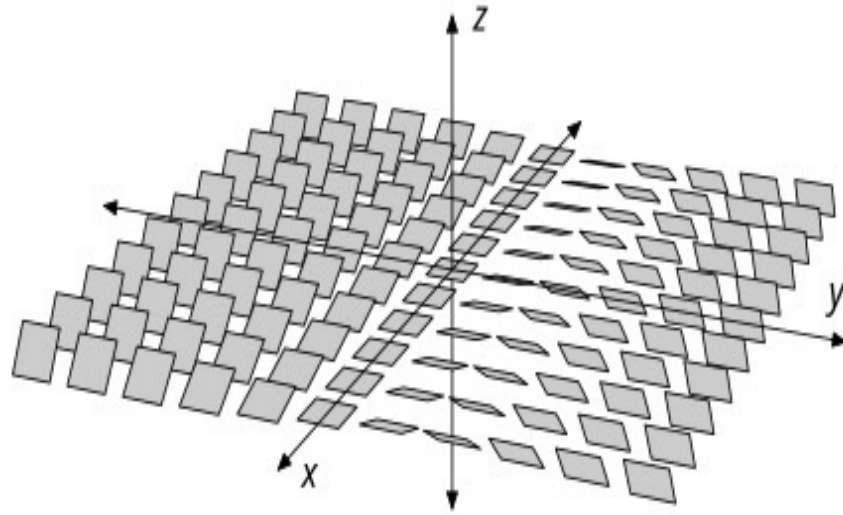
TIGHT
&

OVERTWISTED



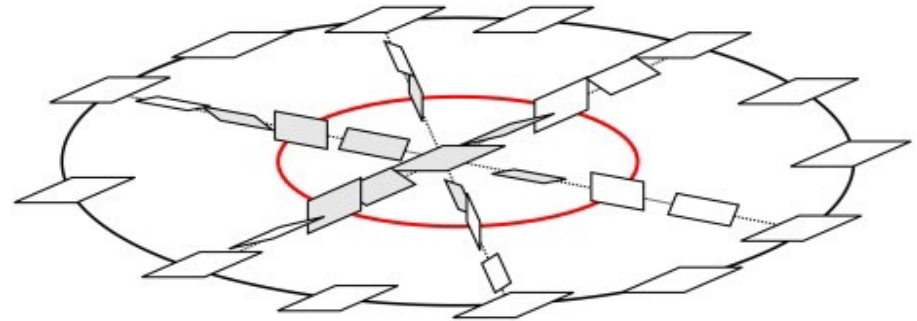
TWO CONTACT STRUCTURES

STANDARD CONTACT STRUCTURE



$$\xi_{st} = \ker(dz - ydx)$$

OVERTWISTED CONTACT STRUCTURE

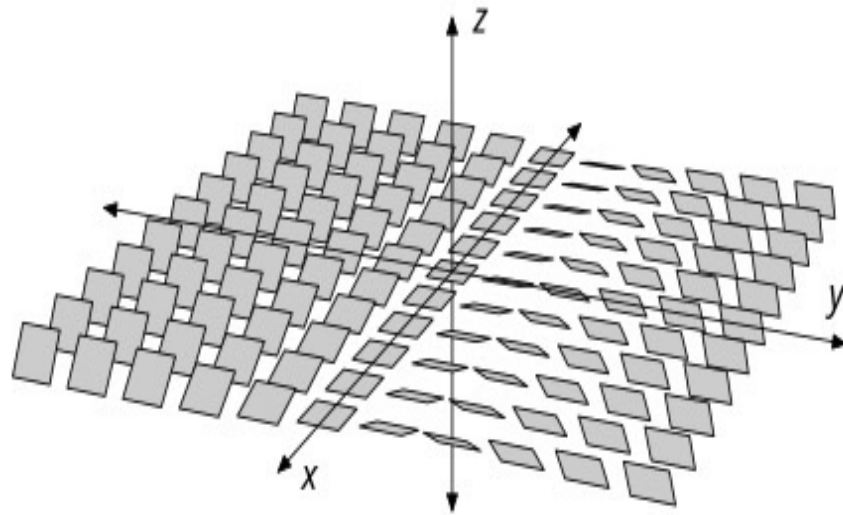


$$\xi_{ot} = \ker(\cos(r)dz + r\sin(r)d\varphi)$$

ARE ξ_{st} & ξ_{ot} ISOTOPIC/CONTACTOMORPHIC?

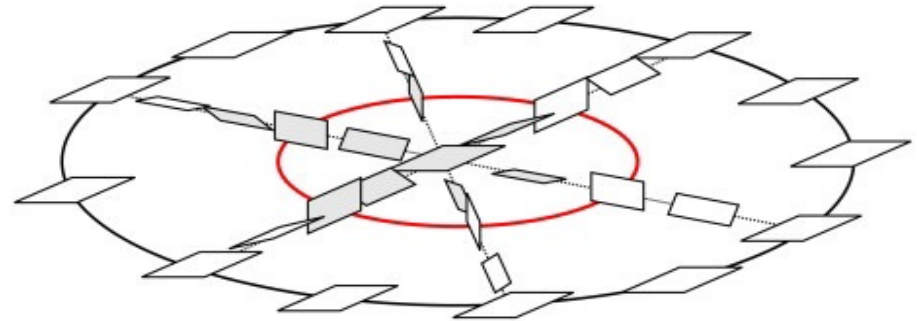
TWO CONTACT STRUCTURES

STANDARD CONTACT STRUCTURE



$$\xi_{st} = \ker(dz - y dx)$$

OVERTWISTED CONTACT STRUCTURE



$$\xi_{ot} = \ker(\cos(r) dz + r \sin(r) d\varphi)$$

ARE ξ_{st} & ξ_{ot} ISOTOPIC/CONTACTOMORPHIC?

BENNEQUIN (1982): NO!

CONSIDER $D = \{r \leq 2\pi\} \times \{0\}$

DEF: $D \hookrightarrow (M, \xi)$ IS AN OVERTWISTED DISK IF $TD|_{\partial D} = \xi|_{\partial D}$

RMK: ENOUGH TWISTING OF ξ ALONG ∂D W.R.T D IS 0

DEF: ξ IS OVERTWISTED IF ξ CONTAINS AN OVERTWISTED DISK

ξ IS TIGHT IF IT IS NOT OVERTWISTED

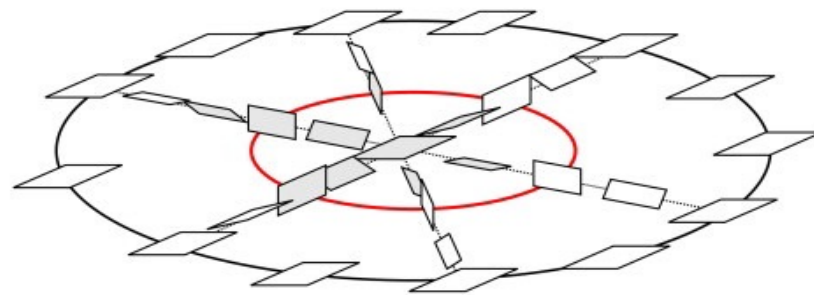
OVERTWISTED CONTACT STRUCTURES

THM (ELIASHBERG) $\mathfrak{F}, \mathfrak{F}'$ OVERTWISTED

CONTACT STRUCTURES &

$\mathfrak{F} \approx \mathfrak{F}'$ AS PLANEFIELDS

$\implies \mathfrak{F} \approx \mathfrak{F}'$



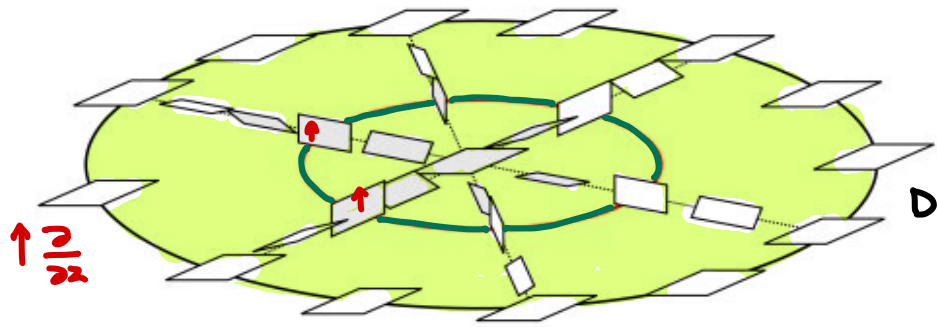
THM (LUTZ-MARTINEZ) ANY HOMOTOPY CLASS OF PLANEFIELDS
IS REPRESENTED BY A(N OVERTWISTED) CONTACT STRUCTURE

OVERTWISTED CONTACT STRUCTURES CAN BE UNDERSTOOD
THROUGH ALGEBRAIC TOPOLOGICAL INVARIANTS : d_2, d_3

RMK: TIGHT CONTACT STRUCTURES ARE HARDER TO CLASSIFY

- ELIASHBERG: S^3 ADMITS A UNIQUE TIGHT CONTACT STRUCTURE
- GIROUX: $\exists$ ∞ -LY MANY 3-MANIFOLDS WITH ∞ -LY MANY TIGHT CONTACT STRUCTURES
- ETNYRE: $-\Sigma(2,3,5)$ ADMITS NO TIGHT CONTACT STRUCTURES
- ...

RECOGNISING OVERTWISTED CONTACT STRUCTURES



D IS CONVEX : $X = \frac{\partial}{\partial z}$ IS A CONTACT VECTORFIELD, $X \neq 0$

$$\Gamma = \left\{ \frac{\partial}{\partial z} \in \xi \right\} = \{r = \pi\} \times \{0\}$$

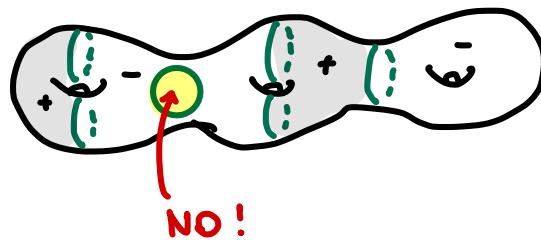
$$\xi_{0r} = \ker (\cos(r) dz + r \sin(r) d\vartheta)$$

THM (GIROUX'S CRITERION): $\Sigma \hookrightarrow (M, \xi)$ CONVEX SURFACE ADMITS A TIGHT NEIGHBOURHOOD $(N(\Sigma), \xi|_{N(\Sigma)})$ IFF

- $\Sigma = S^2$ & $|\Gamma| = 1$
- $\Sigma \neq S^2$ & NO COMPONENT OF Γ BOUNDS A DISC



OR



IDEA OF PROOF

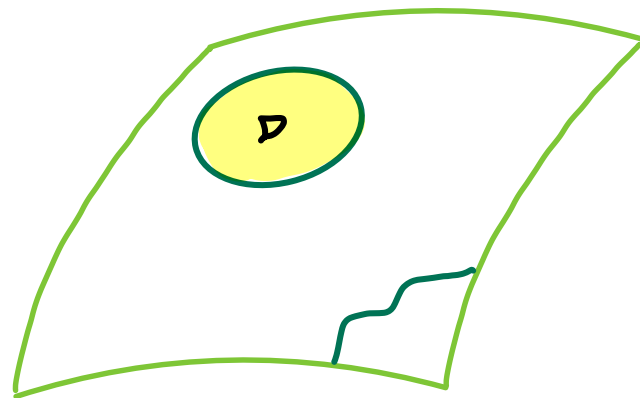
THM (GIROUX'S CRITERION): $\Sigma \hookrightarrow (M, \xi)$ CONVEX SURFACE ADMITS
A TIGHT NEIGHBOURHOOD $(N(\Sigma), \xi|_{N(\Sigma)})$ IFF

- $\Sigma \approx S^2$ & $|\Gamma| = 1$
- $\Sigma \neq S^2$ & NO COMPONENT OF Γ BOUNDS A DISC

$\Rightarrow$ WILL SHOW: Γ HAS A COMPONENT THAT BOUNDS A DISC

& THIS IS NOT THE ONLY COMPONENT

$\Rightarrow \xi|_{N(\Sigma)}$ IS TIGHT



IDEA OF PROOF

THM (GIROUX'S CRITERION): $\Sigma \hookrightarrow (M, \xi)$ CONVEX SURFACE ADMITS
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$\Rightarrow$ WILL SHOW: Γ HAS A COMPONENT THAT BOUNDS A DISC D

& THIS IS NOT THE ONLY COMPONENT

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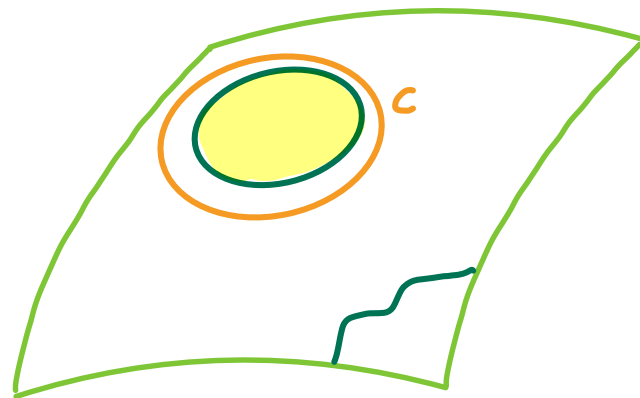
CONSIDER: C ENCAPSULATING D

C IS NON-ISOLATING



LEGENDRIAN REALISATION PRINCIPLE

WE CAN ISOTYPE Σ INSIDE $N(\Sigma)$ SUCH THAT C IS LEGENDRIAN



IDEA OF PROOF

THM (GIROUX'S CRITERION): $\Sigma \hookrightarrow (M, \xi)$ CONVEX SURFACE ADMITS
A TIGHT NEIGHBOURHOOD $(N(\Sigma), \xi|_{N(\Sigma)})$ IFF

- $\Sigma \approx S^2$ & $|\Gamma| = 1$
- $\Sigma \neq S^2$ & NO COMPONENT OF Γ BOUNDS A DISC

$\Rightarrow$ WILL SHOW: Γ HAS A COMPONENT THAT BOUNDS A DISC D

& THIS IS NOT THE ONLY COMPONENT

$\Rightarrow \xi|_{N(\Sigma)}$ IS TIGHT

CONSIDER: C ENCAPSULATING D

C IS NON-ISOLATING



LEGENDRIAN REALISATION PRINCIPLE

WE CAN ISOTOP Σ INSIDE $N(\Sigma)$ SUCH THAT C IS LEGENDRIAN

THEN TWISTING OF ξ ALONG C W.R.T. $\Sigma = \frac{1}{2} |\Gamma \cap C|$

SO D' IS AN OVERTWISTED DISC ✓

$\Leftarrow$ UNIVERSAL COVER OF $\Sigma \dots$ ■

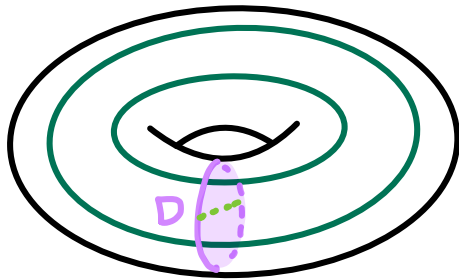
CLASSIFICATION OF TIGHT CONTACT STRUCTURES

THM (ELIASBERG): D^3 ADMITS A UNIQUE TIGHT CONTACT STRUCTURE WITH CONNECTED DIVIDING CURVE ON S^2 :



THIS RESULT ALLOWS US TO PROVE OTHER UNIQUENESS RESULTS

E.G.: $M = D^2 \times S^1$ GIVEN ANY CONTACT STRUCTURE ξ ON M



- STEP 1: BY THE LEGENDRIAN REALISATION PRINCIPLE WE CAN ASSUME ∂D IS LEGENDRIAN

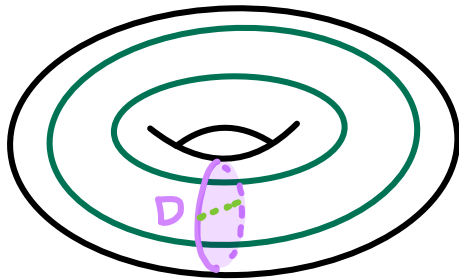
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- STEP 2: ISOTOPE D REL. ∂ TO BE CONVEX: THE DIVIDING CURVE ON D IS A SINGLE ARC

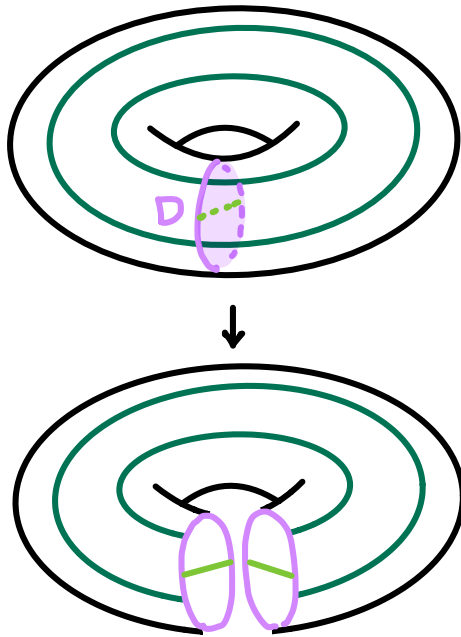
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- STEP 2: ISOTOPE D REL. ∂ TO BE CONVEX: THE DIVIDING CURVE ON D IS A SINGLE ARC
- STEP 3: CUT M ALONG D

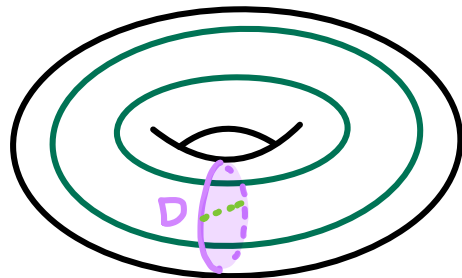
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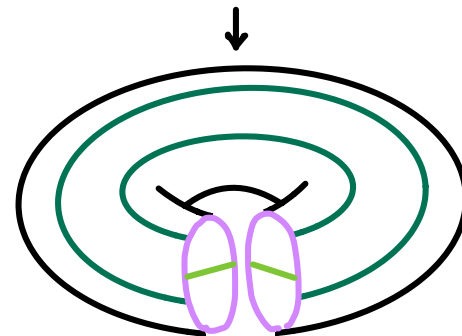


THIS RESULT ALLOWS US TO PROVE OTHER UNIQUENESS RESULTS

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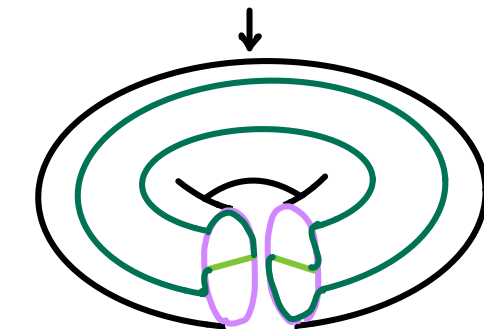


- STEP 1: BY THE LEGENDRIAN REALISATION PRINCIPLE WE CAN ASSUME ∂D IS LEGENDRIAN



- STEP 2: ISOTOPE D REL. ∂ TO BE CONVEX: THE DIVIDING CURVE ON D IS A SINGLE ARC

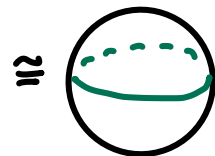
- STEP 3: CUT M ALONG D



- STEP 4: ROUND THE EDGES: WE GET A D^3 WHICH HAS A UNIQUE CONTACT STRUCTURE ξ_0 .

$\Rightarrow$ ANY ξ CAN BE OBTAINED FROM ξ_0 .

BY GLUEING $\Rightarrow \xi$ IS UNIQUE TOO $\blacksquare$

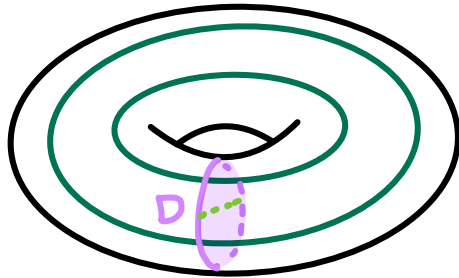


CLASSIFICATION OF TIGHT CONTACT STRUCTURES

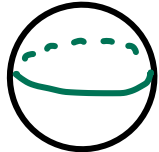
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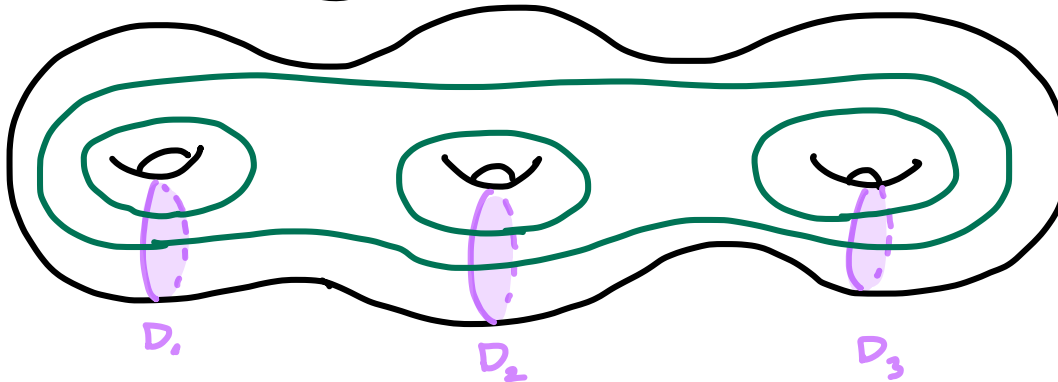
E.G.:



ADMITS A UNIQUE TIGHT CONTACT STRUCTURE

- WHAT DID WE USE IN THE PROOF?
- THAT $|\partial D \cap \Gamma| = 2$ THUS THE DIVIDING CURVE ON D WAS WELL DEFINED
- & THAT WE GOT  AFTER CUTTING AND ROUNDING

SIMILARLY:



ADMITS A UNIQUE TIGHT CONTACT STRUCTURE

DEF: PRODUCT DISC DECOMPOSABLE

TO BE CONTINUED ...

LECTURE 2: DESCRIBING CONTACT STRUCTURES

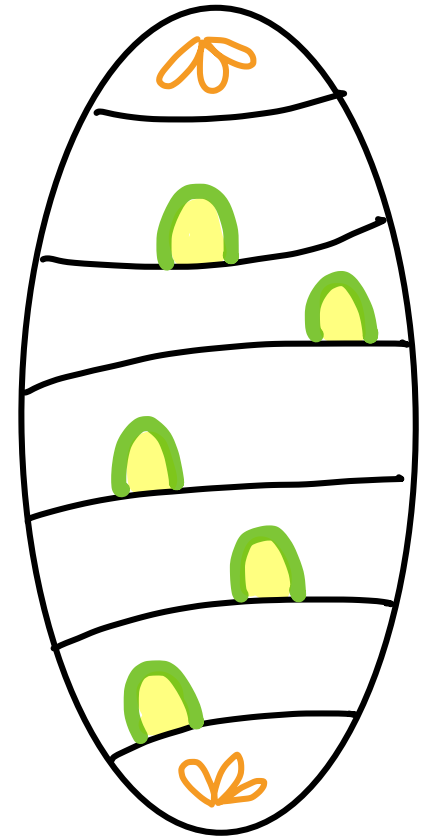
- CONTACT CELL DECOMPOSITIONS
- CONVEX SURFACE THEORY - BYPASSES
- CONTACT HEEGAARD SPLITTINGS (PROOF OF EXISTENCE)
- OPEN BOOK DECOMPOSITIONS
- OPEN BOOK DECOMPOSITIONS & CONTACT HEEGAARD SPLITTINGS

THE GIROUX CORRESPONDENCE VIA CONVEX SURFACES

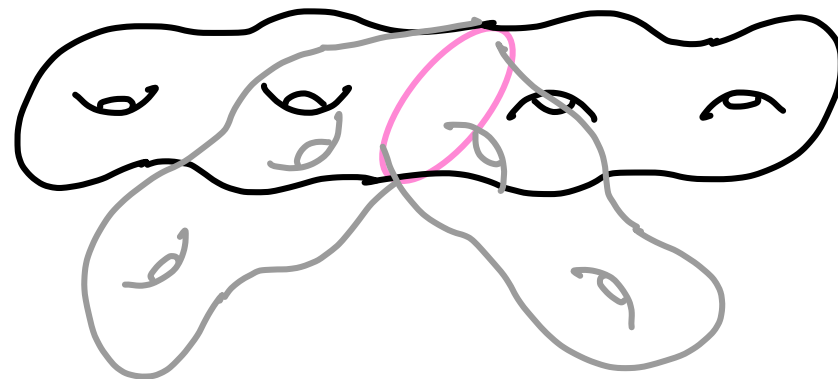
VERA VÉRTESI

JOINT WORK WITH JOAN LICATA

UNIVERSITY OF VIENNA



LECTURE 2



DESCRIBING CONTACT STRUCTURES

- CONTACT CELL DECOMPOSITIONS
- CONVEX SURFACE THEORY - BYPASSES
- CONTACT HEEGAARD SPLITTINGS (PROOF OF EXISTENCE)
- OPEN BOOK DECOMPOSITIONS
- OPEN BOOK DECOMPOSITIONS & CONTACT HEEGAARD SPLITTINGS

LAST TIME (GIRoux)

- $\Sigma \hookrightarrow M$ IS CONVEX IF $\exists$ CONTACT VECTORFIELD $X : \Sigma \nrightarrow X$



$$\Sigma = \Sigma_+ \cup_P \Sigma_-$$

$\alpha(X) > 0$ $\alpha(X) = 0$ $\alpha(X) < 0$

→ GENERIC $\Sigma \hookrightarrow M$ IS CONVEX

→ $\mathfrak{F}|_{N(\Sigma)}$ IS DETERMINED BY $P \subset \Sigma$

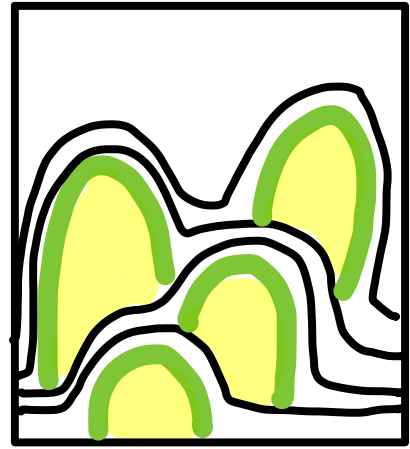
- (M, ξ) OVERTWISTED $\iff \exists \Sigma \hookrightarrow M$ CONVEX w/ Σ_+ OR Σ_- CONTAINING A DISC



TIGHT $\neq$ OVERTWISTED

- $\exists!$ TIGHT CONTACT STRUCTURE ON D^3 WITH CONVEX BDRY: 

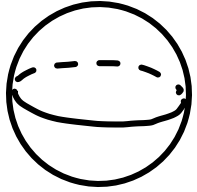
CONTACT CELL- -DECOMPOSITIONS



HANDLE DECOMPOSITIONS OF SMOOTH 3-MANIFOLDS

0-HANDLE

$$h^0 = D^0 \times D^3$$



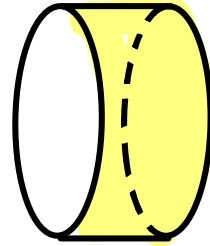
1-HANDLE

$$h^1 = D^1 \times D^2$$



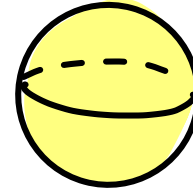
2-HANDLE

$$h^2 = D^2 \times D^1$$



3-HANDLE

$$h^3 = D^3 \times D^0$$

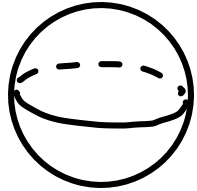


THM (MORSE): ANY 3-MANIFOLD CAN BE OBTAINED VIA
SUCCESSIVELY ATTACHING HANDLES ONTO EACH OTHER
ALONG $\partial D^i \times D^{3-i}$ (HANDLE DECOMPOSITION)

HANDLE DECOMPOSITIONS OF SMOOTH 3-MANIFOLDS

0-HANDLE

$$h^0 = D^0 \times D^3$$



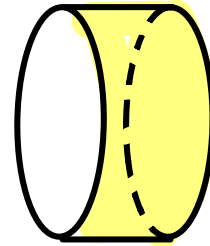
1-HANDLE

$$h^1 = D^1 \times D^2$$



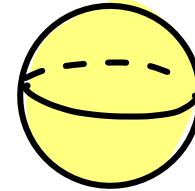
2-HANDLE

$$h^2 = D^2 \times D^1$$



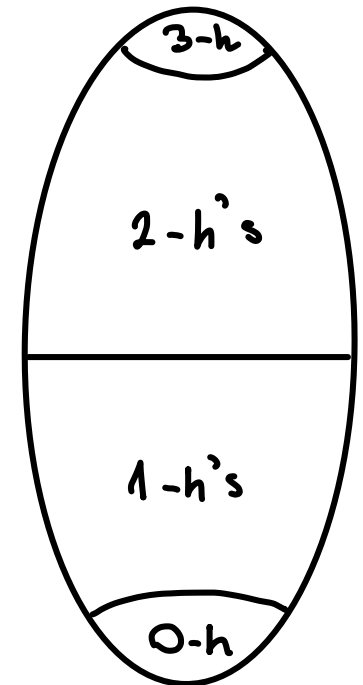
3-HANDLE

$$h^3 = D^3 \times D^0$$



THM (MORSE): ANY 3-MANIFOLD CAN BE OBTAINED VIA
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ALONG $\partial D^i \times D^{3-i}$ (HANDLE DECOMPOSITION)

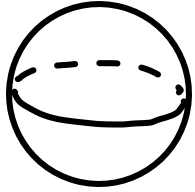
MOREOVER THE INDICES OF HANDLES CAN
BE ASSUMED TO BE IN INCREASING ORDER



HANDLE DECOMPOSITIONS OF SMOOTH 3-MANIFOLDS

0-HANDLE

$$h^0 = D^0 \times D^3$$



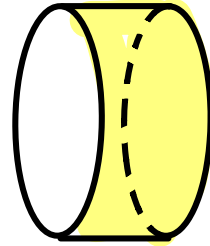
1-HANDLE

$$h^1 = D^1 \times D^2$$



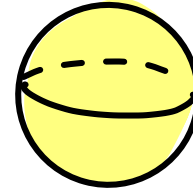
2-HANDLE

$$h^2 = D^2 \times D^1$$



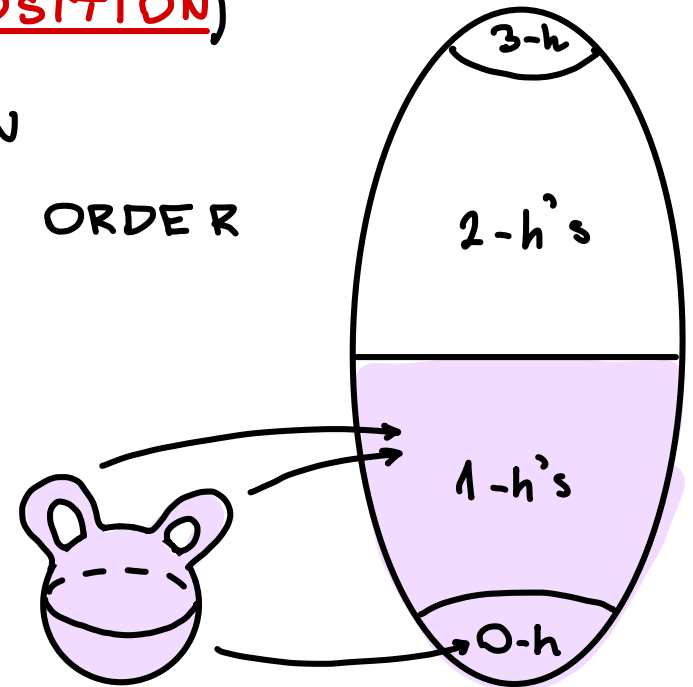
3-HANDLE

$$h^3 = D^3 \times D^0$$



THM (MORSE): ANY 3-MANIFOLD CAN BE OBTAINED VIA
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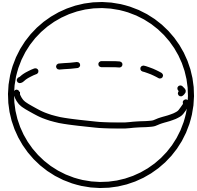


HANDLEBODY

HANDLE DECOMPOSITIONS OF SMOOTH 3-MANIFOLDS

0-HANDLE

$$h^0 = D^0 \times D^3$$



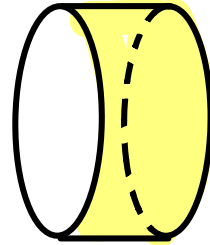
1-HANDLE

$$h^1 = D^1 \times D^2$$



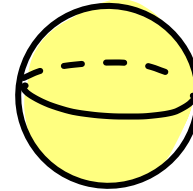
2-HANDLE

$$h^2 = D^2 \times D^1$$



3-HANDLE

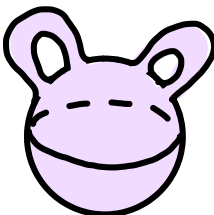
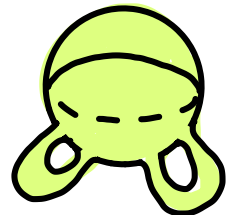
$$h^3 = D^3 \times D^0$$

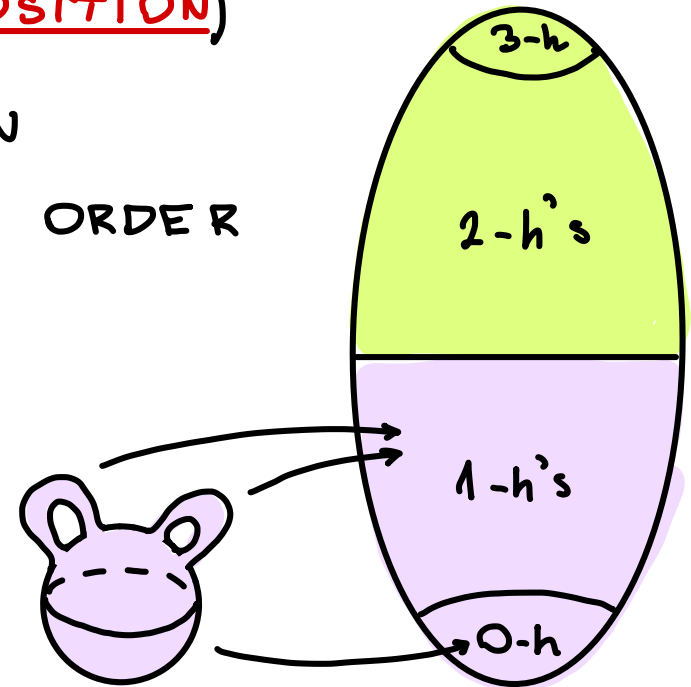


THM (MORSE): ANY 3-MANIFOLD CAN BE OBTAINED VIA
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MOREOVER THE INDICES OF HANDLES CAN
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COR: $M =$  $\cup$ 

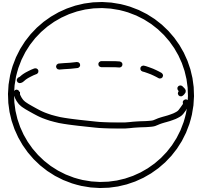


HANDLEBODY

HANDLE DECOMPOSITIONS OF SMOOTH 3-MANIFOLDS

0-HANDLE

$$h^0 = D^0 \times D^3$$



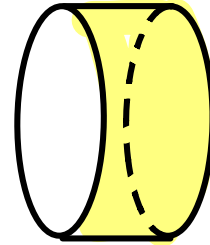
1-HANDLE

$$h^1 = D^1 \times D^2$$



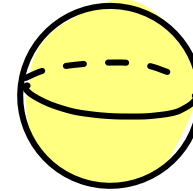
2-HANDLE

$$h^2 = D^2 \times D^1$$



3-HANDLE

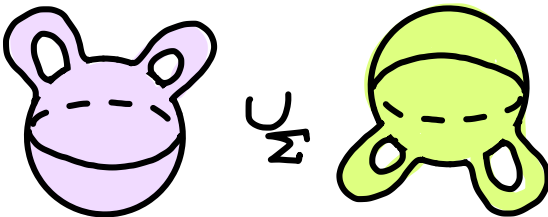
$$h^3 = D^3 \times D^0$$



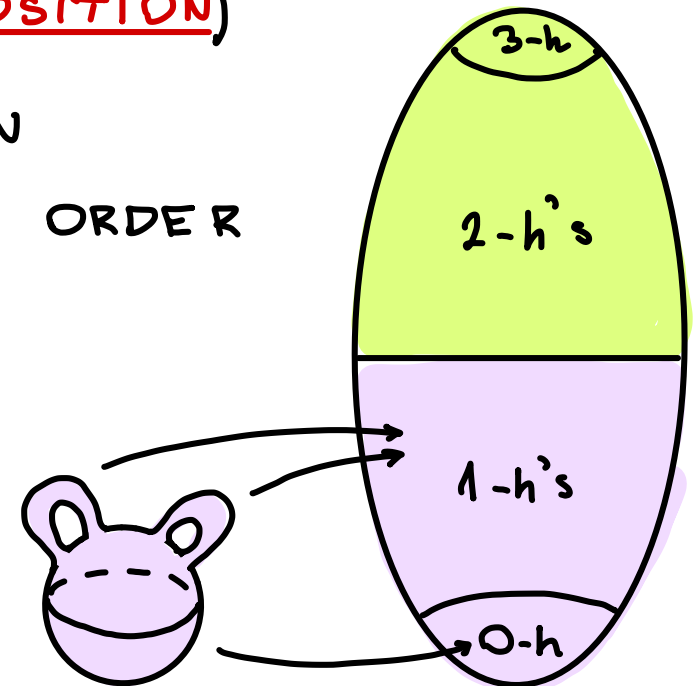
THM (MORSE): ANY (3-)MANIFOLD CAN BE OBTAINED VIA SUCCESSIVELY ATTACHING HANDLES ONTO EACH OTHER ALONG $\partial D^i \times D^{3-i}$ (HANDLE DECOMPOSITION)



MOREOVER THE INDICES OF HANDLES CAN BE ASSUMED TO BE IN INCREASING ORDER

COR: $M =$ 

EVERY 3-MANIFOLD ADMITS A HEEGAARD DECOMPOSITION



HANDLEBODY

HEEGAARD DECOMPOSITIONS

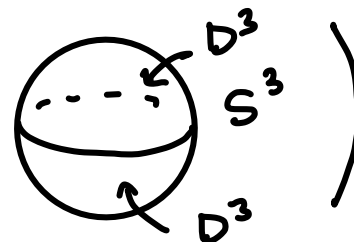
THM (ALEXANDER) EVERY 3-MANIFOLD ADMITS A
HEEGAARD DECOMPOSITION

$$M = U \cup V$$


HANDLEBODIES:

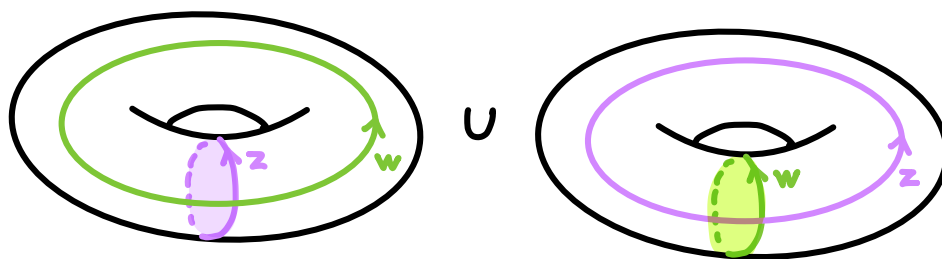


E.G.: • $S^3 = \text{purple sphere} \cup \text{green sphere}$ (SCHEMATIC



• $S^3 = \{|z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$
 $= \{|z|^2 \leq \frac{1}{2}\} \cup \{|w|^2 \leq \frac{1}{2}\}$,

$$\Sigma = \{|z| = |w| = \frac{1}{2}\} \cong S^1 \times S^1$$



ORIGIN - MORSE FUNCTIONS : M SMOOTH (3-)MANIFOLD

DEF.: $f: M \rightarrow \mathbb{R}$ SMOOTH FUNCTION IS MORSE IF ALL

CRITICAL POINTS OF f ARE NONDEGENERATE

$$\nabla f \neq 0$$



$\text{Hess}(f)$ IS
NONDEGENERATE

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FACTS: • MORSE FUNCTIONS ARE C^∞ -GENERIC

• LOCAL MODEL NEAR A MORSE - CRITICAL POINT :

INDEX

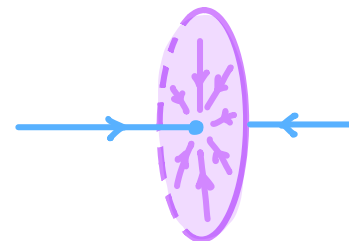
$$-\sum_{j=1}^i x_j^2 + \sum_{j=i+1}^3 x_j^2$$

$\leadsto$ FLOW OF ∇f

$x_{i+1}, \dots, x_3$



$x_1, \dots, x_i$



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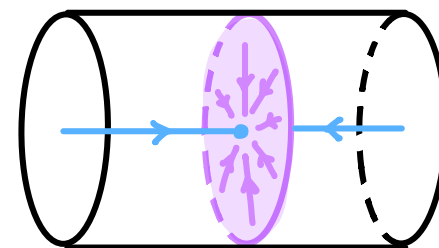
INDEX

$$-\sum_{j=1}^i x_j^2 + \sum_{j=i+1}^n x_j^2$$

$\leadsto$ FLOW OF ∇f

$x_{i+1}, \dots, x_n$

$x_1, \dots, x_i$



CORE

COCORE

• INDEX - i CRITICAL POINTS $\leadsto h^i \approx D^i \times D^{n-i}$ i -HANDLES

ORIGIN - MORSE FUNCTIONS : M SMOOTH (3-)MANIFOLD

DEF.: $f: M \rightarrow \mathbb{R}$ SMOOTH FUNCTION IS MORSE IF ALL

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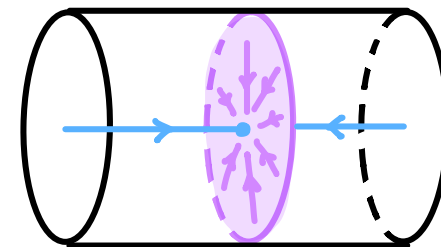
INDEX

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$\leadsto$ FLOW OF ∇f

$x_{i+1}, \dots, x_3$

$x_1, \dots, x_i$



CORE

COCORE

• INDEX - i CRITICAL POINTS $\leadsto h^i = D^i \times D^{n-i}$ i -HANDLES

REARRANGING CRITICAL POINTS : MOVING INDEX i -HANDLES UNDER
INDEX i' -HANDLES

USES TRANSVERSALITY & THAT $\dim(\text{CORE}) + \dim(\text{COCORE}) < 3$

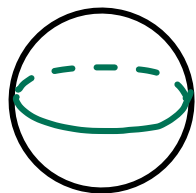


$$i < i'$$

CONTACT HANDLES

0-HANDLE

$$D^0 \times D^3$$

 h^0

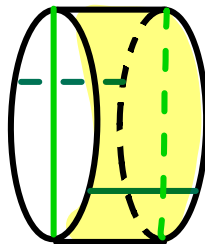
1-HANDLE

$$D^1 \times D^2$$

 h^1

2-HANDLE

$$D^2 \times D^1$$

 h^2

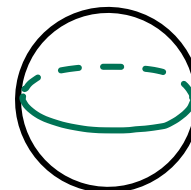
3-HANDLE

$$D^3 \times D^0$$

 h^3

AFTER ROUNDING THE EDGES EACH BECOMES

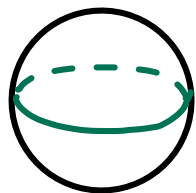
& BY ELIASBERG'S THM THIS ADMITS A
UNIQUE TIGHT CONTACT STRUCTURE



CONTACT HANDLES

0-HANDLE

$$D^0 \times D^3$$



$$(h^0, \xi^0)$$

1-HANDLE

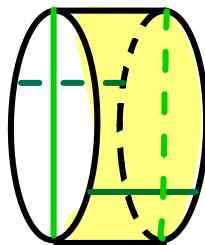
$$D^1 \times D^2$$



$$(h^1, \xi^1)$$

2-HANDLE

$$D^2 \times D^1$$



$$(h^2, \xi^2)$$

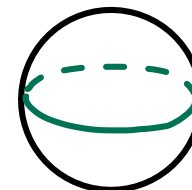
3-HANDLE

$$D^3 \times D^0$$



$$(h^3, \xi^3)$$

AFTER ROUNDING THE EDGES EACH BECOMES



& BY ELIASBERG'S THM THIS ADMITS A
UNIQUE TIGHT CONTACT STRUCTURE

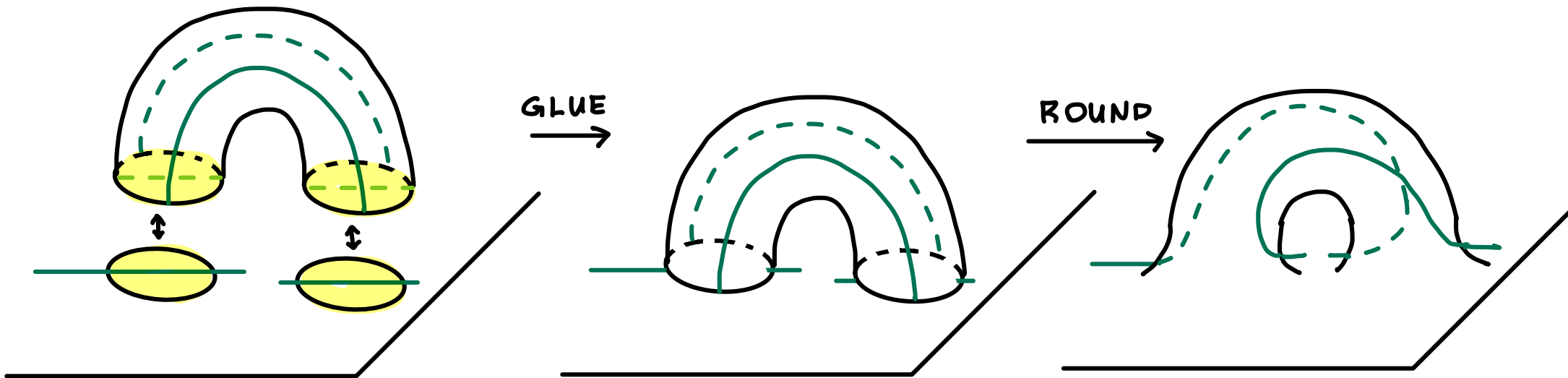
→ UP TO ISOTOPY WE GET WELL DEFINED CONTACT STRUCTURES
ON THE h^i

ATTACHING CONTACT HANDLES

i - HANDLE $h^i = D^i \times D^{3-i}$

$$\partial h^i = \partial D^i \times D^{3-i} \cup D^i \times \partial D^{3-i}$$

WE CAN CONSTRUCT CONTACT MANIFOLDS BY SUCCESSIVELY
GLUING HANDLES ALONG $\partial D^i \times D^{3-i}$:



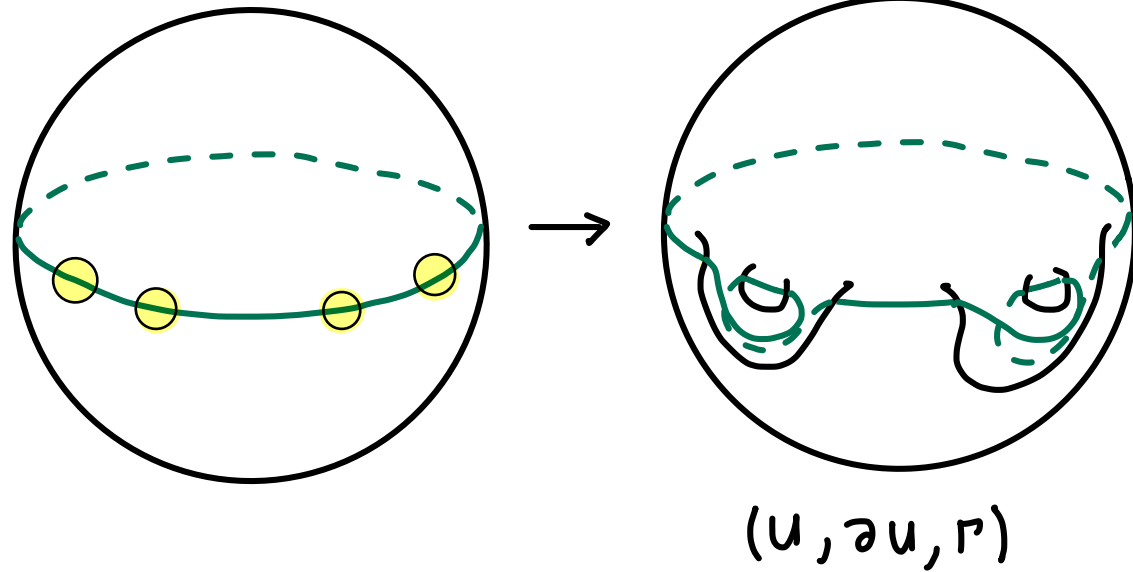
CONTACT HANDLE DECOMPOSITION

THM (GIROUX) ANY CONTACT 3-MANIFOLD ADMITS A
CONTACT HANDLE DECOMPOSITION

CONTACT HANDLEBODY

O-HANDLE $\cup$ SOME 1-HANDLES

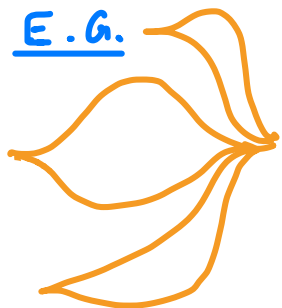
E.G.



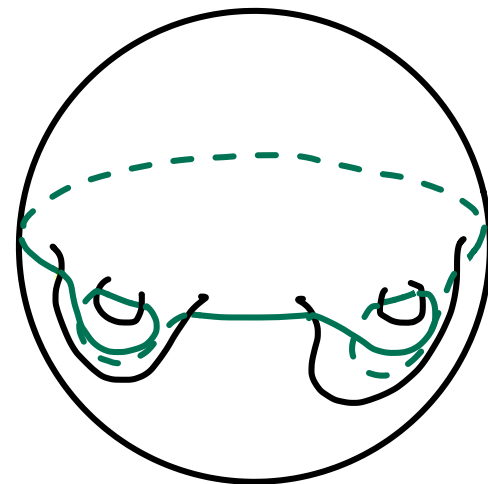
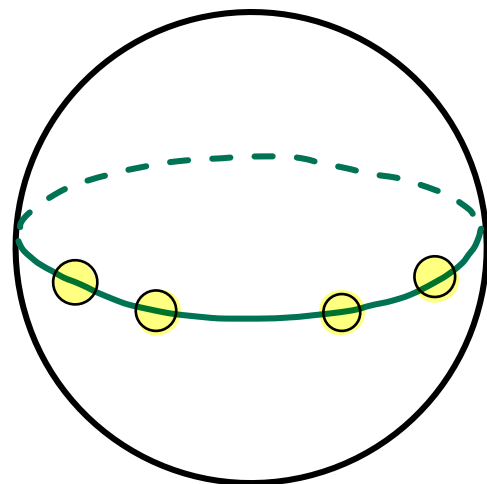
CONTACT HANDLEBODY

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E.G.



NEIGHBOURHOOD OF A
LEGENDRIAN GRAPH:

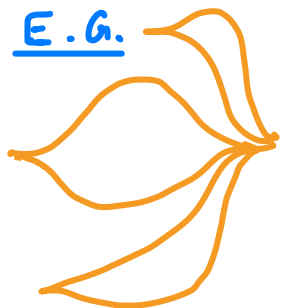


$(U, \partial U, \tau)$

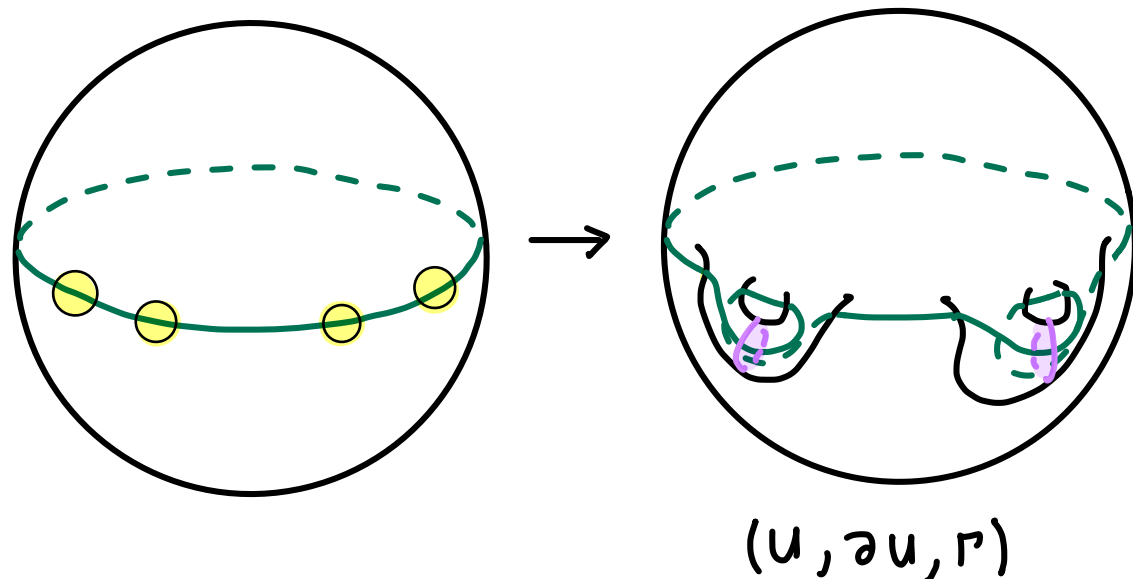
CONTACT HANDLEBODY

O-HANDLE U SOME 1-HANDLES

E.G.



NEIGHBOURHOOD OF A
LEGENDRIAN GRAPH:



NOTE: CONTACT HANDLEBODIES ARE PRODUCT DISC DECOMPOSABLE

RECALL: A PRODUCT DISC DECOMPOSABLE HANDLEBODY U ADMITS
A UNIQUE TIGHT CONTACT STRUCTURE WITH DIVIDING
CURVE Γ ON ∂U .

THUS

THM: A PRODUCT DISC DECOMPOSABLE HANDLEBODY U WITH
TIGHT CONTACT STRUCTURE ξ IS A
CONTACT HANDLEBODY

REARRANGING CONTACT HANDLES

JUST AS IN THE SMOOTH CASE

IN A CONTACT HANDLE DECOMPOSITION ONE CAN ASSUME THAT:

- CONTACT 0-h's ARE ATTACHED FIRST
- CONTACT 3-h's ARE ATTACHED LAST

REARRANGING CONTACT HANDLES

JUST AS IN THE SMOOTH CASE

IN A CONTACT HANDLE DECOMPOSITION ONE CAN ASSUME THAT:

- CONTACT 0-h's ARE ATTACHED FIRST
- CONTACT 3-h's ARE ATTACHED LAST

BUT! CONTACT 1-h's CANNOT ALWAYS BE ATTACHED
BEFORE CONTACT 2-h's

INTERLUDE - CONTACT MORSE FUNCTIONS

ONLY FOR FUN

(M^3, ξ) CONTACT MANIFOLD

DEF: $f: M \rightarrow \mathbb{R}$ MORSE FUNCTION IS CONTACT IF $\exists$ X CONTACT VECTORFIELD THAT IS AN ALMOST GRADIENT FOR f

RMK: NONCRITICAL LEVELSETS ARE CONVEX

↖

⊥ TO LEVEL SETS
& "LIKE" ∇f NEAR
CRITICAL POINTS

DEF: CRITICAL SUBMANIFOLD: $C^2 = \{x \in \xi\}$

($\approx$ UNION OF DIVIDING CURVES OF LEVELSETS)

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($\approx$ UNION OF DIVIDING CURVES OF LEVELSETS)

THM (GIROUX): $f|_C: C \rightarrow \mathbb{R}$ IS ALSO MORSE WITH THE SAME CRITICAL POINTS AS f WITH INDICES

f	$f _C$
0	0
1	1
2	1
3	2

INTERLUDE - CONTACT MORSE FUNCTIONS

ONLY FOR FUN

(M^3, ξ) CONTACT MANIFOLD

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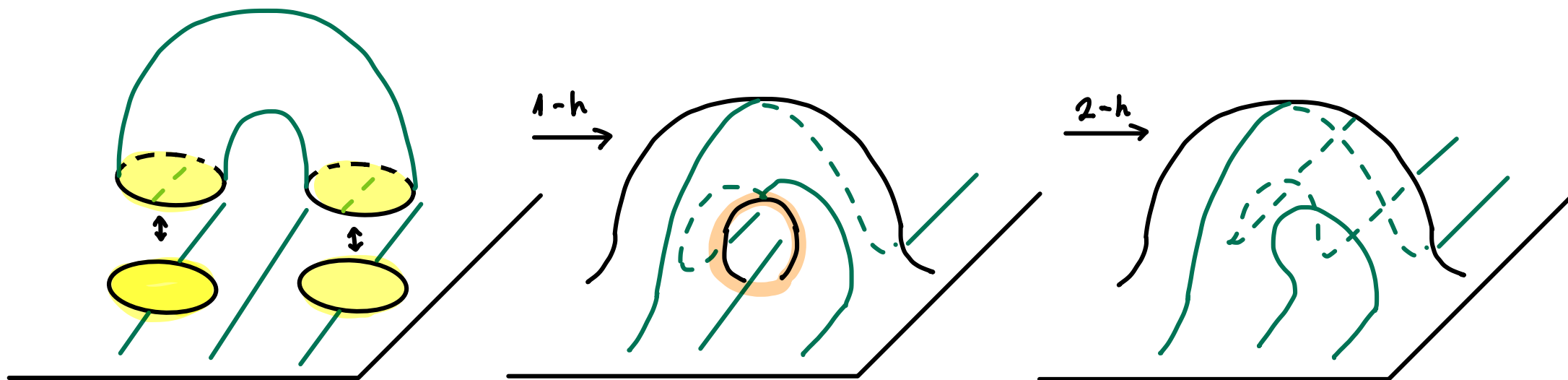
THM (GIROUX): $f|_C: C \rightarrow \mathbb{R}$ IS ALSO MORSE WITH THE SAME CRITICAL POINTS AS f WITH INDICES

THIS EXPLAINS WHY WE CANNOT MOVE
1-HANDLES UNDER 2-HANDLES

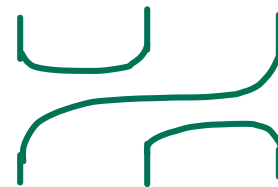
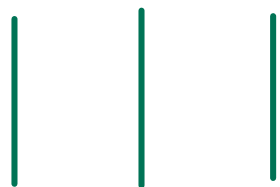
(THEY ARE BOTH 1-HANDLES FOR C)

f	$f _C$
0	0
1	1
2	1
3	2

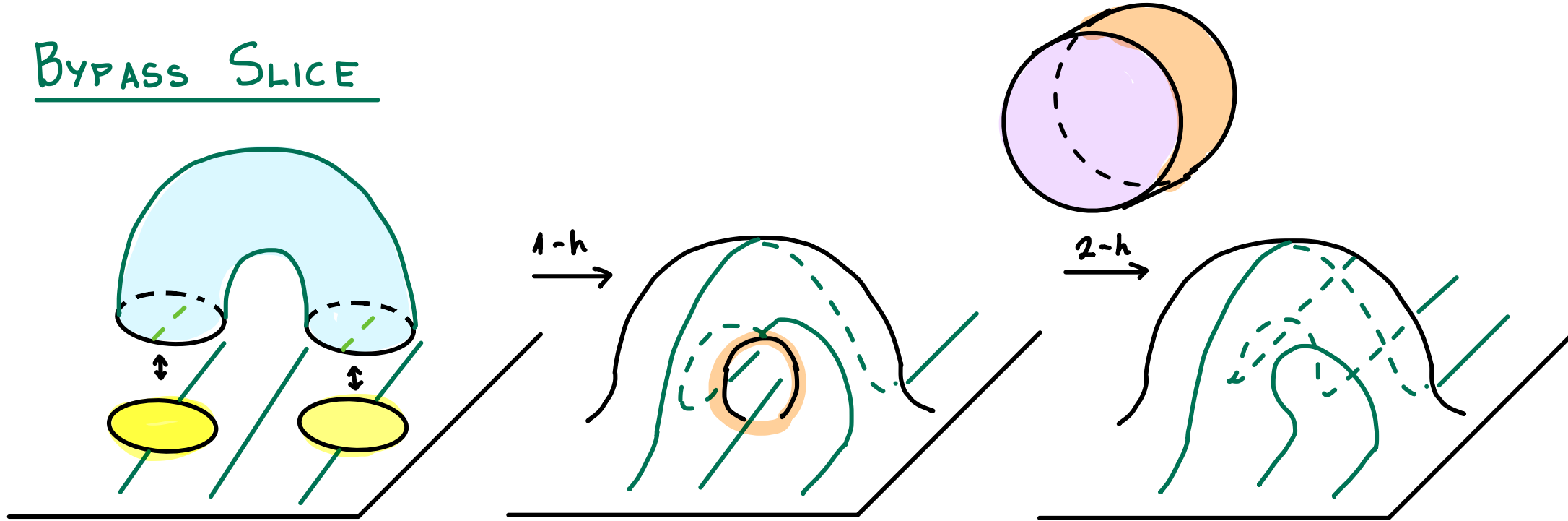
BYPASS SLICE



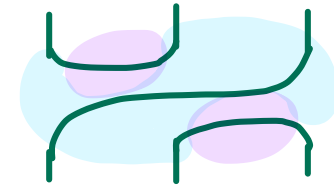
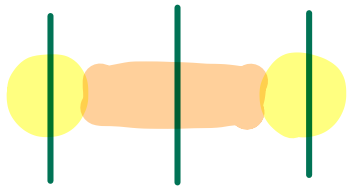
FROM THE TOP



BYPASS SLICE



FROM THE TOP



THE ABOVE PAIR OF CONTACT 1- & 2-h CAN BE ATTACHED TO ANY CONVEX SURFACE (Σ, Γ) ALONG ANY ARC c INTERSECTING Γ AS 

WILL SEE: BYPASSES ARE BASIC BUILDING BLOCKS OF CONTACT STRUCTURES ON $\Sigma \times I$

CONVEX SURFACE THEORY

0-PARAMETER

RECALL (GIROUX): CONVEX SURFACES ARE C^∞ -GENERIC

1-PARAMETER

THM (GIROUX, COLIN, REPHRASED BY HONDA) ANY 1-PARAMETER FAMILY OF SURFACES $(\Sigma_t)_{t \in [0,1]}$ WITH Σ_0, Σ_1 CONVEX CAN BE ISOTOPED TO $(\Sigma_t^?)_{t \in [0,1]}$ SO THAT

- $\Sigma_t = \Sigma_t^?$ NEAR $t = 0$ & 1

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- $\Sigma'_{t_i - \varepsilon}$ & $\Sigma'_{t_i + \varepsilon}$ COBOUND A **BYPASS SLICE** ($i = 1 \dots k$)

CONVEX SURFACE THEORY

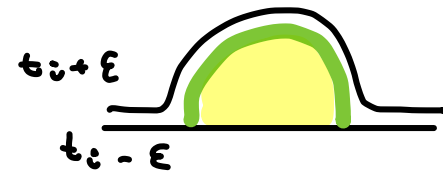
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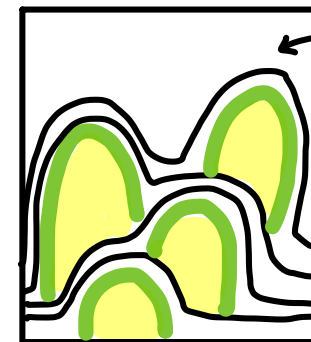
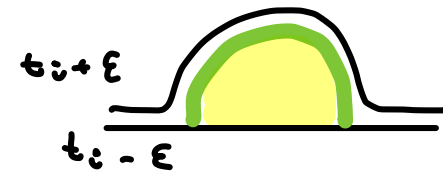
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- $\Sigma'_t = \Sigma'_1$ NEAR $t=0$ & 1
- Σ'_t IS CONVEX EXCEPT AT DISCRETE TIMES $\{t_1, \dots, t_k\} \in [0,1]$
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THM (GIROUX REPHRASED BY HONDA)

ANY CONTACT STRUCTURE ON $\Sigma \times I$ IS

CONTACTOMORPHIC TO A STACK OF
BYPASS SLICES

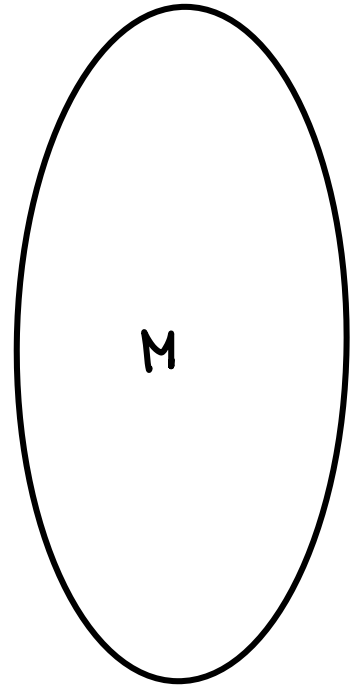


I-INVARIANT

EXISTENCE OF CONTACT HEEGAARD DECOMPOSITIONS

THM (GIROUX): ANY CONTACT 3-MANIFOLD (M, ξ) ADMITS A
CONTACT HEEGAARD DECOMPOSITION

PROOF (LICATA - V)

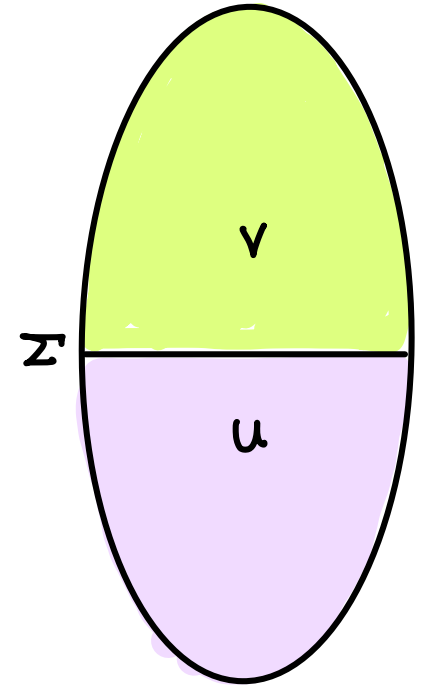


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PROOF (LICATA - V)

STEP 1: TAKE ANY SMOOTH HEEGAARD DECOMPOSITION OF M : $M = U \cup_{\Sigma} V$



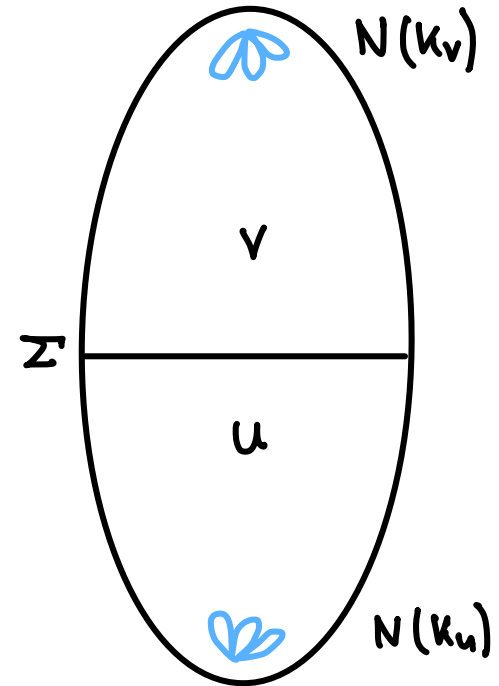
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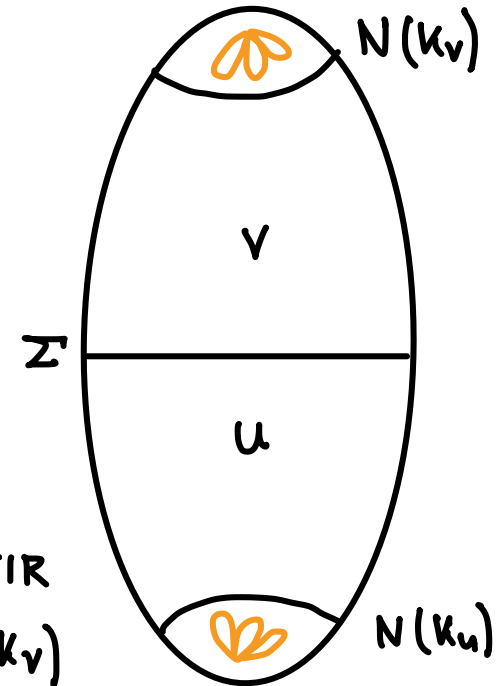
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STEP 3: LEGENDRIAN REALISE K_U & K_V & TAKE THEIR STANDARD CONTACT NEIGHBOURHOODS $N(K_U)$ & $N(K_V)$



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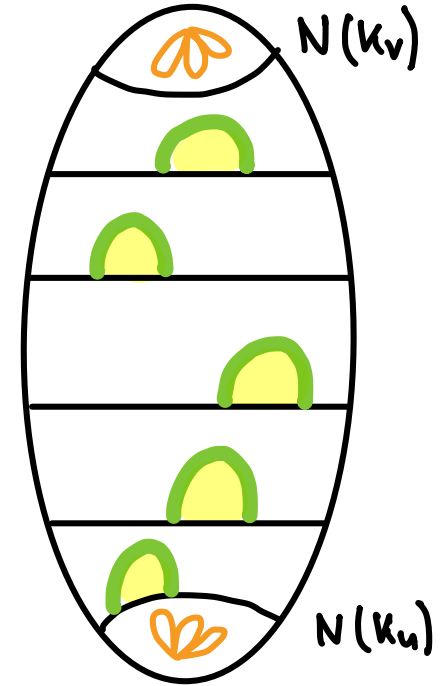
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STEP 4: $X = M - (N(K_U) \cup N(K_V)) \overset{\text{DIFFEO}}{\cong} \Sigma \times I \Rightarrow \xi|_X$ CAN BE WRITTEN AS A STACK OF BYPASS-SLICES $\cong h^1 \cup h^2$

(AWAY FROM THE HANDLES ξ IS I-INVARIANT)

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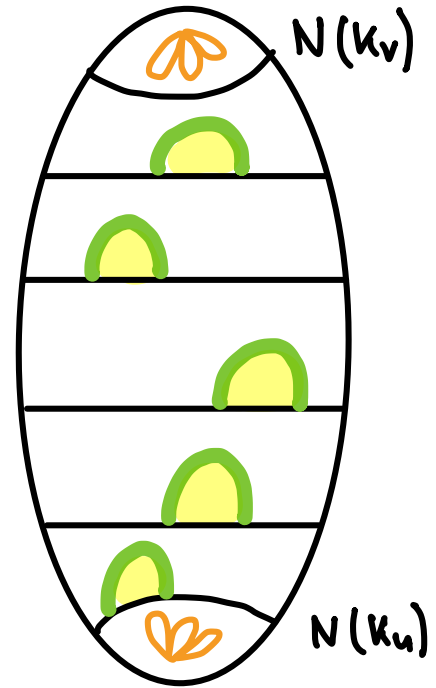
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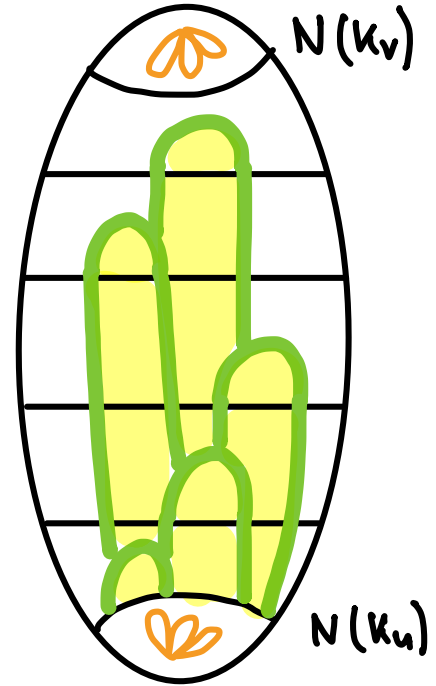
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STEP 5: USE THIS FLOW TO EXTEND THE HANDLES



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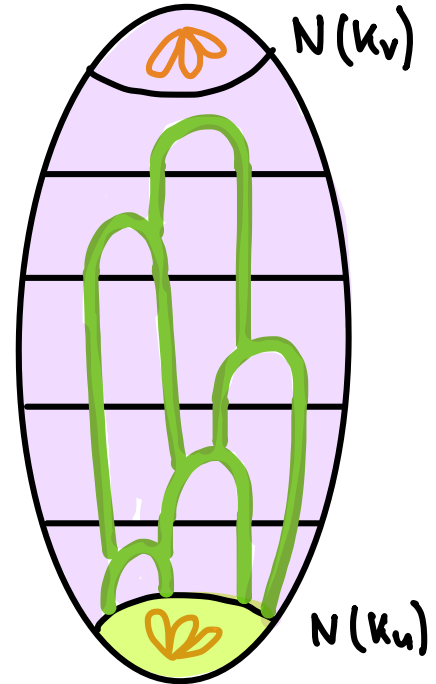
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STEP 5: USE THIS FLOW TO EXTEND THE HANDLES

CONSIDER: $\hat{U} = N(K_u) \cup (\cup h_i^1)$

$\uparrow$ $\nwarrow$
 CONTACT HANDLEBODY CONTACT 1-HANDLES

$\Rightarrow \hat{U}$ IS A CONTACT HANDLEBODY



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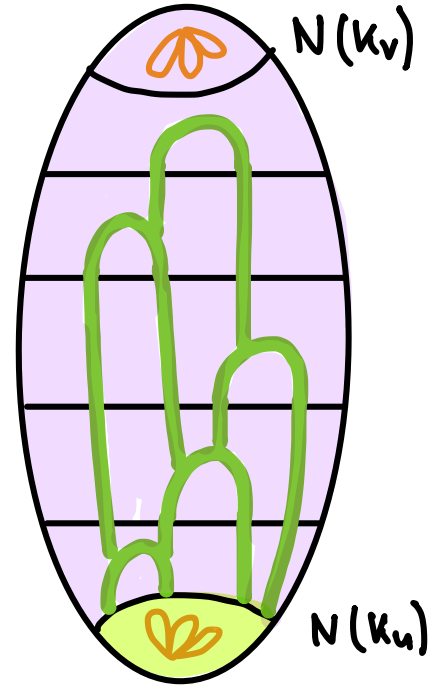
STEP 5: USE THIS FLOW TO EXTEND THE HANDLES

CONSIDER: $\hat{U} = N(K_u) \cup (\cup h_i^1)$
 $\uparrow$ $\nwarrow$
 CONTACT HANDLEBODY CONTACT 1-HANDLES

$\Rightarrow \hat{U}$ IS A CONTACT HANDLEBODY

UPSIDE DOWN: $\hat{V} = M \setminus \hat{U} \cong N(K_v) \cup (\cup h_i^2)$ IS A CONTACT HANDLEBODY

$\Rightarrow M = \hat{U} \cup \hat{V}$ IS A CONTACT HEEGAARD DECOMPOSITION



BRIDGING

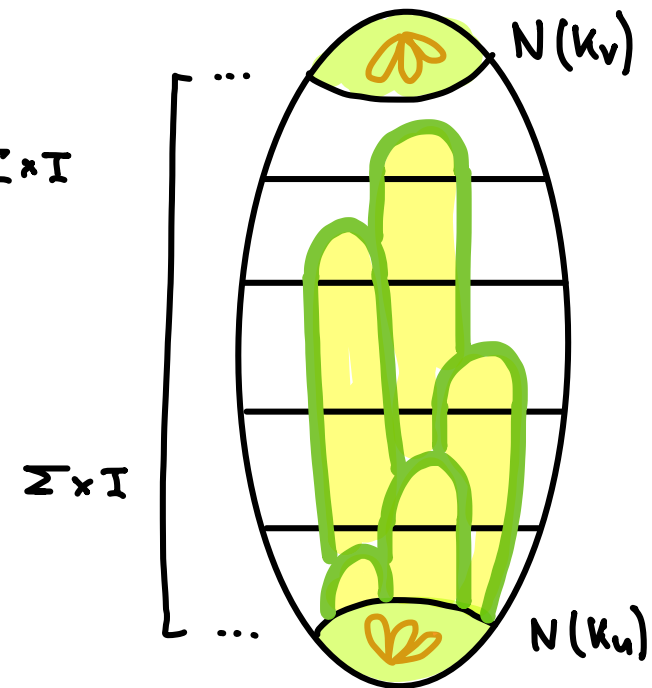
$$M = \underset{\mathcal{H}}{U \cup V} \xrightarrow{\text{BRIDGING}} M = \underset{\hat{\mathcal{H}}}{\hat{U} \cup \hat{V}} \quad \text{BRIDGE} \quad \text{WHERE}$$

- $\hat{U} = N(k_u) \cup (\cup h_i^a)$
- $\hat{V} = \overline{M \setminus \hat{U}}$

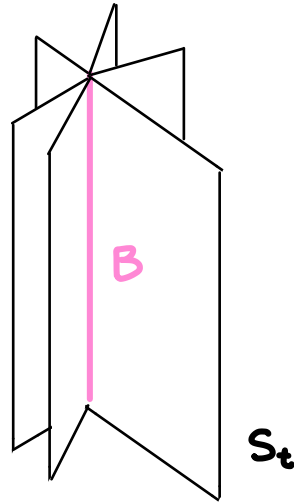
NOTE: THE BRIDGE DEPENDS ON

- $\mathcal{H}$
- k_u & k_v
- THE BYPASSES $\mathcal{B}$ BUILDING UP $\mathcal{B}|_{\Sigma \times I}$

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}(k_u, k_v, \mathcal{B})$$



OPEN BOOK DECOMPOSITIONS



OPEN BOOK DECOMPOSITIONS

DEF: PAIR (B, π) , WHERE

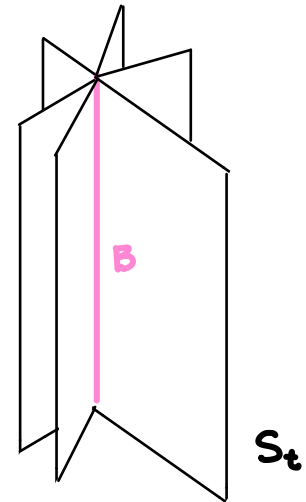
- $B \hookrightarrow M$ EMBEDDED 1-MANIFOLD: BINDING

- $\pi: M - B \longrightarrow S^1$ FIBRATION SUCH THAT

$\rightarrow \forall t \in S^1 \quad S_t := \pi^{-1}(t)$ IS A SEIFERT SURFACE FOR B

$\rightarrow \& \text{ ON } N(B) \cong B \times D^2 \quad \pi = \text{ANGLE}$

DEF: $S_t := \pi^{-1}(t)$ ARE THE PAGES OF (B, π)



OPEN BOOK DECOMPOSITIONS

DEF: PAIR (B, π) , WHERE

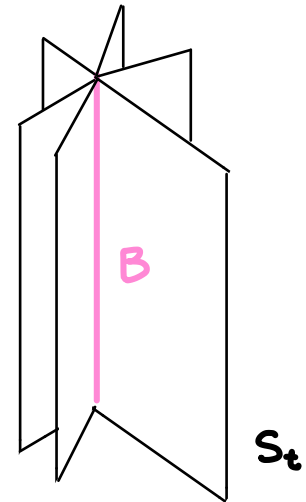
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DEF: $S_t := \pi^{-1}(t)$ ARE THE PAGES OF (B, π)

E.G.: $M = S^3 = \{|z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$

$$B = \{|z| = 0\} \cong S^1, \quad \pi: S^3 \setminus B \longrightarrow S^1$$

$$(z, w) \longmapsto \frac{z}{|z|}$$



OPEN BOOK DECOMPOSITIONS

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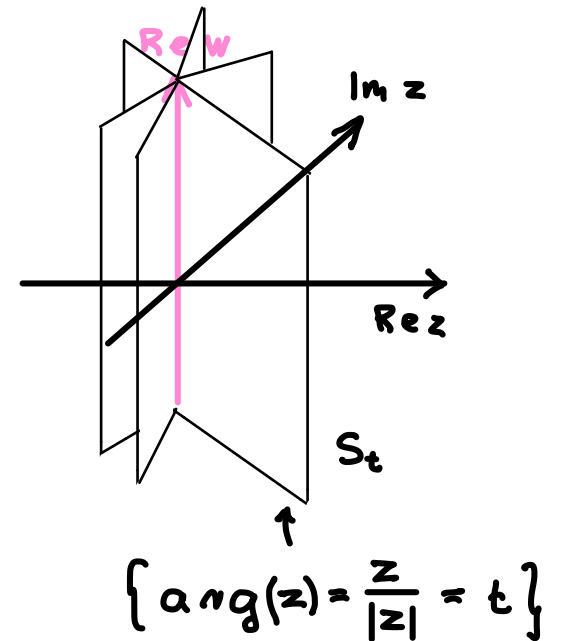
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$$(z, w) \longmapsto \frac{z}{|z|}$$

ON $\mathbb{R}^3$ WITH COORDINATE $S: \frac{1}{1 - \operatorname{Im} w} (\underline{z}, \operatorname{Re} w)$

HERE $S_t \cong D^2 \quad \forall t$



OPEN BOOK DECOMPOSITIONS - ANOTHER EXAMPLE

E.G.: $M = S^3 = \{|z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$

$$B = \{|z|=0\} \sqcup \{|w|=0\}$$

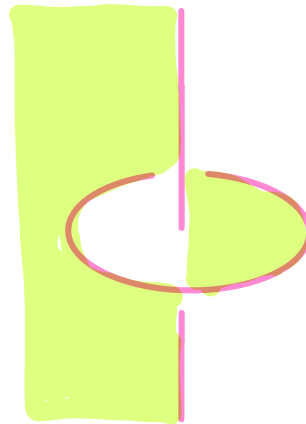
$$\pi: S^3 \setminus B \longrightarrow S^1$$

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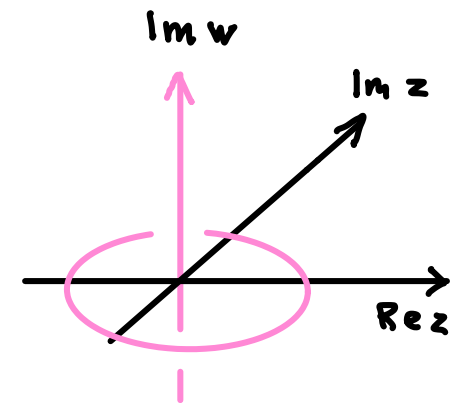
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$$S_t \cong S^1 \times I \quad \forall t$$



[Video](#)



THM (ALEXANDER): ANY 3-MANIFOLD ADMITS AN OPEN BOOK DECOMPOSITION

PROOF LATER

OPEN BOOK DECOMPOSITIONS - ANOTHER EXAMPLE

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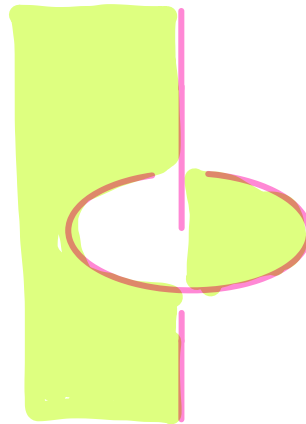
$$\pi: S^3 \setminus B \longrightarrow S^1$$

$$(z, w) \longmapsto \frac{zw}{|zw|}$$

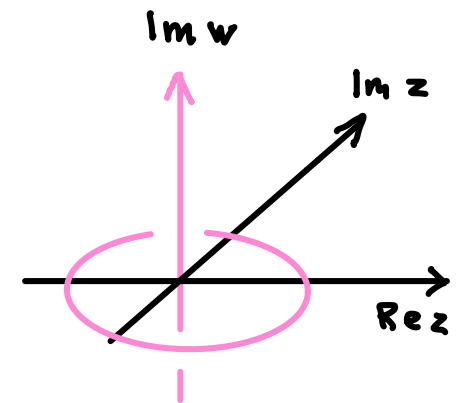
ON $\mathbb{R}^3$ WITH COORDINATE $S: \frac{1}{1 - \operatorname{Im} w} (z, \operatorname{Re} w)$

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$$S_t \cong S^1 \times I \quad \forall t$$



[Video](#)



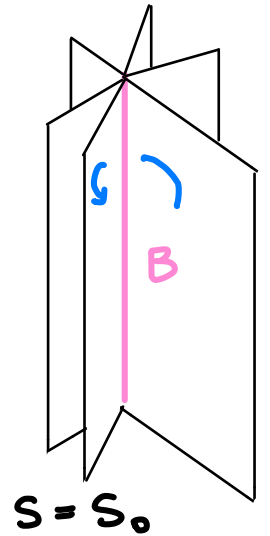
THM (ALEXANDER): ANY 3-MANIFOLD ADMITS AN OPEN BOOK DECOMPOSITION

PROOF LATER

ABSTRACT OPEN BOOKS

FIX $S := S_0$ & LOOK AT THE FIRST RETURN-MAP
OF $\pi: M-B \rightarrow S' \rightsquigarrow$ GET (S, ψ) WHERE

- S IS AN ORIENTED SURFACE WITH BOUNDARY
- & $\psi: S \rightarrow S$ HOMEOMORPHISM THAT FIXES $N(\partial S)$

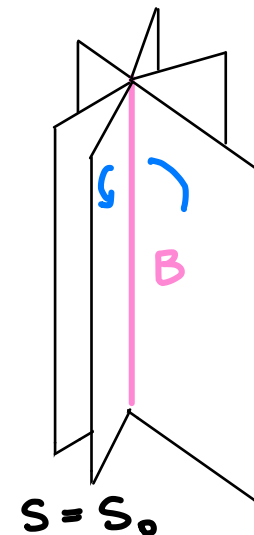


DEF: THE PAIR (S, ψ) IS AN ABSTRACT OPEN BOOK

ABSTRACT OPEN BOOKS

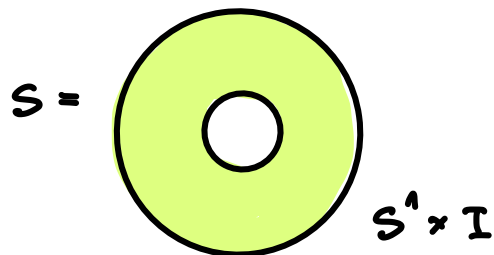
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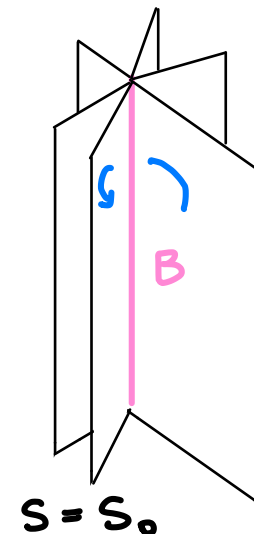
E.G.: THE PREVIOUS EXAMPLE GIVES



ABSTRACT OPEN BOOKS

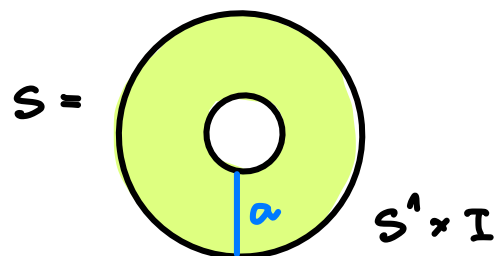
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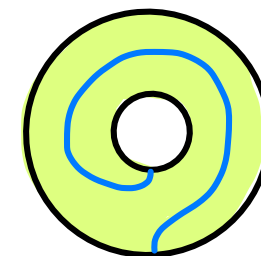


DEF: THE PAIR (S, ψ) IS AN ABSTRACT OPEN BOOK

E.G.: THE PREVIOUS EXAMPLE GIVES

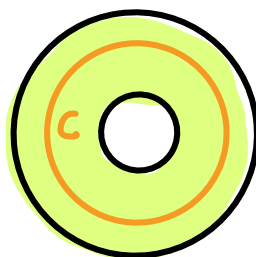


& ψ MAPPING a TO $\psi(a)$



(THIS DETERMINES ψ UP TO ISOTOPY)

DEF: THE ABOVE MAP IS A RIGHT HANDED DEHN-TWIST
ALONG c



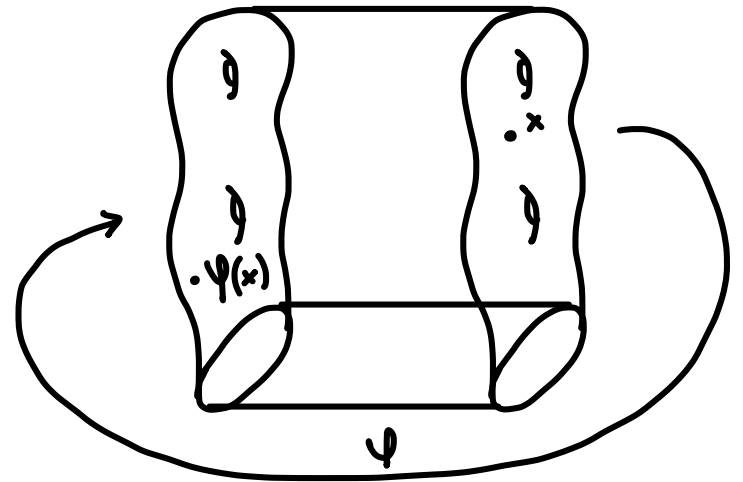
ABSTRACT OPEN BOOKS

CONVERSALY: AN ABSTRACT OB (S, ψ) DETERMINES A 3-MANIFOLD M TOGETHER WITH AN OPEN BOOK DECOMPOSITION

PROOF: • TAKE THE MAPPING TORUS OF ψ :

$$M_\psi = S \times I / (x, 1) \sim (\psi(x), 0)$$

• (AS ψ FIXES ∂S) $\partial M_\psi = \partial S \times S^1$



ABSTRACT OPEN BOOKS

CONVERSALY: AN ABSTRACT OB (S, ψ) DETERMINES A 3-MANIFOLD M TOGETHER WITH AN OPEN BOOK DECOMPOSITION

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$$M_\psi = S \times I / (x, 1) \sim (\psi(x), 0)$$

- (AS ψ FIXES ∂S) $\partial M_\psi = \partial S \times S^1$
- & WE COLLAPSE EACH CIRCLE $x \times S^1$ ($x \in \partial S$)

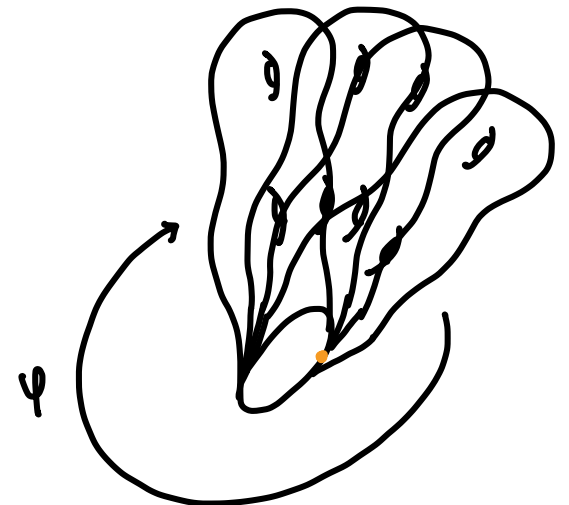
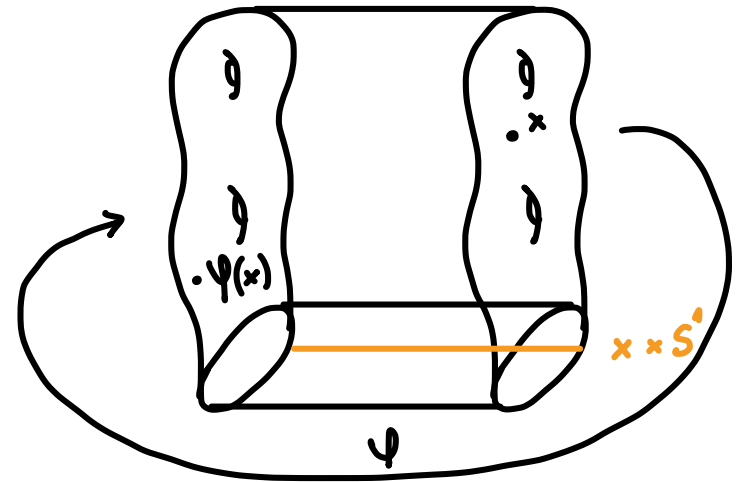
$$M := M_\psi / (x, t) \sim (x, t') \quad x \in \partial S$$

THEN: $B = \partial S / \sim$

$$\pi: M \rightarrow B$$

$$(x, t) \mapsto t$$

GIVES AN OBD



INTERLUDE - DESCRIBING MONODROMIES

FOR FUN ONLY

POWER OF OPEN BOOKS: TO DESCRIBE CTCT STRUCTURES

WE NEED TO UNDERSTAND $\{\psi: S \rightarrow S : \psi|_{\partial S} = \text{id}\} / \text{ISOTOPY} =: \text{MCG}(S)$

(MAPPING CLASS GROUP)

INTERLUDE - DESCRIBING MONODROMIES

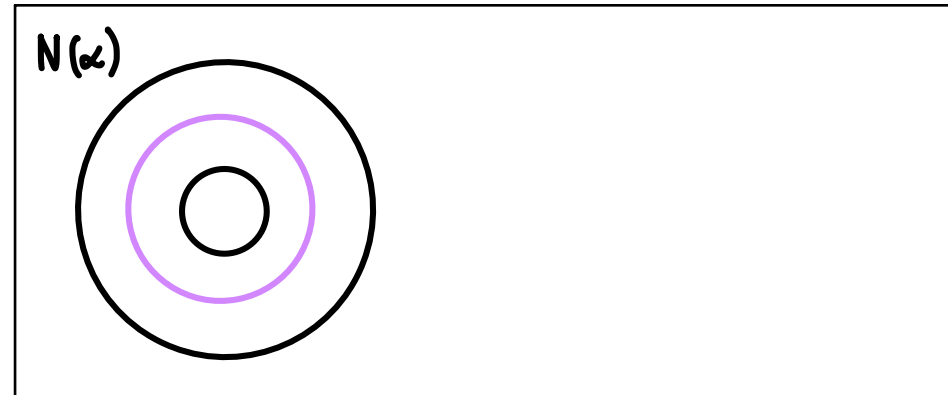
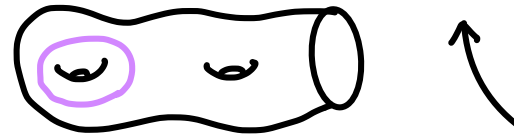
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THM (DEHN): $\text{MCG}(S)$ IS GENERATED BY DEHN TWISTS ALONG
SIMPLE CLOSED CURVES $\propto$



INTERLUDE - DESCRIBING MONODROMIES

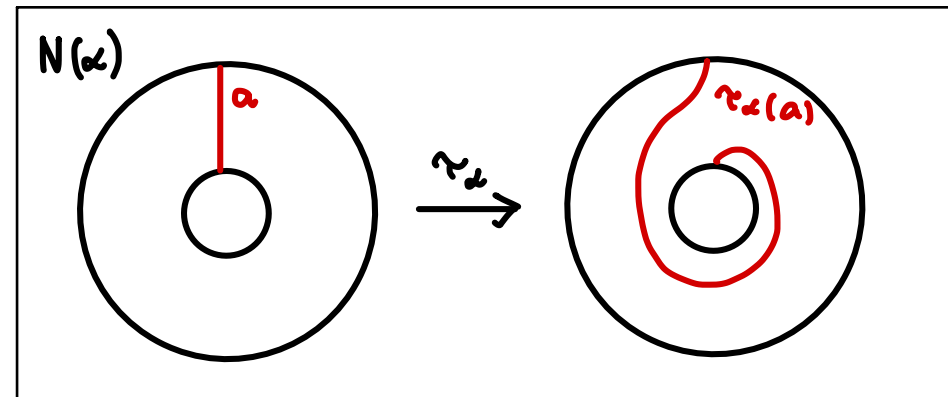
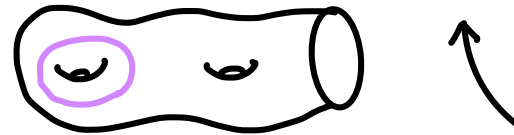
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SIMPLE CLOSED CURVES α



INTERLUDE - DESCRIBING MONODROMIES

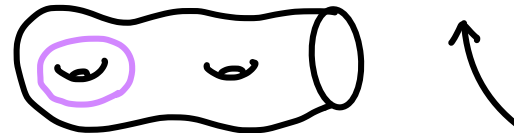
FOR FUN ONLY

POWER OF OPEN BOOKS: TO DESCRIBE CTCT STRUCTURES

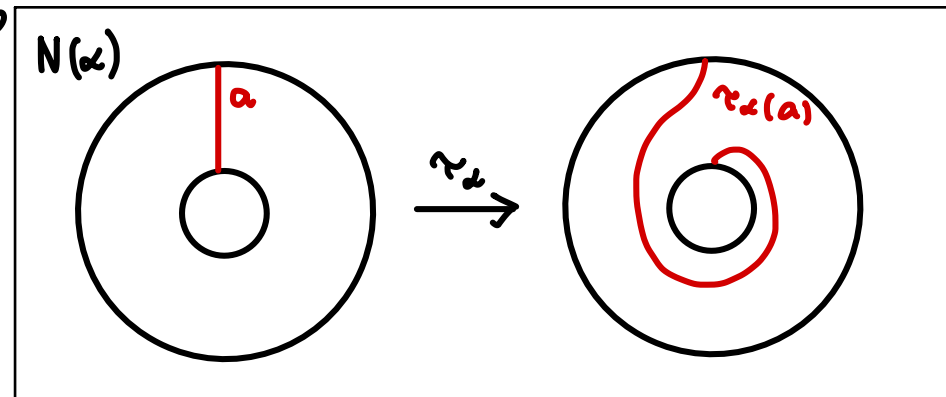
WE NEED TO UNDERSTAND $\{ \psi: S \rightarrow S : \psi|_{\partial S} = \text{id} \} / \text{ISOTOPY} =: \text{MCG}(S)$

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THM (LICKORISH) $\text{MCG}(S)$ IS GENERATED
BY DEHN TWISTS ALONG



→ WE CAN REPHRASE/SOLVE PROBLEMS ABOUT CTC STRUCTURES
COMBINATORIALLY (CURVES ON A SURFACE)

INTERLUDE - DESCRIBING MONODROMIES

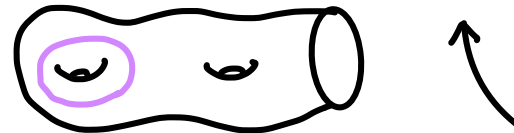
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POWER OF OPEN BOOKS: TO DESCRIBE CTCT STRUCTURES

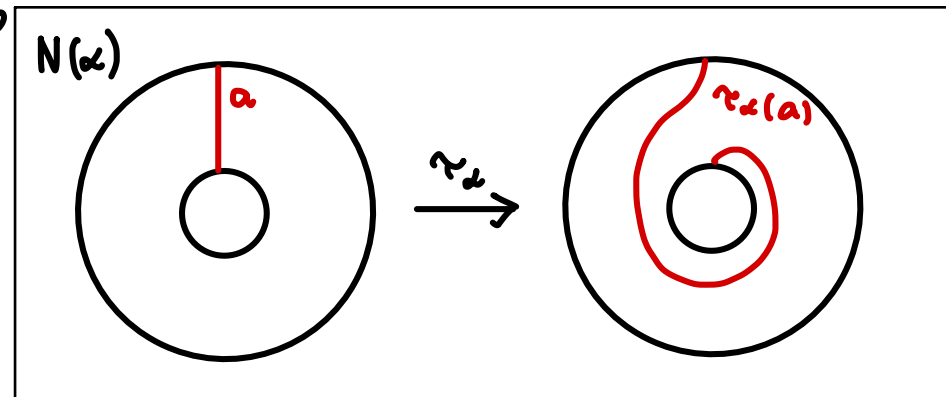
WE NEED TO UNDERSTAND $\{\psi: S \rightarrow S : \psi|_{\partial S} = \text{id}\} / \text{ISOTOPY} =: \text{MCG}(S)$

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THM (DEHN): $\text{MCG}(S)$ IS GENERATED BY DEHN TWISTS ALONG
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→ WE CAN REPHRASE/SOLVE PROBLEMS ABOUT CTC STRUCTURES
COMBINATORIALLY (CURVES ON A SURFACE)

GIRoux CORRESPONDANCE ALLOWS MS TO USE OPEN BOOKS
TO DEFINE INVARIANTS OF CTC STRUCTURES

OPEN BOOKS & CONTACT STRUCTURES

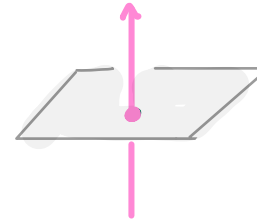
DEF: AN OBD (B, π) SUPPORTS A CONTACT STRUCTURE IF

- B IS TRANSVERSE ($TB \not\subset \xi$)

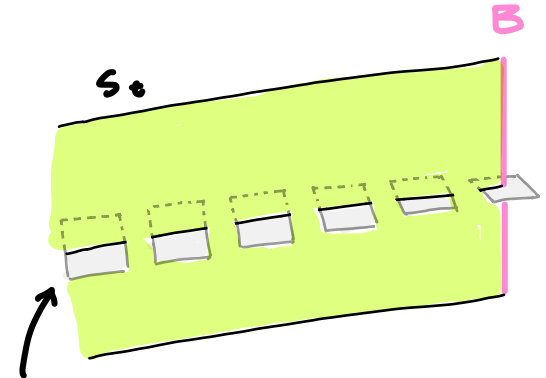
- $d\alpha|_{T\dot{S}_t} > 0$

$\pi^{-1}(t)$

POSITIVE AREA FORM ON S_t :



$$\alpha|_B > 0$$



ξ NEVER GETS $= TS_t$

OPEN BOOKS & CONTACT STRUCTURES

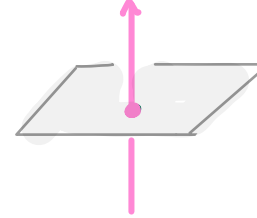
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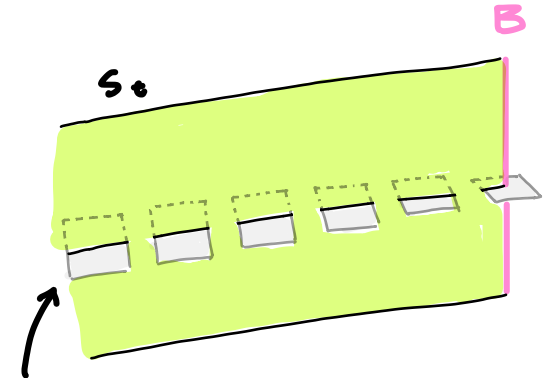
- $d\alpha|_{TS_t} > 0$

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ξ NEVER GETS $= TS_t$

WE WILL HAVE A MORE TOPOLOGICAL DEF LATER

CONSTRUCTION (THURSTON-VINKENKEMPLEY): ANY OBD SUPPORTS A CONTACT STRUCTURE THAT IS UNIQUE UP TO ISOTOPY

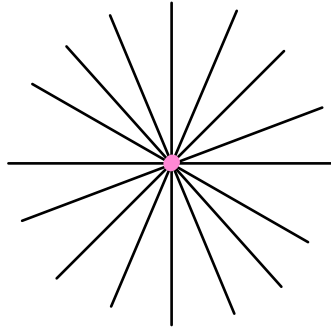
IDEA: • 1-PARAMETER FAMILY OF AREA FORM $\{\beta_t\}$

$$\rightsquigarrow \alpha = \beta_t + K dt \quad \text{WHERE } K \gg 0$$

- + LOCAL CONSTRUCTION NEAR BINDING

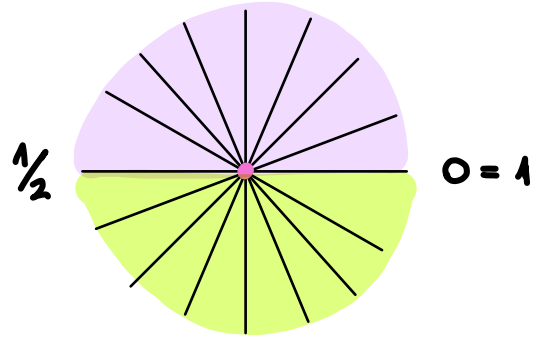
OPEN BOOK - & CONTACT HEEGAARD DECOMPOSITIONS

GIVEN AN OBD (B, π)



OPEN BOOK DECOMPOSITIONS $\leadsto$ HEEGAARD DECOMPOSITIONS

GIVEN AN OBD $(B, \pi) \leadsto$ CONSIDER $M = U \cup_{\Sigma} V$ WHERE

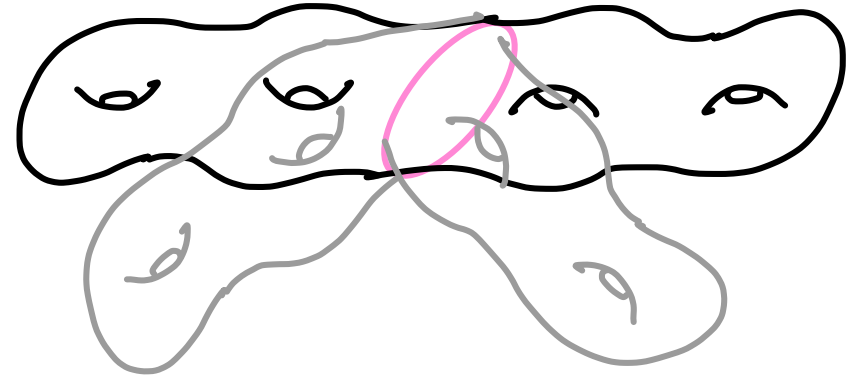


$$U = \overline{\pi^{-1}([0, 1/2])}$$

$$V = \overline{\pi^{-1}([1/2, 1])}$$

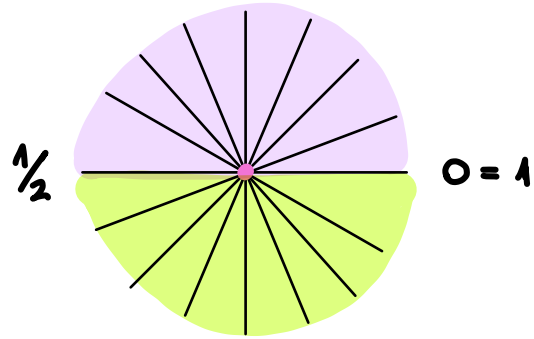
HERE $\Sigma = S_0 \cup S_1$

PROP: U & V ARE HANDLEBODIES



OPEN BOOK DECOMPOSITIONS $\leadsto$ HEEGAARD DECOMPOSITIONS

GIVEN AN OBD $(B, \pi) \leadsto$ CONSIDER $M = U \cup_{\Sigma} V$ WHERE



$$U = \overline{\pi^{-1}([0, 1/2])}$$

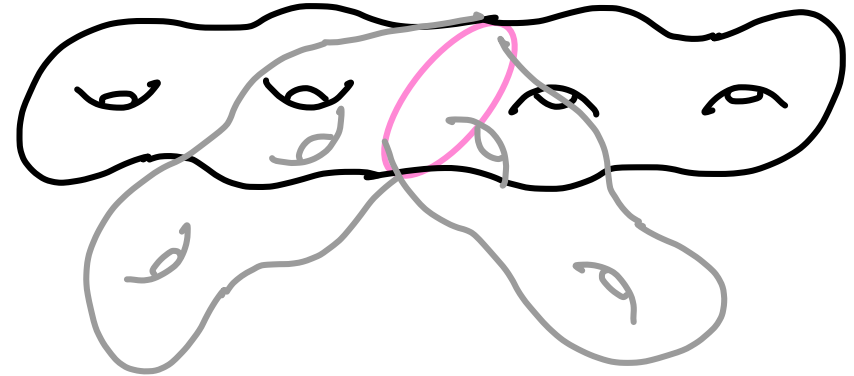
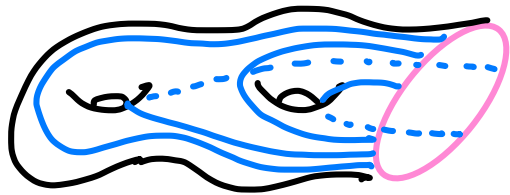
$$V = \overline{\pi^{-1}([1/2, 1])}$$

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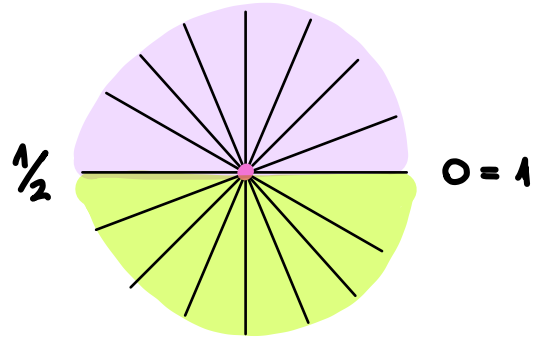
PROOF: FOR U :

- LET $a_1, \dots, a_{2g+b}$ BE ARCS ON S SUCH THAT $S - \cup a_i \cong D^2$



OPEN BOOK DECOMPOSITIONS $\leadsto$ HEEGAARD DECOMPOSITIONS

GIVEN AN OBD $(B, \pi) \leadsto$ CONSIDER $M = U \cup_{\Sigma} V$ WHERE



$$U = \overline{\pi^{-1}([0, 1/2])}$$

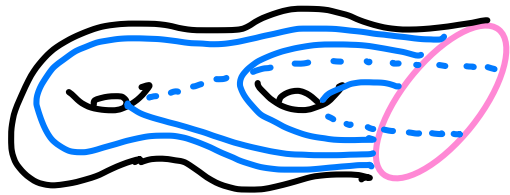
$$V = \overline{\pi^{-1}([1/2, 1])}$$

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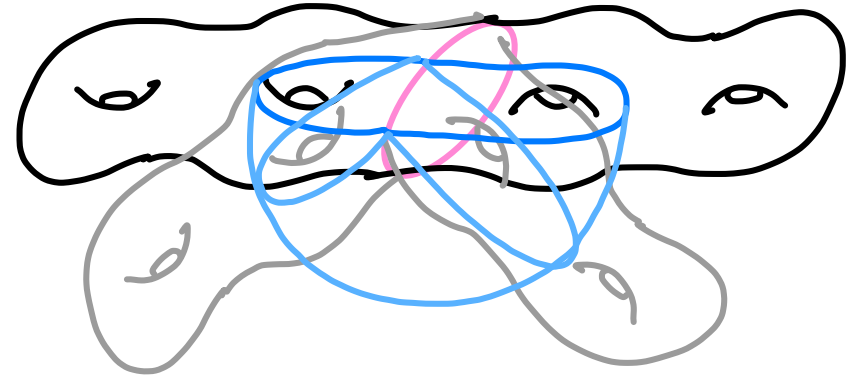
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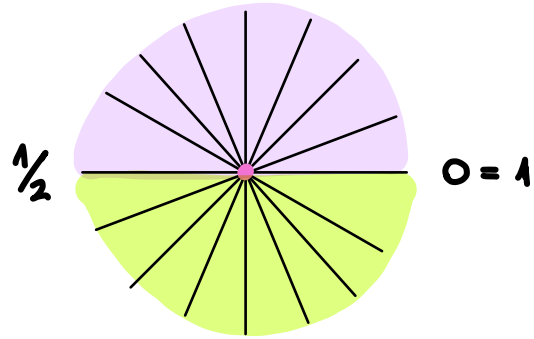


$$\text{THEN } D_i := a_i \times [0, 1/2] / \sim = \begin{array}{|c|} \hline \text{rectangle with horizontal blue lines} \\ \hline \end{array} \cong \begin{array}{c} \text{blue oval with two pink dots} \\ \hline \end{array} \cong D^2$$



OPEN BOOK DECOMPOSITIONS $\leadsto$ HEEGAARD DECOMPOSITIONS

GIVEN AN OBD $(B, \pi) \leadsto$ CONSIDER $M = U \cup_{\Sigma} V$ WHERE



$$U = \overline{\pi^{-1}([0, 1/2])}$$

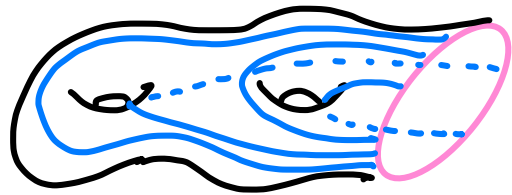
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HERE $\Sigma = S_0 \cup S_1$

PROP: U & V ARE HANDLEBODIES

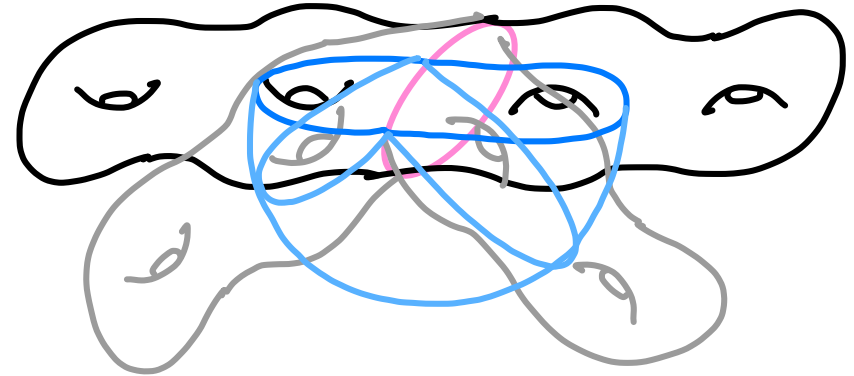
PROOF: FOR U :

- LET $\alpha_1, \dots, \alpha_{2g+b}$ BE ARCS ON S SUCH THAT $S - \cup \alpha_i \cong D^2$



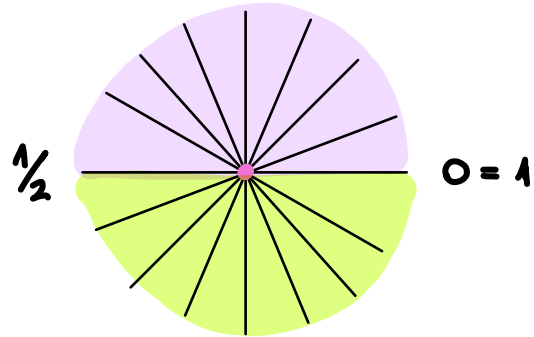
$$\text{THEN } D_i := \alpha_i \times [0, 1/2] / \sim = \begin{array}{|c|} \hline \text{rectangle with horizontal lines} \\ \hline \end{array} \cong \begin{array}{c} \text{blue oval with two points} \\ \hline \end{array} \cong D^2$$

- & $U - \cup D_i = (S - \cup \alpha_i) \times [0, 1/2] / \sim \cong D^2 \times [0, 1/2] / \sim \cong D^3$



OPEN BOOK DECOMPOSITIONS $\leadsto$ HEEGAARD DECOMPOSITIONS

GIVEN AN OBD $(B, \pi) \leadsto$ CONSIDER $M = U \cup_{\Sigma} V$ WHERE



$$U = \overline{\pi^{-1}([0, 1/2])}$$

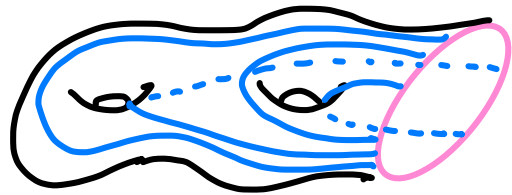
$$V = \overline{\pi^{-1}([1/2, 1])}$$

HERE $\Sigma = S_0 \cup S_1$

PROP: U & V ARE HANDLEBODIES

PROOF: FOR U :

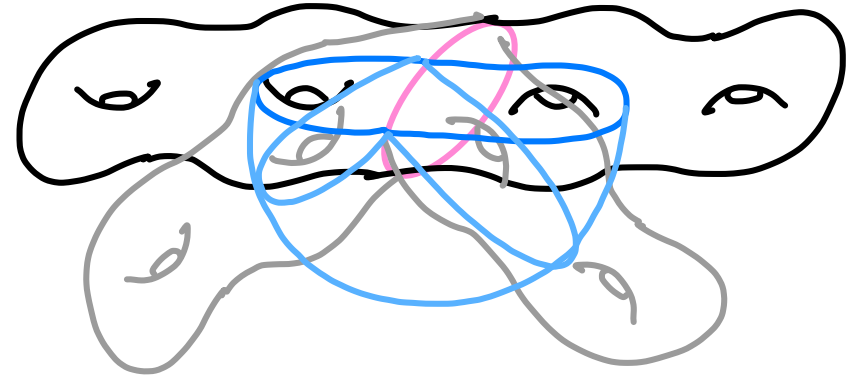
- LET $a_1, \dots, a_{2g+b}$ BE ARCS ON S SUCH THAT $S - \cup a_i \cong D^2$



$$\text{THEN } D_i := a_i \times [0, 1/2] / \sim = \begin{array}{|c|} \hline \text{rectangle with horizontal lines} \\ \hline \end{array} \cong \begin{array}{c} \text{blue oval with two points} \\ \hline \end{array} \cong D^2$$

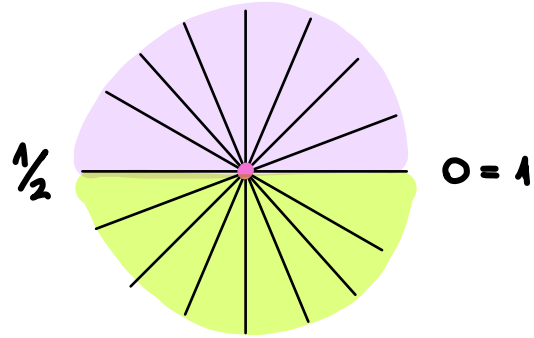
$$\bullet \text{ \& } U - \cup D_i = (S - \cup a_i) \times [0, 1/2] / \sim \cong D^2 \times [0, 1/2] / \sim \cong D^3$$

$\Rightarrow U$ IS A HANDLEBODY $\blacksquare$



OPEN BOOK DECOMPOSITIONS $\leadsto$ HEEGAARD DECOMPOSITIONS

GIVEN AN OBD $(B, \pi) \leadsto$ CONSIDER $M = U \cup_{\Sigma} V$ WHERE



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HERE $\Sigma = S_0 \cup S_1$

PROP: U & V ARE HANDLEBODIES

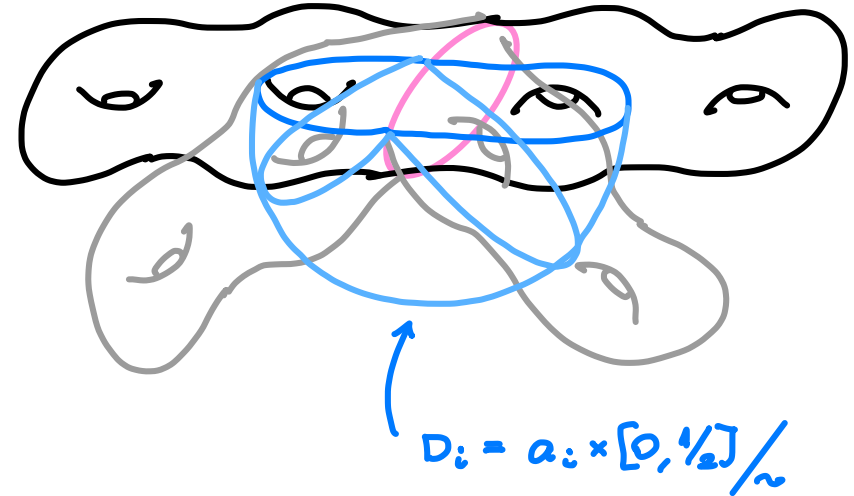
MOREOVER: $(U, \Sigma, \Gamma = B)$ IS

PRODUCT DISC DECOMPOSABLE

$\exists$ DISCS D_i SUCH THAT

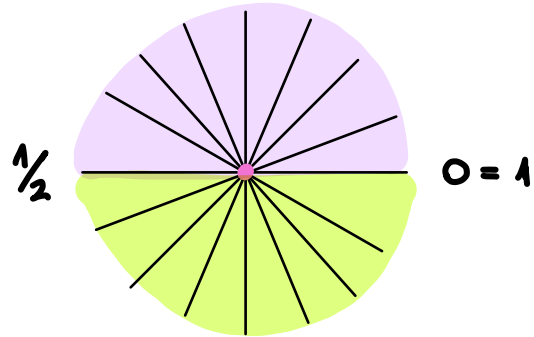
- $U - \cup D_i \cong D^3$

- $\& |\partial D_i \cap \Gamma| = 2 \quad \forall i$



OPEN BOOK DECOMPOSITIONS $\leadsto$ HEEGAARD DECOMPOSITIONS

GIVEN AN OBD $(B, \pi) \leadsto$ CONSIDER $M = U \cup_{\Sigma} V$ WHERE



$$U = \pi^{-1}([0, 1/2])$$

$$V = \pi^{-1}([1/2, 1])$$

HERE $\Sigma = S_0 \cup S_1$

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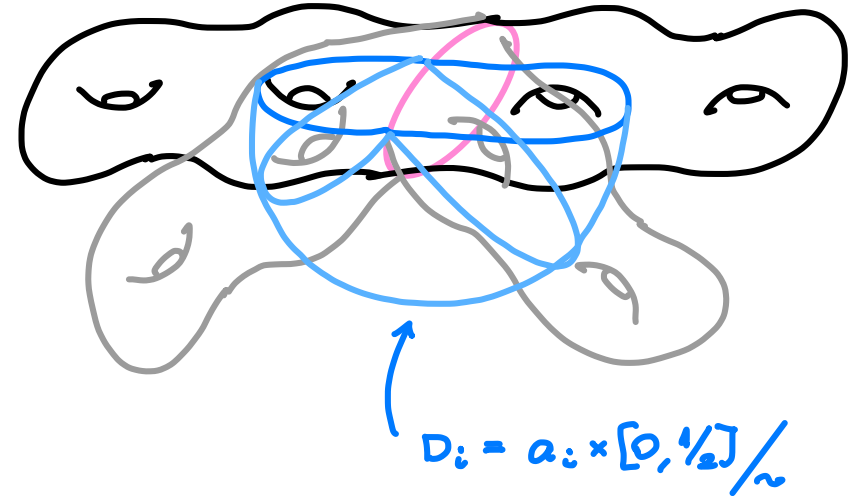
MOREOVER: $(U, \Sigma, \Gamma = B)$ IS

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$\exists$ DISCS D_i SUCH THAT

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WE HAVE SEEN: AN OPEN BOOK DECOMPOSITION DEFINES
A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE
HANDLEBODIES

WARNING: NEED Γ ON Σ

OPEN BOOK - & CONTACT HEEGAARD DECOMPOSITIONS

THM: AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

$$(B, \pi) \rightsquigarrow M = U \cup_{(\Sigma, \Gamma)} V$$

LET'S LOOK AT THE CONTACT STRUCTURE SUPPORTED BY (B, π)

THM (TORISU): THE SURFACE Σ IS CONVEX WITH DIVIDING CURVE Γ . THE CONTACT STRUCTURES $\mathfrak{g}|_U$ & $\mathfrak{g}|_V$ ARE TIGHT.

OPEN BOOK - & CONTACT HEEGAARD DECOMPOSITIONS

THM: AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

$$(B, \pi) \rightsquigarrow M = U \cup_{(\Sigma, \Gamma)} V$$

LET'S LOOK AT THE CONTACT STRUCTURE SUPPORTED BY (B, π)

THM (TORISU): THE SURFACE Σ IS CONVEX WITH DIVIDING CURVE Γ . THE CONTACT STRUCTURES $\mathfrak{z}|_U$ & $\mathfrak{z}|_V$ ARE TIGHT.

SO: $(U, \mathfrak{z}_U)$ & $(V, \mathfrak{z}_V)$ ARE CONTACT HANDLEBODIES

THUS $M = U \cup_{(\Sigma, \Gamma)} V$ IS A CONTACT HEEGAARD DECOMPOSITION

OPEN BOOK - & CONTACT HEEGAARD DECOMPOSITIONS

THM: AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

$$(B, \pi) \rightsquigarrow M = U \cup_{(\Sigma, \Gamma)} V$$

LET'S LOOK AT THE CONTACT STRUCTURE SUPPORTED BY (B, π)

THM (TORISU): THE SURFACE Σ IS CONVEX WITH DIVIDING CURVE Γ . THE CONTACT STRUCTURES $\mathfrak{z}|_U$ & $\mathfrak{z}|_V$ ARE TIGHT.

SO: $(U, \mathfrak{z}_U)$ & $(V, \mathfrak{z}_V)$ ARE CONTACT HANDLEBODIES

THUS $M = U \cup_{(\Sigma, \Gamma)} V$ IS A CONTACT HEEGAARD DECOMPOSITION

THIS GIVES RISE TO AN EQUIVALENT DEFINITION:

DEF: $\mathfrak{z}$ IS SUPPORTED BY THE OPEN BOOK (B, π) IF THE HEEGAARD DECOMPOSITION DEFINED BY (B, π) IS A CONTACT HEEGAARD DECOMPOSITION

HEEGAARD DECOMPOSITIONS $\leadsto$ OPEN BOOK DECOMPOSITIONS

PROP. (U, Γ) PRODUCT DISC DECOMPOSABLE HANDLEBODY

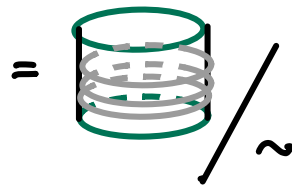
$$\Rightarrow U = S \times I / (x, t) \sim (x, t') \quad x \in \partial S, t, t' \in I$$

SUCH THAT $\partial U = S \times 0 \cup_{\Gamma} S \times 1$

$$\& \Gamma = \partial S / \sim$$

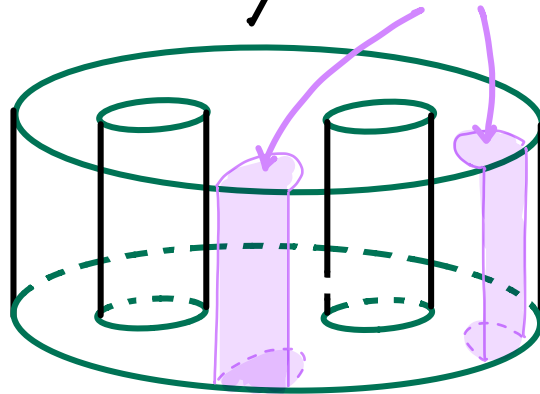
IDEA: INDUCTION ON THE # OF PRODUCT DISCS

► $n=0$



IDENTIFIED

► $n \leadsto n+1$



HEEGAARD DECOMPOSITIONS $\leadsto$ OPEN BOOK DECOMPOSITIONS

PROP. (U, Γ) PRODUCT DISC DECOMPOSABLE HANDLEBODY

$$\Rightarrow U = S \times I / (x, t) \sim (x, t') \quad x \in \partial S, t, t' \in I$$

$$\text{SUCH THAT } \partial U = S \times 0 \cup_{\Gamma} S \times 1$$

$$\& \Gamma = \partial S / \sim$$

GIVEN A CONTACT HEEGAARD DECOMPOSITION $M = U \cup_{(\Sigma, \Gamma)} V$

U

$$\partial U = \Sigma = R_+ \cup_{\Gamma} R_-$$

V

$$\partial V = -\Sigma = -R_+ \cup_{\Gamma} R_-$$

HEEGAARD DECOMPOSITIONS $\leadsto$ OPEN BOOK DECOMPOSITIONS

PROP. (U, Γ) PRODUCT DISC DECOMPOSABLE HANDLEBODY

$$\Rightarrow U = S \times I / (x, t) \sim (x, t') \quad x \in \partial S, t, t' \in I$$

$$\text{SUCH THAT } \partial U = S \times 0 \cup_{\Gamma} S \times 1$$

$$\& \Gamma = \partial S / \sim$$

GIVEN A CONTACT HEEGAARD DECOMPOSITION $M = U \cup_{(\Sigma, \Gamma)} V$

$$\text{BY } \underline{\text{PROP.}} \quad U = S \times [0, \frac{1}{2}] \quad \partial U = \Sigma = \overset{S \times 0}{=} R_+ \cup_{\Gamma} \overset{-S \times \frac{1}{2}}{=} R_-$$

$$V = S \times [\frac{1}{2}, 1] \quad \partial V = -\Sigma = \overset{\downarrow}{=} -R_+ \cup_{\Gamma} \overset{\downarrow}{=} R_-$$

$$\qquad \qquad \qquad \overset{=}{\parallel} \quad \overset{=}{\parallel}$$

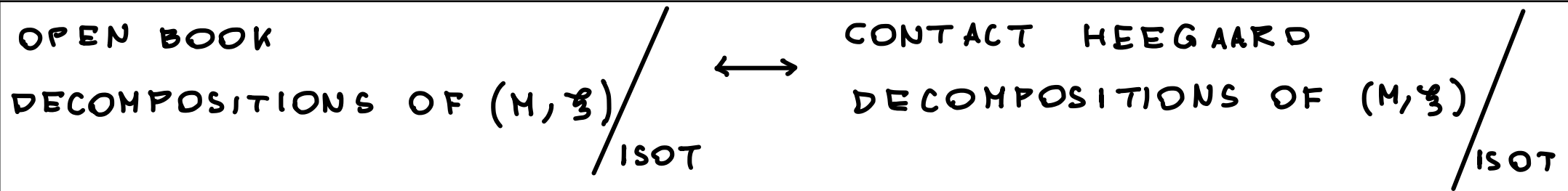
$$\qquad \qquad \qquad -S \times 1 \quad S \times \frac{1}{2}$$

$\Rightarrow$ GLUES TO A FULL FIBRATION (B, π)

GIVEN BY PROJECTION ONTO $[0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$

OPEN BOOK - & CONTACT HEEGAARD DECOMPOSITIONS

SO WE GET A ONE-TO-ONE CORRESPONDANCE



SO WE CAN WORK WITH WHICHEVER IS MORE CONVENIENT

RECALL :

THM: EVERY CONTACT MANIFOLD (M, ξ) ADMITS A
CONTACT HEEGAARD DECOMPOSITION

COR: EVERY CONTACT MANIFOLD (M, ξ) ADMITS AN
OPEN BOOK DECOMPOSITION

COMING UP:

- STABILISATION
- STATEMENT OF GIROUX CORRESPONDENCE
- IDEA OF PROOF

SRNI - LECTURE 3

PROOF OF GIROUX CORRESPONDENCE

- STABILISATION
- STATEMENT OF GIROUX CORRESPONDENCE
- IDEA OF PROOF
- FURTHER DIRECTIONS

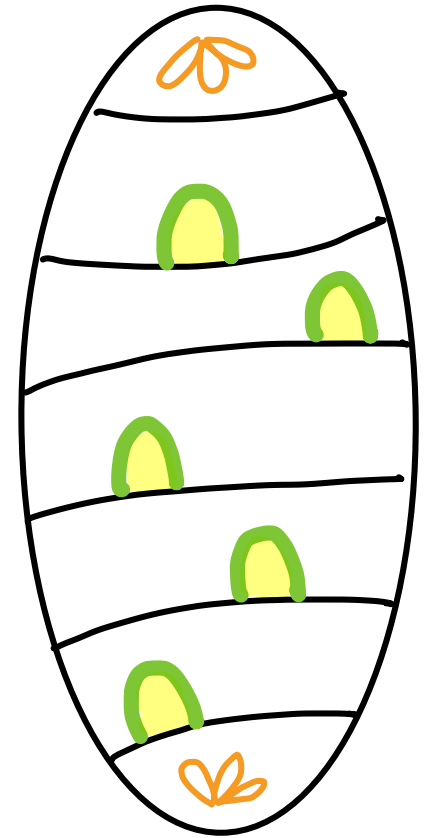


THE GIROUX CORRESPONDENCE VIA CONVEX SURFACES

VERA VÉRTESI

JOINT WORK WITH JOAN LICATA

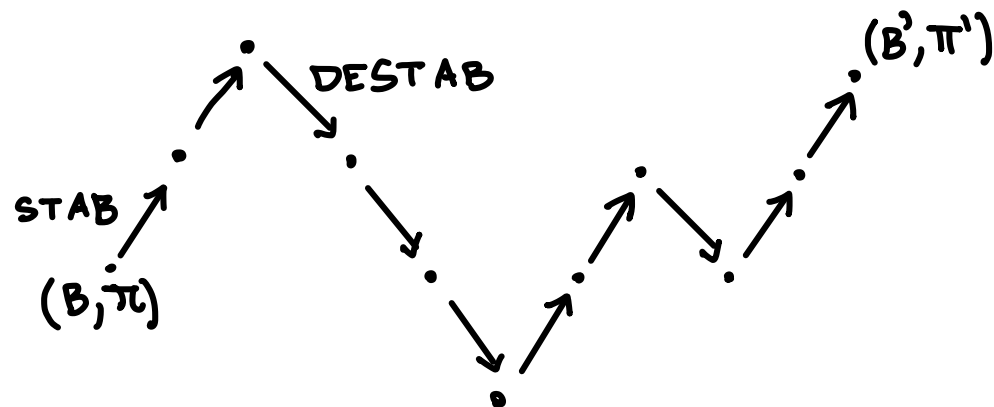
UNIVERSITY OF VIENNA



LECTURE 3

PROOF OF GIROUX'S CORRESPONDENCE

- STABILISATION
- STATEMENT OF GIROUX CORRESPONDENCE
- IDEA OF PROOF
- FURTHER DIRECTIONS



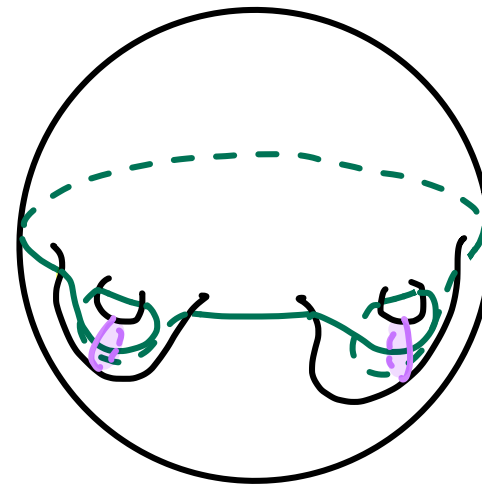
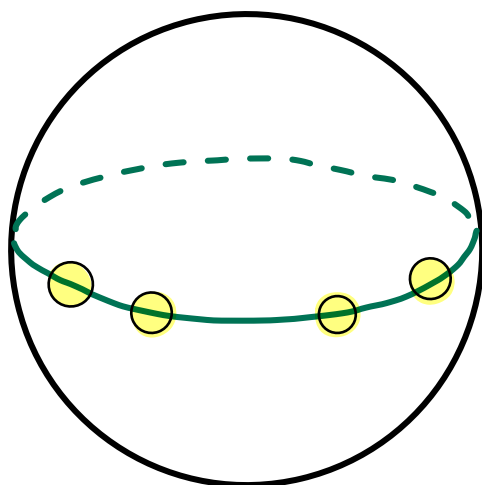
LAST TIME

- CONTACT HANDLEBODY:

- CTCT $D-h \cup 1-h's$
- $N(\text{LEGENDRIAN GRAPH})$

→ TIGHT
+

PRODUCT DISC DECOMPOSABLE



- CONTACT HEEGAARD DECOMPOSITION

$$M = U \cup_{\Sigma} V$$

- Σ CONVEX
- U & V CONTACT HANDLEBODY

- OPEN BOOK DECOMPOSITION



- OPEN BOOK

DECOMPOSITIONS OF (M, ξ)

/ ISOT



CONTACT HEEGAARD

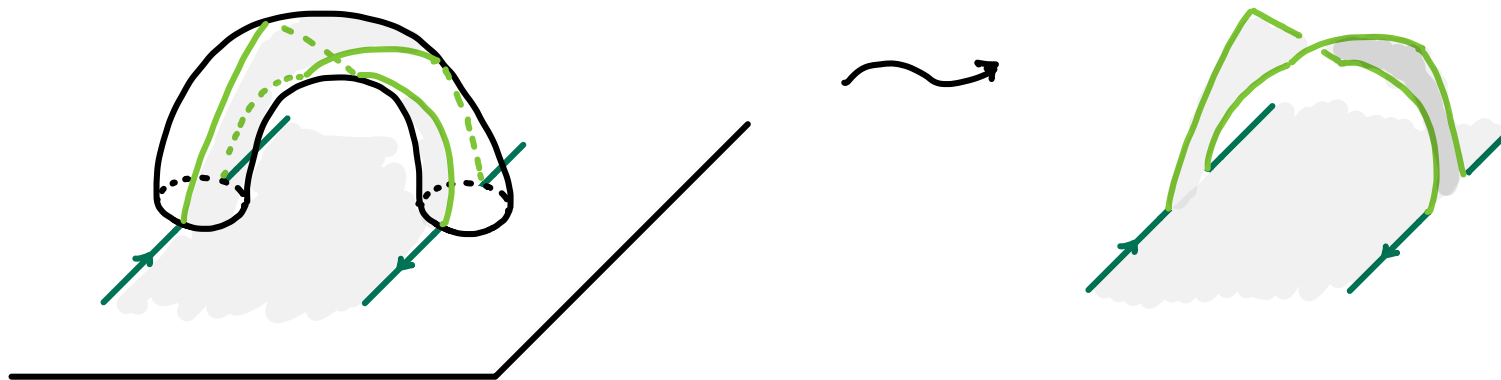
DECOMPOSITIONS OF (M, ξ)

/ ISOT

STABILISATION

OF HEEGAARD DECOMPOSITIONS
- SMOOTH
- CONTACT

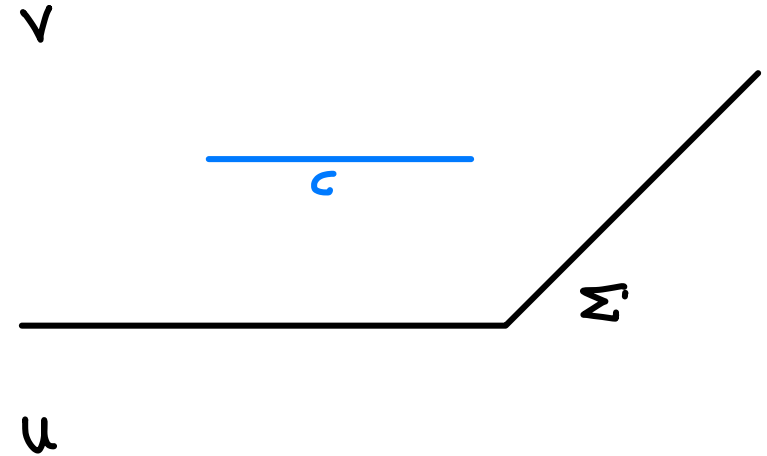
OPEN BOOK DECOMPOSITIONS



STABILISATION OF HEEGAARD DECOMPOSITIONS

$M = U \cup_{\Sigma} V$ HEEGAARD DECOMPOSITION

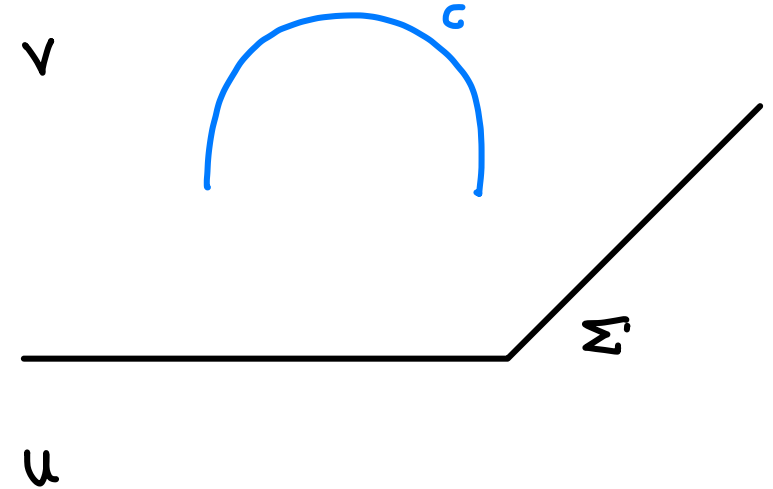
- c ARC ON Σ



STABILISATION OF HEEGAARD DECOMPOSITIONS

$M = U \cup_{\Sigma} V$ HEEGAARD DECOMPOSITION

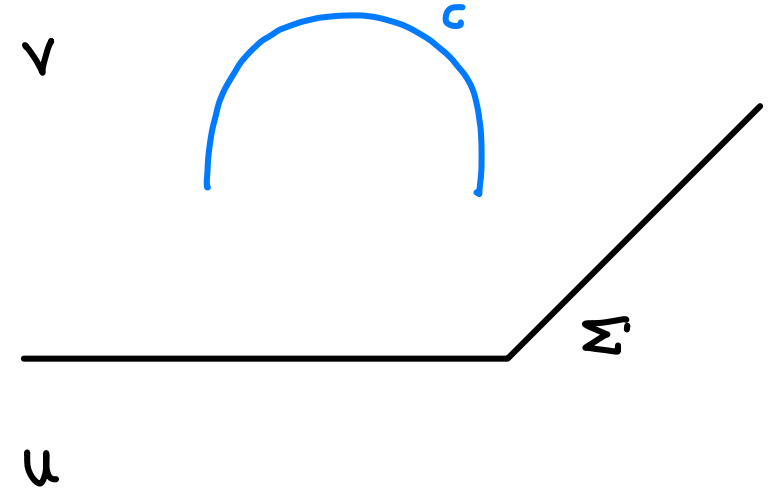
- c ARC ON Σ
- ISOTOPE $\dot{c}$ INTO V



STABILISATION OF HEEGAARD DECOMPOSITIONS

$M = U \cup_{\Sigma} V$ HEEGAARD DECOMPOSITION

- c ARC ON Σ
- ISOTOPE $\dot{c}$ INTO V

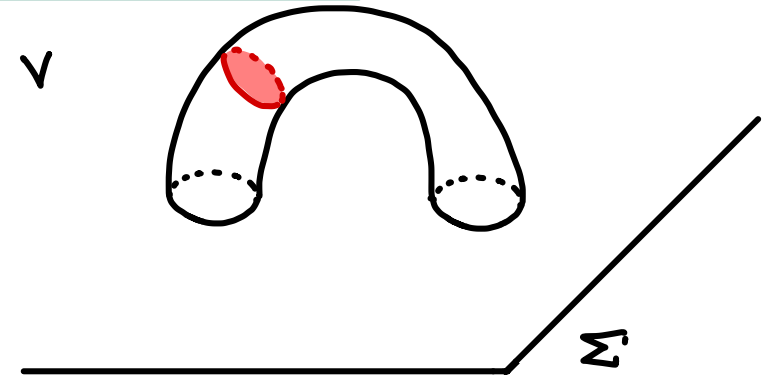


STABILISATION OF HEEGAARD DECOMPOSITIONS

$M = U \cup_{\Sigma} V$ HEEGAARD DECOMPOSITION

- c ARC ON Σ
- ISOTOPE $\dot{c}$ INTO V

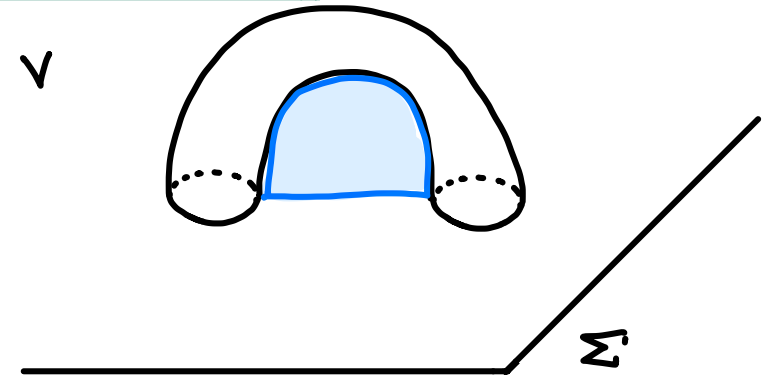
$\rightsquigarrow - U^{\sharp} := U \cup N(c)$ IS A HANDLEBODY U



STABILISATION OF HEEGAARD DECOMPOSITIONS

$M = U \cup_{\Sigma} V$ HEEGAARD DECOMPOSITION

- c ARC ON Σ
- ISOTOPE $\dot{c}$ INTO V

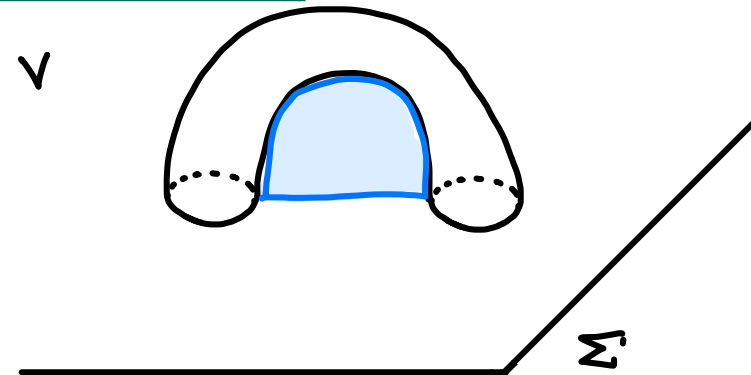


- $\rightsquigarrow$
- $U' := U \cup N(c)$ IS A HANDLEBODY U
 - $V' := V \setminus N(c)$ IS ALSO A HANDLEBODY
 - $\Sigma' := \partial U'$

STABILISATION OF HEEGAARD DECOMPOSITIONS

$$M = U \underset{Z}{V} \quad \text{H E E G A A R D D E C O M P O S I T I O N}$$

- c ARC ON Σ
- ISOTOPE $\dot{c}$ INTO V



$\rightsquigarrow$ - $U' := U \cup N(c)$ IS A HANDLEBODY U
 - $V' := V \setminus N(c)$ IS ALSO A HANDLEBODY
 - $\Sigma' := \partial U'$

DEF:

$M = U U_{(\varepsilon, \tau)} V$ $M = U' U_{(\varepsilon', \tau')} V'$

STABILISATION DESTABILISATION

The diagram illustrates the relationship between two matrix factorizations, M and M' . On the left, $M = U U_{(\varepsilon, \tau)} V$ is written. On the right, $M' = U' U_{(\varepsilon', \tau')} V'$ is written. A curved arrow points from M to M' , labeled STABILISATION in red. A curved arrow points from M' back to M , labeled DESTABILISATION in red.

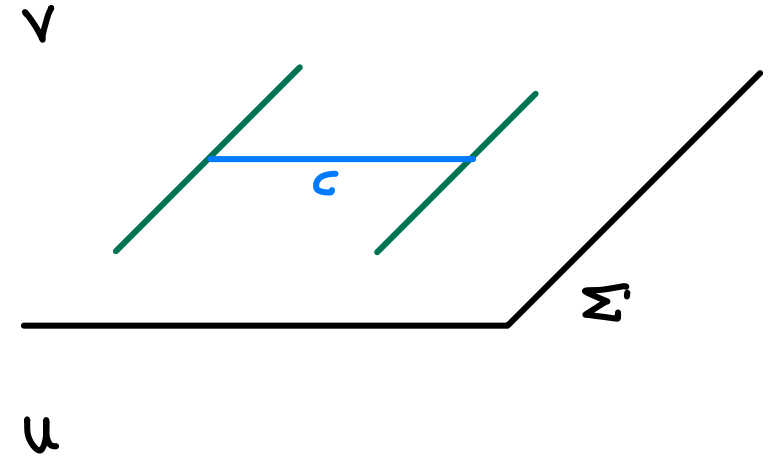
REIDEMEISTER-SINGER THEOREM: ANY TWO HD-S OF M^3 ARE RELATED BY STABILISATIONS & DESTABILISATIONS

STABILISATION OF CONTACT HEEGAARD DECOMPOSITIONS

(M, ξ) CONTACT 3-MANIFOLD

$M = U \cup_{(\Sigma, \Gamma)} V$ CONTACT HD

- c LEGENDRIAN ARC ON Σ
WITH $TW_c(\xi, T\Sigma) = -\frac{1}{2}$



STABILISATION OF CONTACT HEEGAARD DECOMPOSITIONS

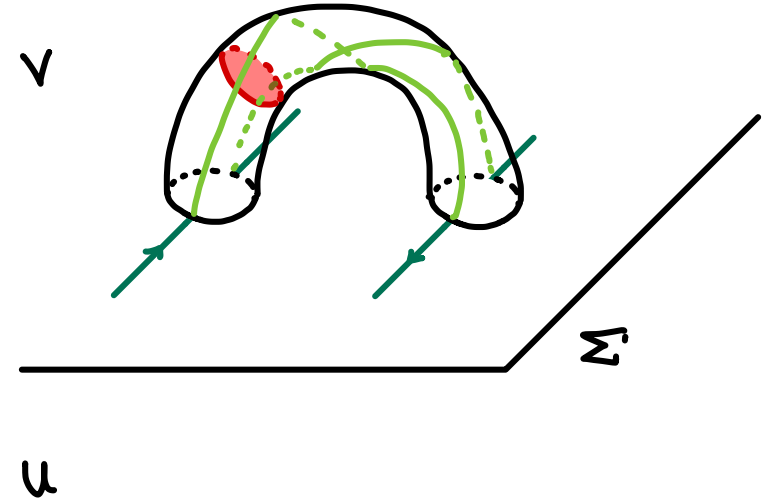
(M, ξ) CONTACT 3-MANIFOLD

$M = U \cup_{(\Sigma, \Gamma)} V$ CONTACT HD

- c LEGENDRIAN ARC ON Σ
WITH $TW_c(\xi, T\Sigma) = -\frac{1}{2}$

- LEGENDRIAN ISOTOPE $\dot{c}$ INTO V

$\rightsquigarrow - U' := U \cup N(c)$ IS A CONTACT HANDLEBODY



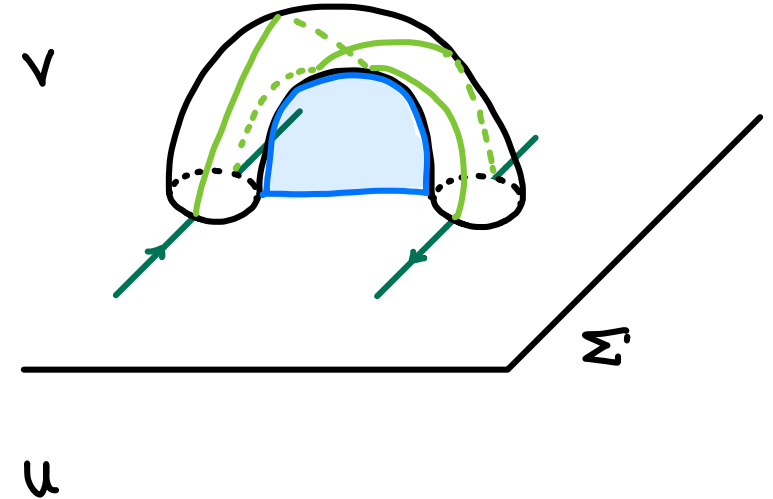
STABILISATION OF CONTACT HEEGAARD DECOMPOSITIONS

(M, ξ) CONTACT 3-MANIFOLD

$M = U \cup_{(\Sigma, \Gamma)} V$ CONTACT HD

- c LEGENDRIAN ARC ON Σ
WITH $Tw_c(\xi, T\Sigma) = -\frac{1}{2}$

- LEGENDRIAN ISOTOPE $\dot{c}$ INTO V



- $\rightsquigarrow$
- $U' := U \cup N(c)$ IS A CONTACT HANDLEBODY
 - $V' := V \setminus N(c)$ IS ALSO A CONTACT HANDLEBODY
 - $\Sigma' := \partial U'$

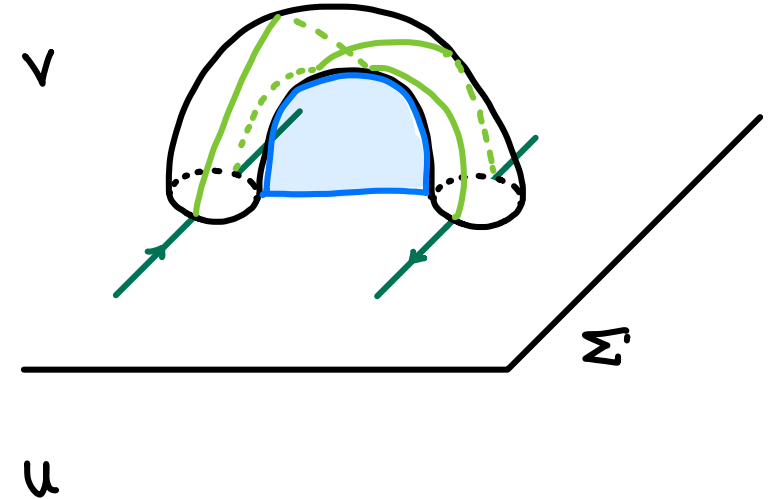
STABILISATION OF CONTACT HEEGAARD DECOMPOSITIONS

(M, ξ) CONTACT 3-MANIFOLD

$M = U \cup_{(\Sigma, \Gamma)} V$ CONTACT HD

- c LEGENDRIAN ARC ON Σ
WITH $Tw_c(\xi, T\Sigma) = -\frac{1}{2}$

- LEGENDRIAN ISOTOPE $\dot{c}$ INTO V



- $\rightsquigarrow$
- $U' := U \cup N(c)$ IS A CONTACT HANDLEBODY
 - $V' := V \setminus N(c)$ IS ALSO A CONTACT HANDLEBODY
 - $\Sigma' := \partial U'$

CONTACT STABILISATION

DEF:

$$M = U \cup_{(\Sigma, \Gamma)} V$$

$$M = U' \cup_{(\Sigma', \Gamma')} V'$$

CONTACT DESTABILISATION

STABILISATION OF OPEN BOOKS

RECALL: CONTACT HEEGAARD
DECOMPOSITION



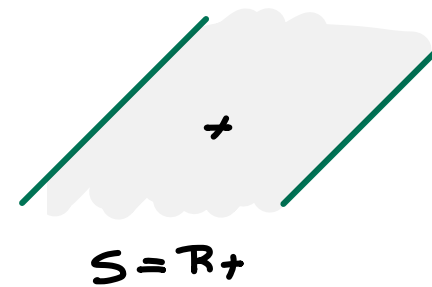
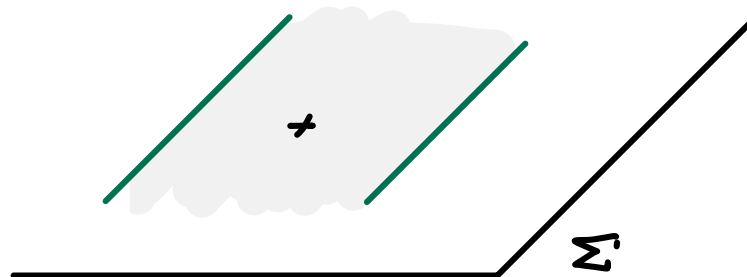
OPEN BOOK
DECOMPOSITION

$$M = U \cup_{(\Sigma, \tau)} V$$



$$(S = R_+, h)$$

V



STABILISATION OF OPEN BOOKS

RECALL: CONTACT HEEGAARD
DECOMPOSITION



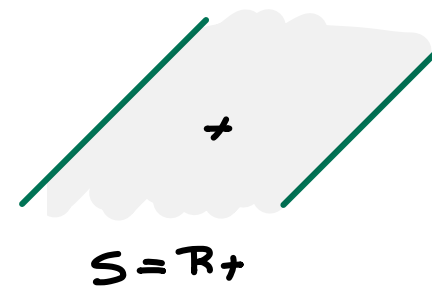
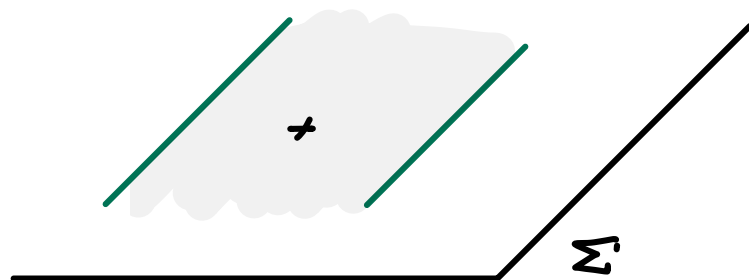
OPEN BOOK
DECOMPOSITION

$$M = U \cup_{(\Sigma, \tau)} V$$



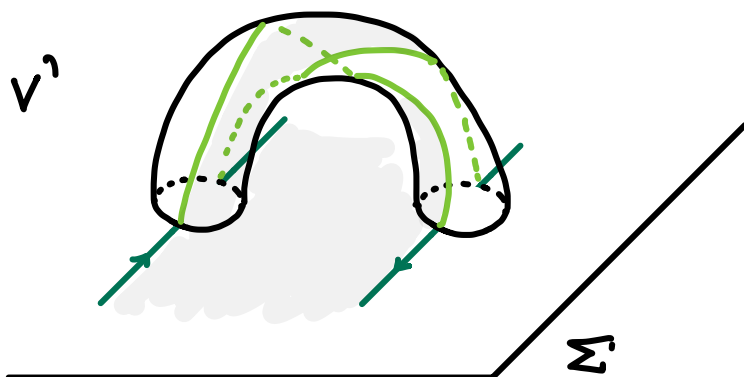
$$(S = R_+, h)$$

V



U

↓ STABILISATION



U'

STABILISATION OF OPEN BOOKS

RECALL: CONTACT HEEGAARD
DECOMPOSITION



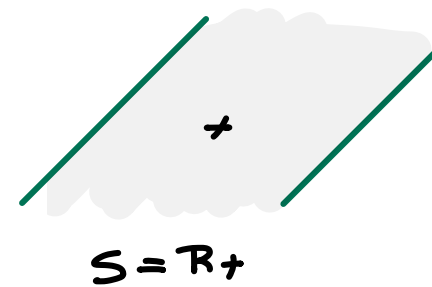
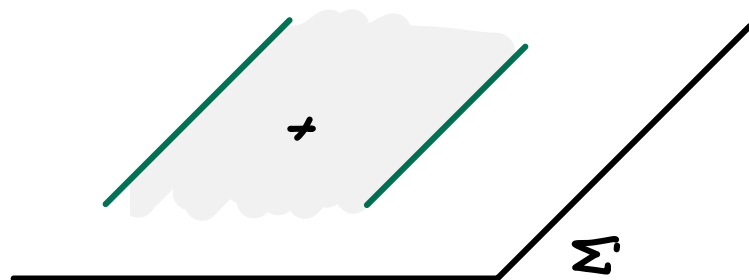
OPEN BOOK
DECOMPOSITION

$$M = U \cup_{(\Sigma, \tau)} V$$



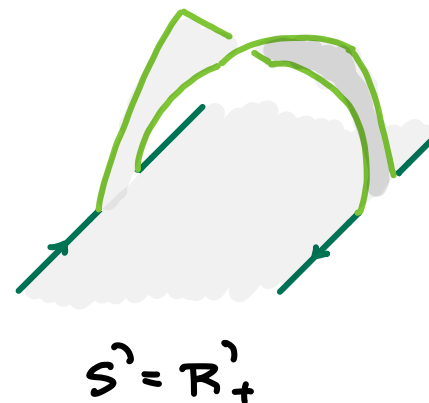
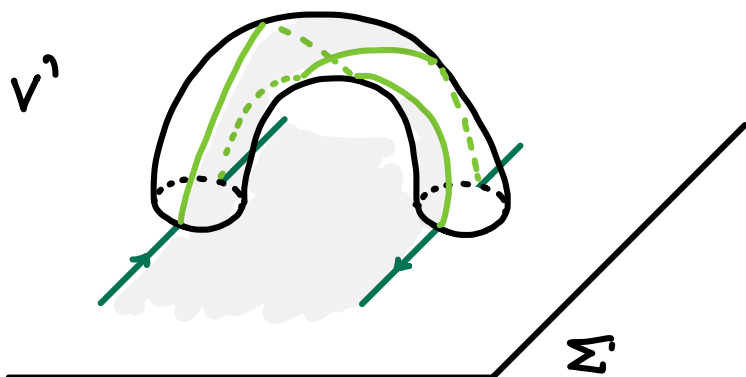
$$(S = R_+, \psi)$$

V



U

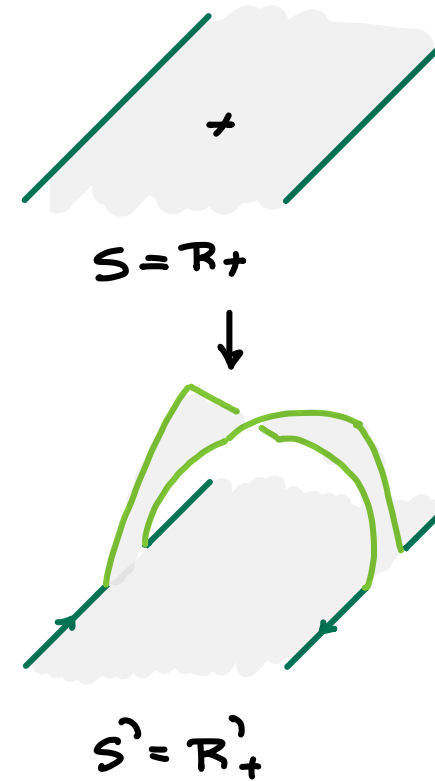
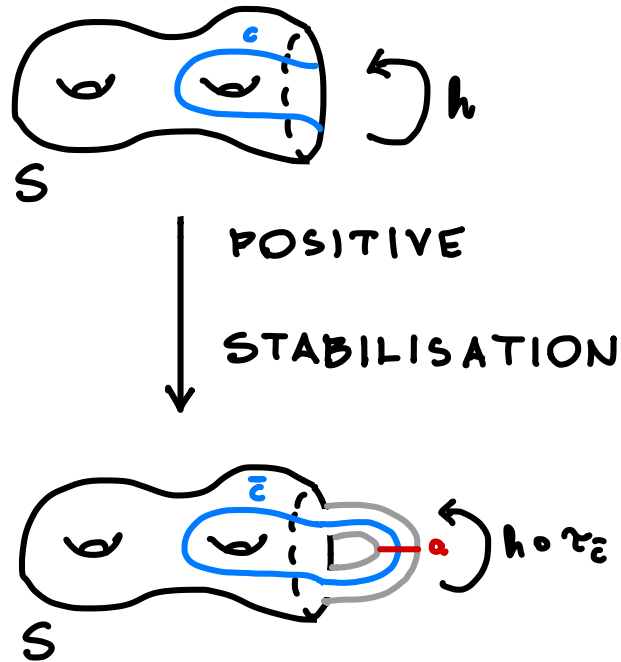
↓ STABILISATION



U'

STABILISATION OF OPEN BOOKS

THE MONODROMIES CAN ALSO BE READ OFF & WE GET



SO UNDER THE CORRESPONDENCE

CONTACT HEEGAARD
DECOMPOSITION



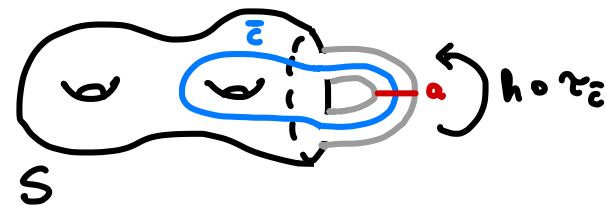
OPEN BOOK
DECOMPOSITION

STABILISATION



POSITIVE STABILISATION

(RE)STATEMENT OF GIROUX CORRESPONDENCE



GIROUX CORRESPONDENCE - REPHRASED

GIROUX CORRESPONDANCE

OPEN BOOKS

POSITIVE

STABILISATION



CONTACT

3-MANIFOLDS

CONTACTOMORPHISM

GIROUX CORRESPONDANCE (REPHRASED)

CONTACT HEEGAARD

DECOMPOSITIONS

CONTACT

STABILISATION



CONTACT

3-MANIFOLDS

CONTACTOMORPHISM

GIROUX CORRESPONDENCE - REPHRASED

GIROUX CORRESPONDANCE

OPEN BOOKS
/ POSITIVE
STABILISATION

$\leftrightarrow$

CONTACT
3-MANIFOLDS
/ CONTACTOMORPHISM

GIROUX CORRESPONDANCE (REPHRASED)

CONTACT HEEGAARD
DECOMPOSITIONS / CONTACT
STABILISATION

$\leftrightarrow$

CONTACT
3-MANIFOLDS
/ CONTACTOMORPHISM

FROM SMOOTH TOPOLOGY:

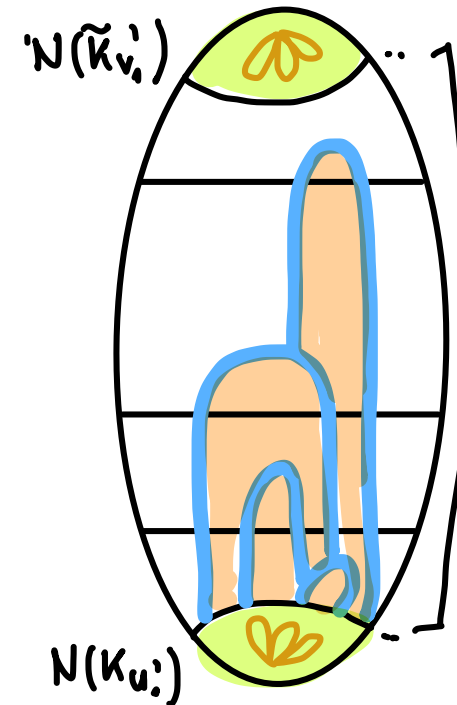
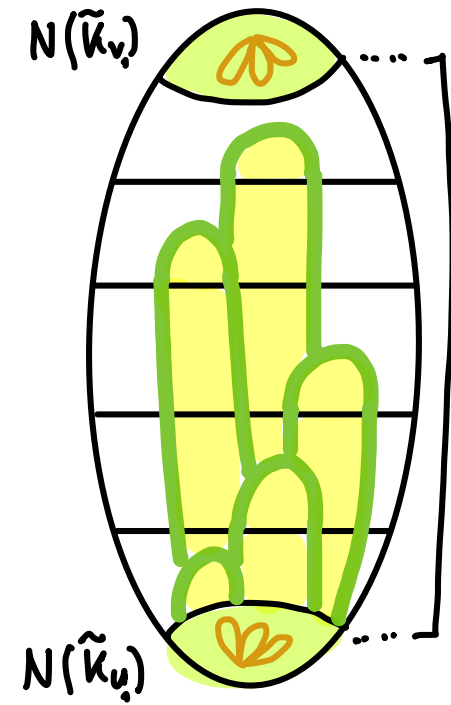
REIDEMEISTER-SINGER THEOREM

HEEGAARD
DECOMPOSITIONS /
STABILISATION

$\leftrightarrow$

SMOOTH
3-MANIFOLDS
/ DIFFEOMORPHISM

IDEA OF PROOF



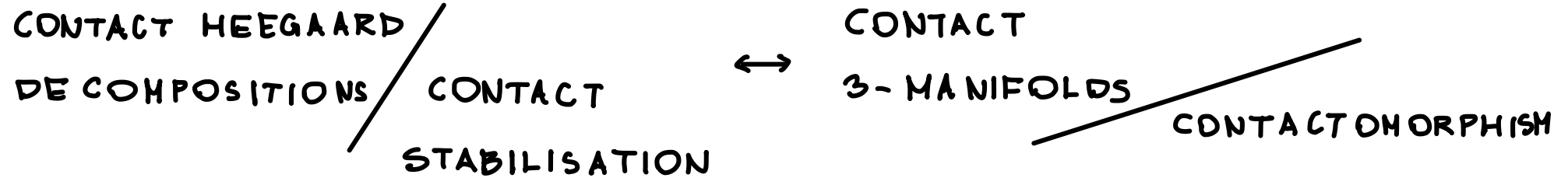
WHAT DO WE WANT TO PROVE?

GIROUX CORRESPONDANCE (REPHRASED)

CONTACT HEEGAARD
DECOMPOSITIONS / CONTACT
STABILISATION $\leftrightarrow$ CONTACT
3-MANIFOLDS / CONTACT OMORPHISM

WHAT DO WE WANT TO PROVE?

GIROUX CORRESPONDANCE (REPHRASED)



MORE PRECISELY : GIVEN TWO CONTACT HEEGAARD DECOMPOSITIONS

$\mathcal{H}^3$

$\mathcal{H}^1$

WHAT DO WE WANT TO PROVE?

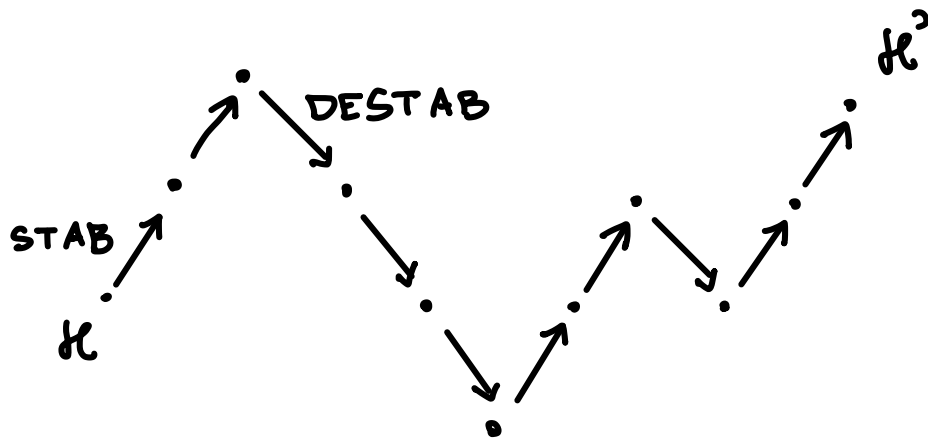
GIROUX CORRESPONDANCE (REPHRASED)

CONTACT HEEGAARD
DECOMPOSITIONS / CONTACT
STABILISATION

$\leftrightarrow$

CONTACT
3-MANIFOLDS / CONTACTOMORPHISM

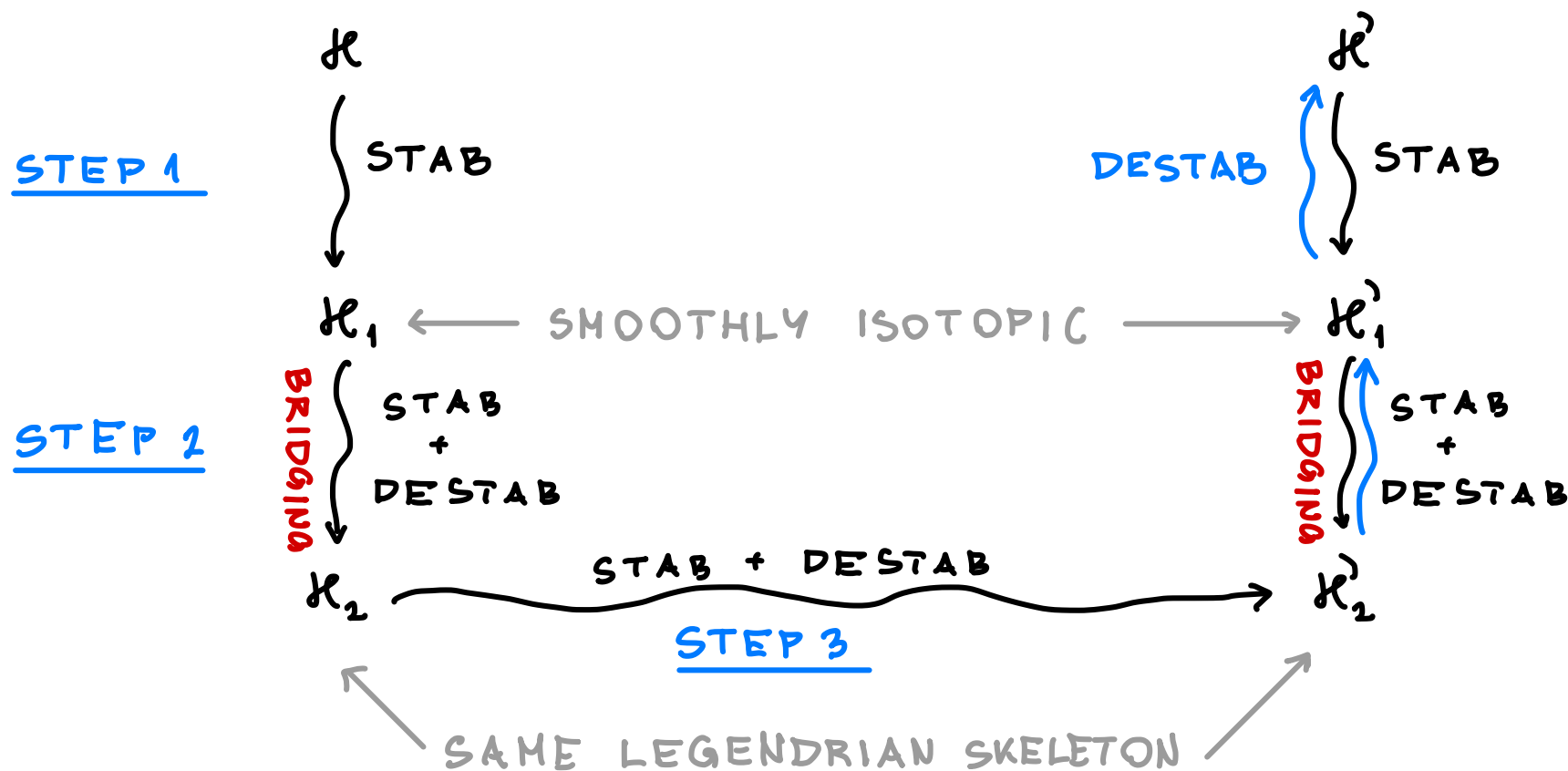
MORE PRECISELY : GIVEN TWO CONTACT HEEGAARD DECOMPOSITIONS



THEN THERE IS A SEQUENCE
OF CONTACT STABILISATIONS
& CONTACT DESTABILISATIONS
CONNECTING THEM

IDEA OF PROOF

IN THREE STEPS WE MAKE THEM MORE & MORE SIMILAR



STEP 1

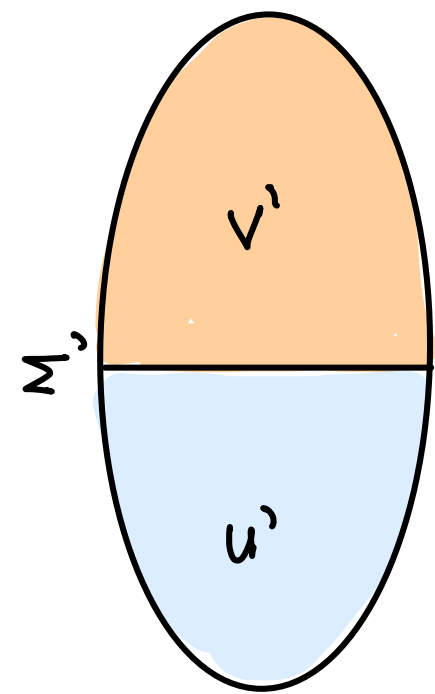
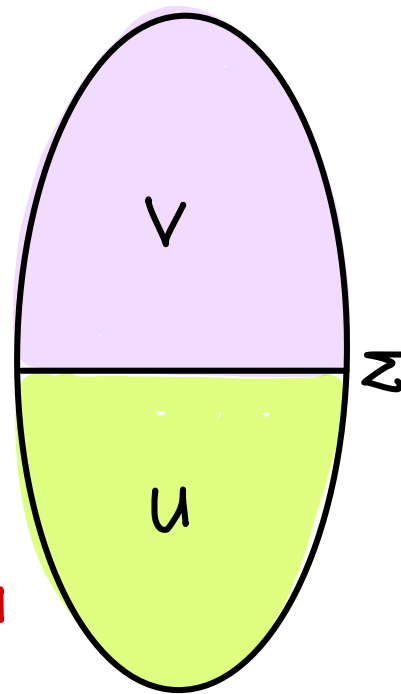
GIVEN TWO CTCT HD'S

$$\underbrace{M = U \cup_Z V}_{\mathcal{H}} \quad \& \quad \underbrace{M' = U' \cup_{Z'} V'}_{\mathcal{H}'}$$

REIDEMEISTER

SINGER

$\mathcal{H}$ & $\mathcal{H}'$ ADMITS
A COMMON **SMOOTH**
STABILISATION



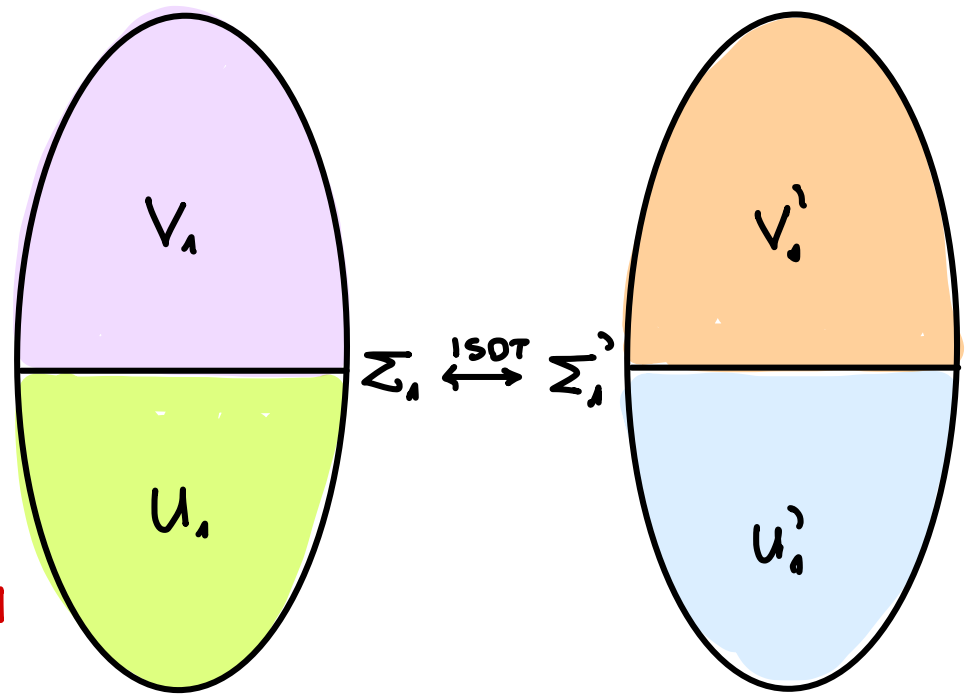
STEP 1

GIVEN TWO CTCT HD'S

$$\underbrace{M = U \cup_{\Sigma} V}_{\mathcal{H}} \quad \& \quad \underbrace{M' = U' \cup_{\Sigma'} V'}_{\mathcal{H}'}$$

REIDEMEISTER
SINGER

$\mathcal{H}$ & $\mathcal{H}'$ ADMITS
A COMMON SMOOTH
STABILISATION



REALISE THESE STABILISATIONS WITH CTCT STABILISATIONS
ON EACH

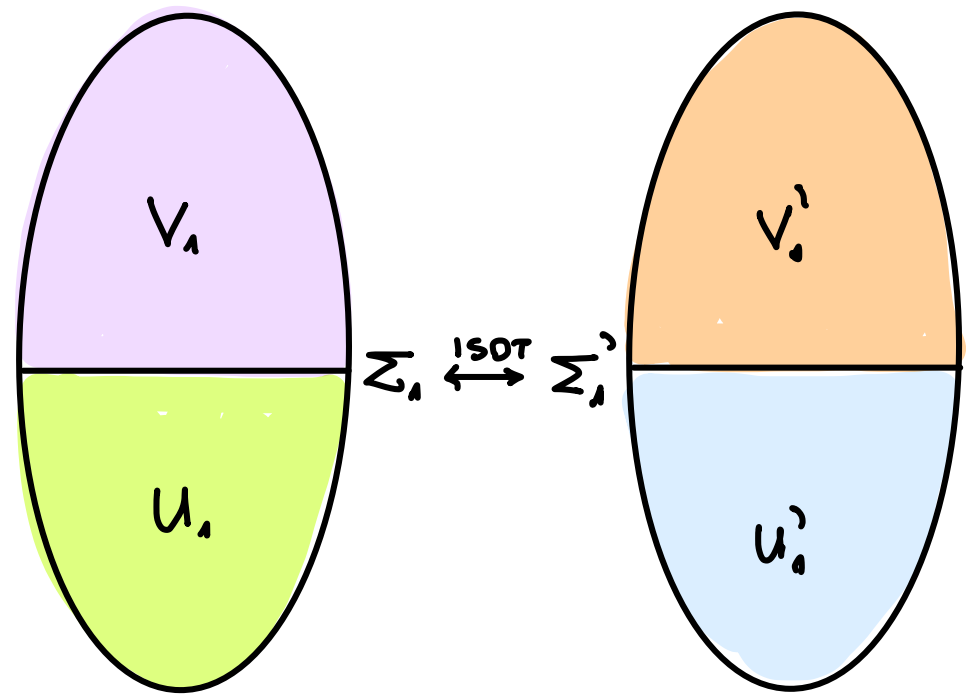
$$\rightsquigarrow \text{GET } \underbrace{M = U_1 \cup_{\Sigma_1} V_1}_{\mathcal{H}_1} \quad \& \quad \underbrace{M = U_1' \cup_{\Sigma_1'} V_1'}_{\mathcal{H}_2}, \text{ WHERE } \Sigma_1 \text{ \& \& } \Sigma_1'$$

ARE SMOOTHLY ISOTOPIC

STEP 2

$\mathcal{H}_1$
 $M = U_1 \cup_{\Sigma_1} V_1$ & $\mathcal{H}_2$
 $M = U'_1 \cup_{\Sigma'_1} V'_1$
 WHERE Σ_1 & Σ'_1 ARE **SMOOTHLY**
 ISOTOPIC

APPLY THE EXISTENCE PROOF

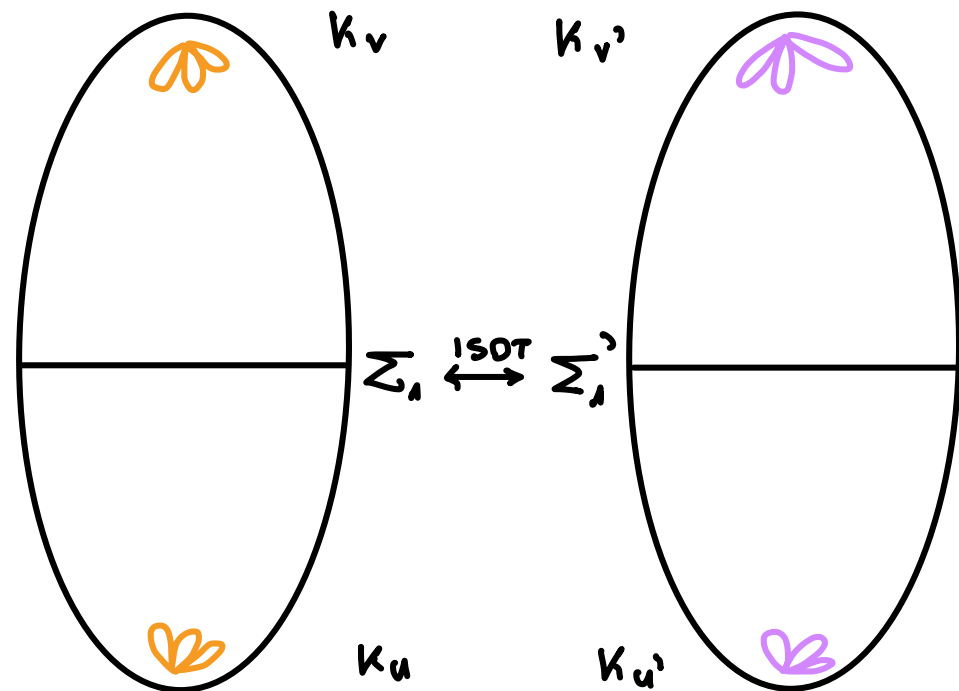


STEP 2

$\mathcal{H}_1$
 $M = U_1 \cup_{\Sigma_1} V_1$ & $\mathcal{H}_2$
 $M = U'_1 \cup_{\Sigma'_1} V'_1$
 WHERE Σ_1 & Σ'_1 ARE **SMOOTHLY**
 ISOTOPIC

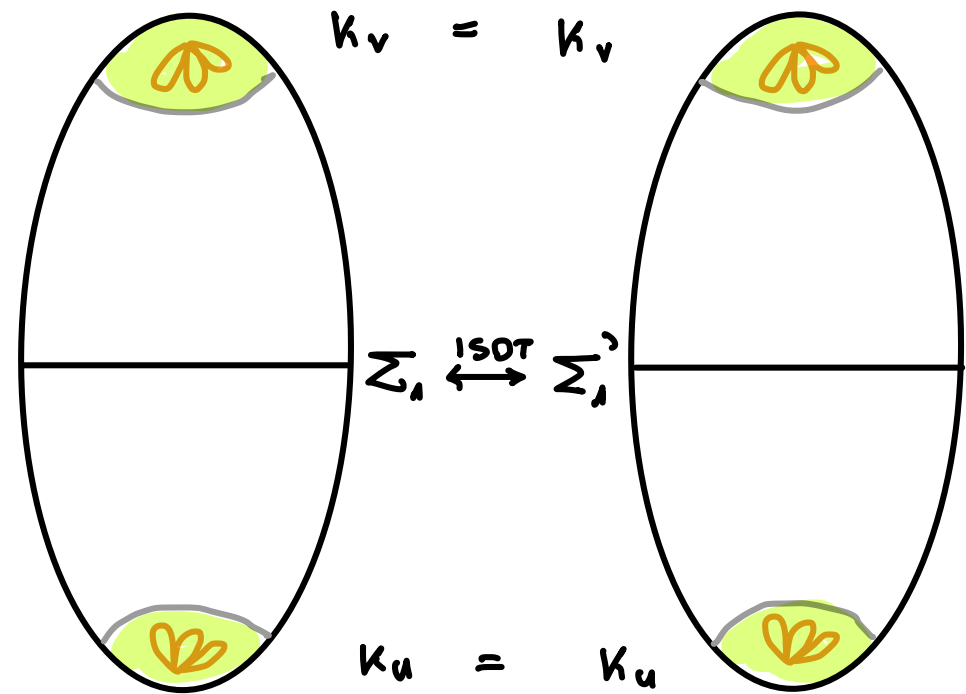
APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES



STEP 2

$\mathcal{H}_1$
 $M = U_1 \cup_{\Sigma_1} V_1$ & $\mathcal{H}_2$
 $M = U'_1 \cup_{\Sigma'_1} V'_1$
 WHERE Σ_1 & Σ'_1 ARE **SMOOTHLY**
 ISOTOPIC



APPLY THE EXISTENCE PROOF

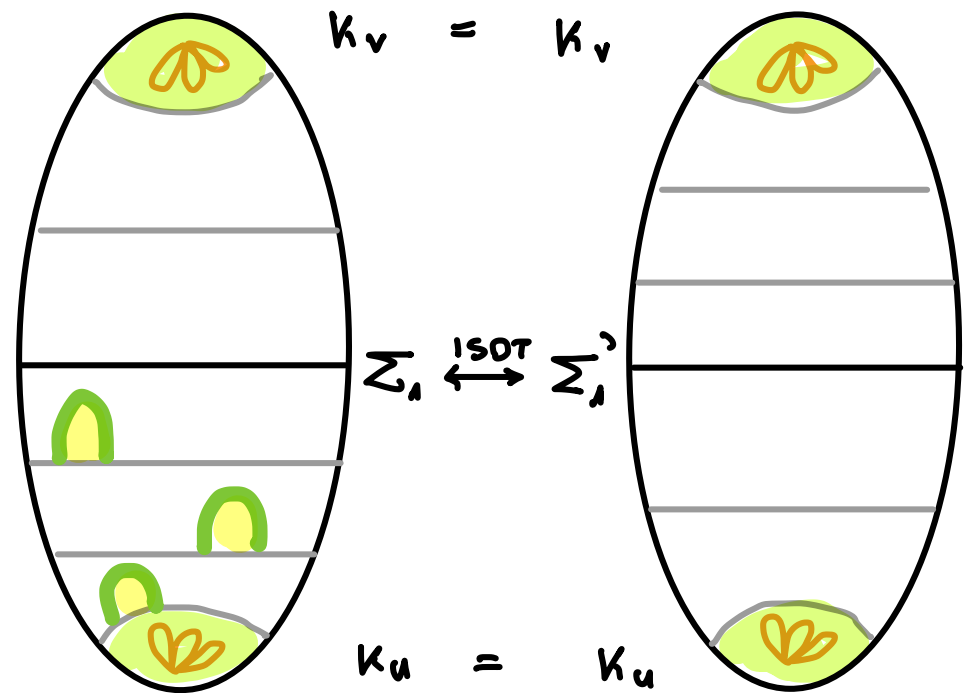
- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

FUCHS-TABACHNIKOV: AFTER SUFFICIENTLY MANY LEGENDRIAN
 STAB. WE CAN ASSUME $K_u = K_u'$ & $K_v = K_v'$

- TAKE THE STANDARD NEIGHBOURHOOD $N(K_u)$, $N(K_v)$

STEP 2

$\mathcal{H}_1$ $\mathcal{H}_2$
 $M = U_1 \cup_{\Sigma_1} V_1$ & $M = U'_1 \cup_{\Sigma'_1} V'_1$
 WHERE Σ_1 & Σ'_1 ARE **SMOOTHLY**
 ISOTOPIC



APPLY THE EXISTENCE PROOF

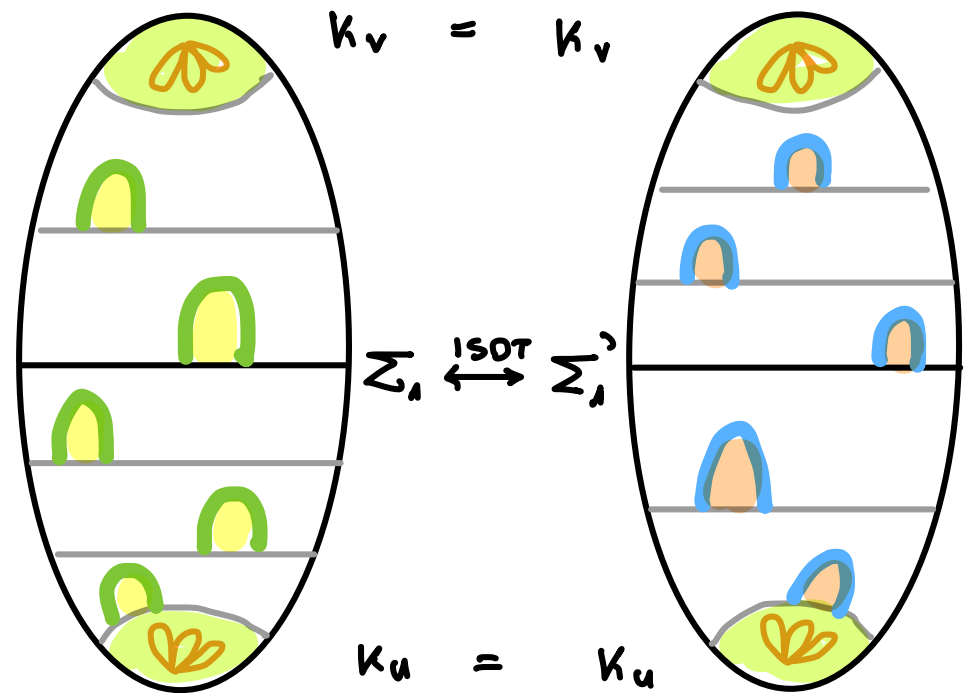
- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

FUCHS-TABACHNIKOV: AFTER SUFFICIENTLY MANY LEGENDRIAN
 STAB. WE CAN ASSUME $K_u = K'_u$ & $K_v = K'_v$

- TAKE THE STANDARD NEIGHBOURHOOD $N(K_u), N(K_v)$
- $U'_1 \setminus N(K_u)$ & $V'_1 \setminus N(K_v)$ ARE $\cong \Sigma_1 \times I \xRightarrow{\text{HONDA}} =$ STACKS OF BYPASSES

STEP 2

$\mathcal{H}_1$ $\mathcal{H}_2$
 $M = U_1 \cup_{\Sigma_1} V_1$ & $M = U'_1 \cup_{\Sigma'_1} V'_1$
 WHERE Σ_1 & Σ'_1 ARE **SMOOTHLY**
 ISOTOPIC



APPLY THE EXISTENCE PROOF

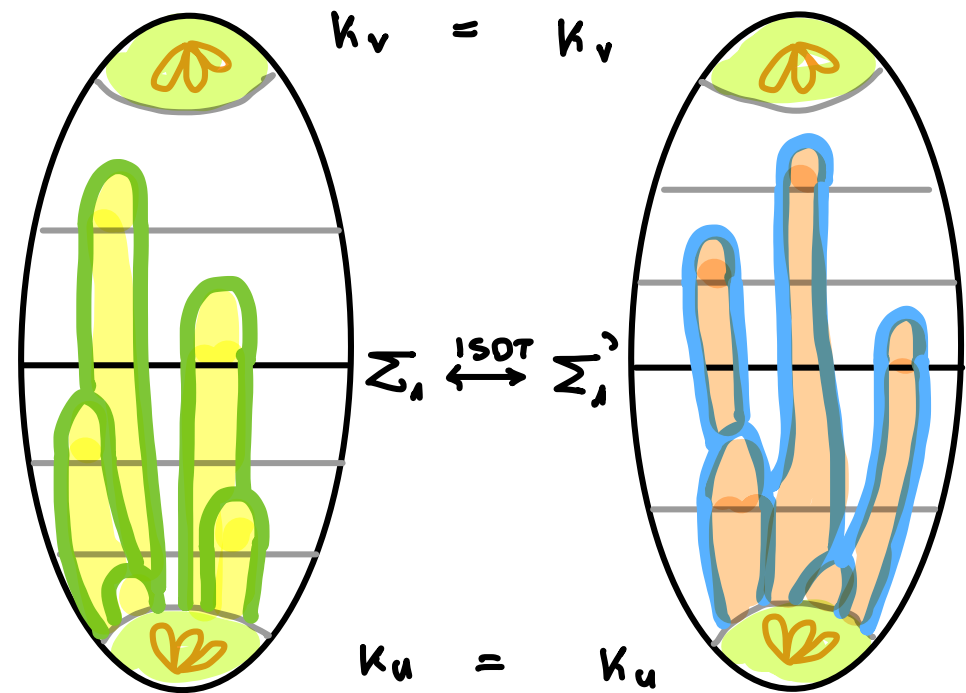
- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

FUCHS-TABACHNIKOV: AFTER SUFFICIENTLY MANY LEGENDRIAN
 STAB. WE CAN ASSUME $K_u = K_u'$ & $K_v = K_v'$

- TAKE THE STANDARD NEIGHBOURHOOD $N(K_u)$, $N(K_v)$
- $U'_1 \setminus N(K_u)$ & $V'_1 \setminus N(K_v)$ ARE $\cong \Sigma_1 \times I \xRightarrow{\text{HONDA}} =$ STACKS OF BYPASSES

STEP 2

$\mathcal{H}_1$
 $M = U_1 \cup_{\Sigma_1} V_1$ & $\mathcal{H}_2$
 $M = U'_1 \cup_{\Sigma'_1} V'_1$
 WHERE Σ_1 & Σ'_1 ARE **SMOOTHLY**
 ISOTOPIC



APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

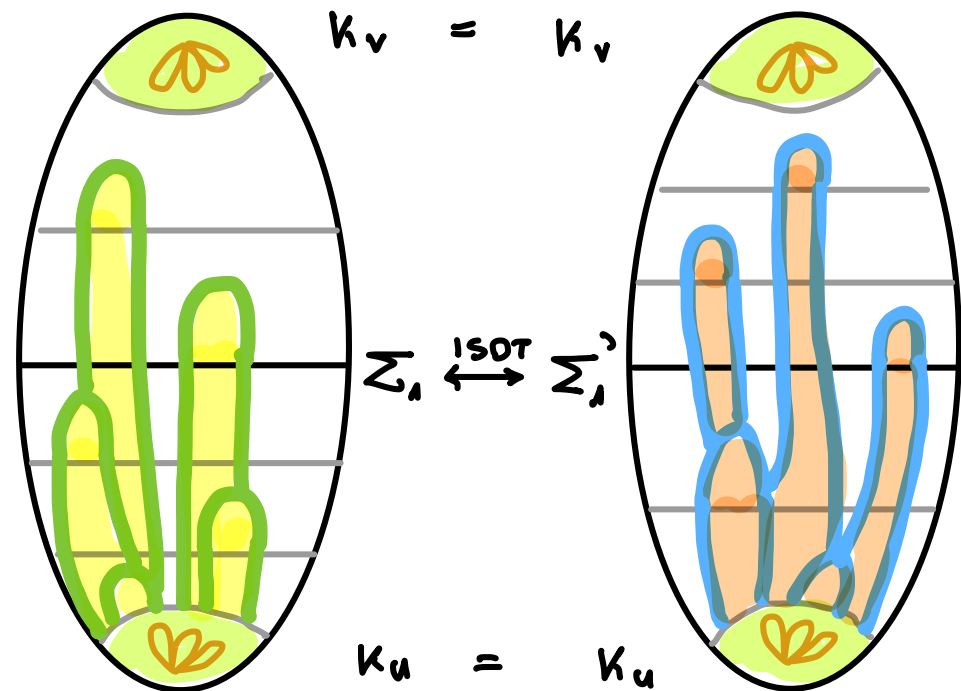
FUCHS-TABACHNIKOV: AFTER SUFFICIENTLY MANY LEGENDRIAN
 STAB. WE CAN ASSUME $K_u = K'_u$ & $K_v = K'_v$

- TAKE THE STANDARD NEIGHBOURHOOD $N(K_u), N(K_v)$
- $U'_1 \setminus N(K_u)$ & $V'_1 \setminus N(K_v)$ ARE $\cong \Sigma_1 \times I \xRightarrow{\text{HONDA}} =$ STACKS OF BYPASSES
- EXTEND THE HANDLES

STEP 2

$$\underbrace{M}_{\mathcal{H}_1} = U_1 \cup_{\Sigma_1} V_1 \quad \& \quad \underbrace{M}_{\mathcal{H}_2} = U_1' \cup_{\Sigma_1'} V_1'$$

WHERE Σ_1 & Σ_1' ARE **SMOOTHLY** ISOTOPIC



APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

FUCHS-TABACHNIKOV: AFTER SUFFICIENTLY MANY LEGENDRIAN STAB. WE CAN ASSUME $K_u = K_u'$ & $K_v = K_v'$

- TAKE THE STANDARD NEIGHBOURHOOD $N(K_u), N(K_v)$
- $U_1' \setminus N(K_u)$ & $V_1' \setminus N(K_v)$ ARE $\cong \Sigma_1 \times I \xRightarrow{\text{HONDA}} =$ STACKS OF BYPASSES
- EXTEND THE HANDLES

$$U_2' := N(U_K) \cup (1-h's) \quad V_2' = M \setminus U_2'$$

$$\rightsquigarrow \underbrace{M = U_2 \cup V_2}_{\mathcal{H}_2} \quad \& \quad \underbrace{M = U_2' \cup V_2'}_{\mathcal{H}_2'}$$

STEP 2

- $U' \setminus N(K_U)$ & $V' \setminus N(K_V)$ ARE $\cong \Sigma_1 \times I$

HONDA

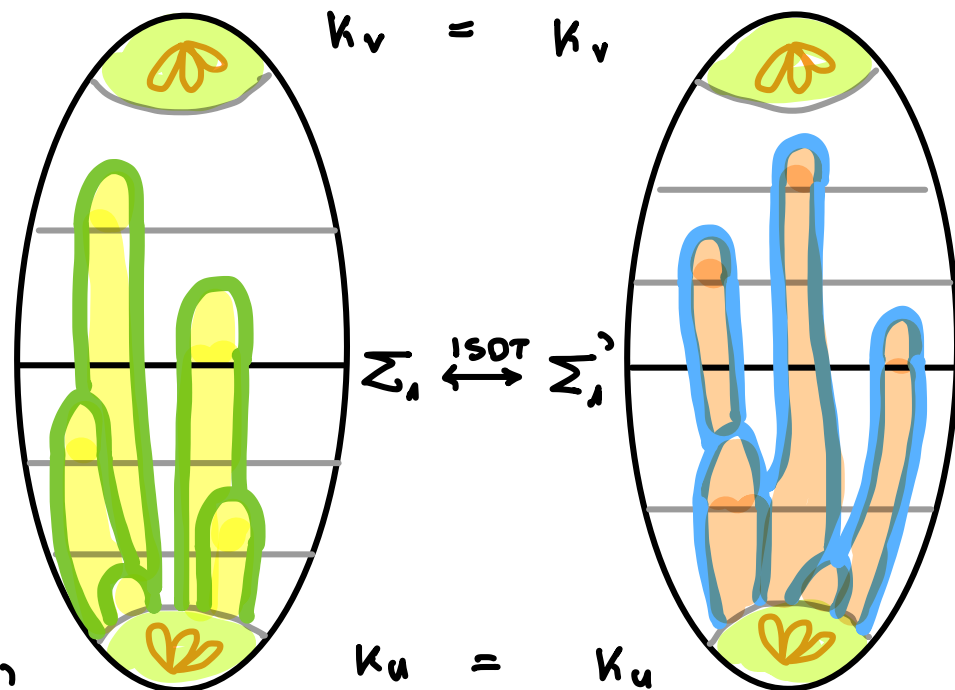
$\implies$ = STACKS OF BYPASSES

- EXTEND THE HANDLES

$$U'_2 := N(U_K) \cup (1-h's)$$

$$V'_2 = M \setminus U'_2$$

$$\rightsquigarrow \underbrace{M = U_2 \cup V_2}_{\mathcal{H}_2} \quad \& \quad \underbrace{M = U'_2 \cup V'_2}_{\mathcal{H}'_2}$$



PROP (L-V) $\mathcal{H}'_2$ CAN BE OBTAINED FROM $\mathcal{H}'_1$ VIA A SEQUENCE OF CONTACT STAB & DESTAB

STEP 2

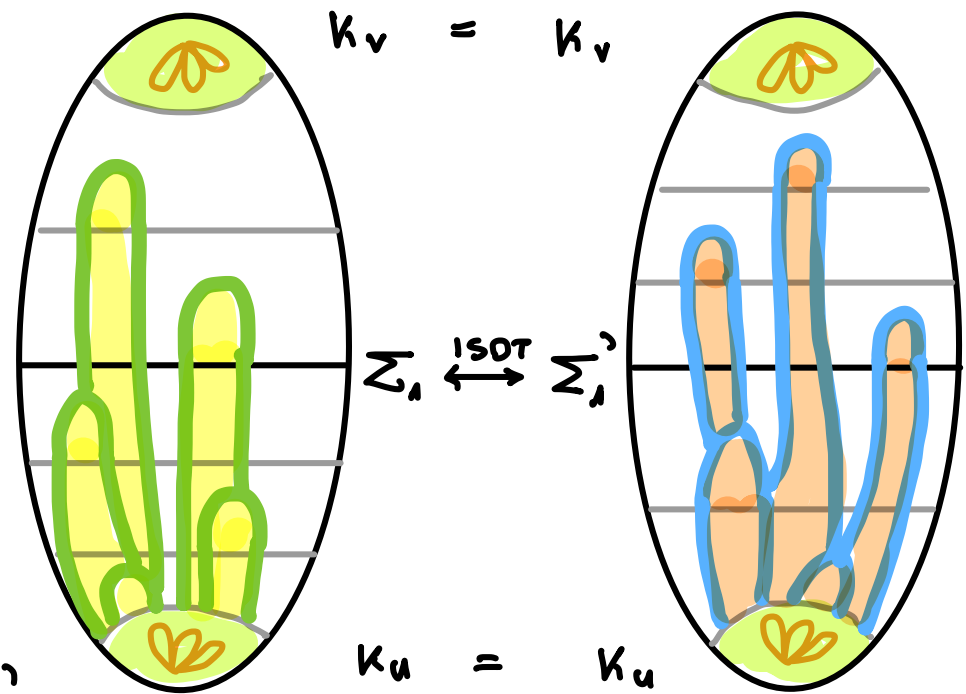
- $U \setminus N(K_u)$ & $V \setminus N(K_v)$ ARE $\cong \Sigma_1 \times I$

HONDA $\implies$ = STACKS OF BYPASSES

- EXTEND THE HANDLES

$$U_2 := N(U_k) \cup (1-h's) \quad V_2 = M \setminus U_2$$

$$\rightsquigarrow \underbrace{M = U_2 \cup V_2}_{\mathcal{H}_2} \quad \& \quad \underbrace{M = U_2' \cup V_2'}_{\mathcal{H}_2'}$$



PROP (L-V) $\mathcal{H}_2'$ CAN BE OBTAINED FROM $\mathcal{H}_1'$ VIA A SEQUENCE OF CONTACT STAB & DESTAB

BEFORE NEXT STEP NOTICE

$$\mathcal{H}_2 = \hat{\mathcal{H}}(U_k, V_k, \mathcal{B}) \quad \mathcal{H}_2' = \hat{\mathcal{H}}(U_k, V_k, \mathcal{B}')$$

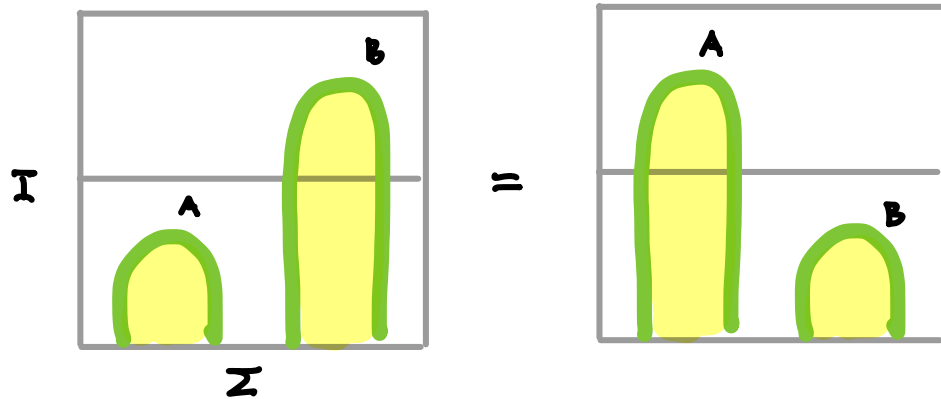
WHERE $\mathcal{B}$ & $\mathcal{B}'$ ARE DIFFERENT DECOMPOSITIONS OF $\mathbb{S}^1 \setminus M - (N(U_k) \cup N(V_k))$ AS BYPASS STACKS

INTERLUDE - STACKING BYPASSES

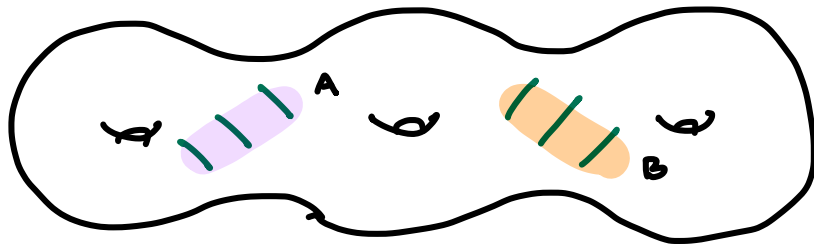


THE FOLLOWING MOVES DO NOT CHANGE $\mathfrak{Z}$

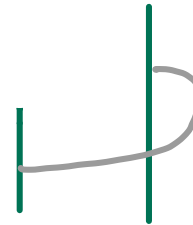
COMMUTATION



FROM TOP



TRIVIAL BYPASS

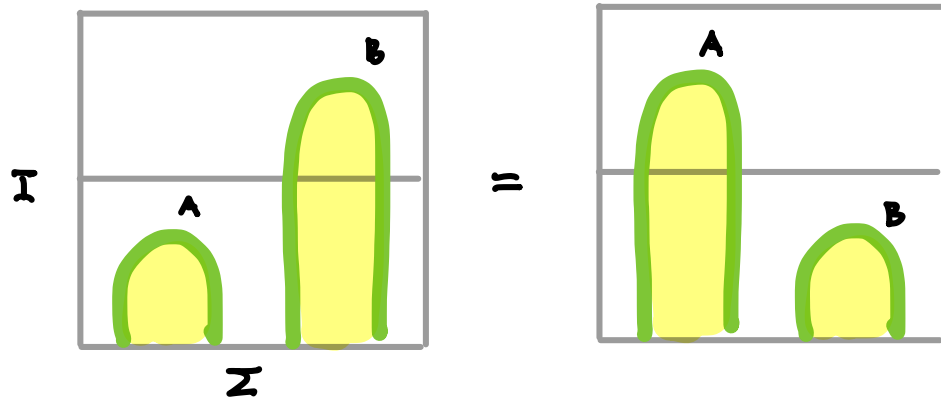


INTERLUDE - STACKING BYPASSES

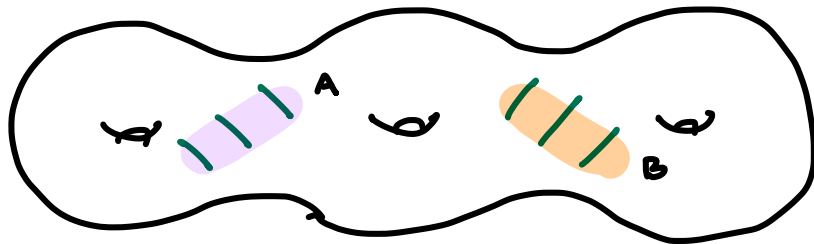


THE FOLLOWING MOVES DO NOT CHANGE $\mathfrak{Z}$

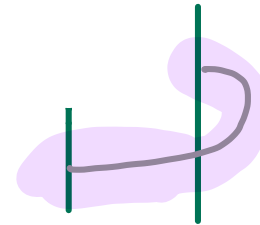
COMMUTATION



FROM TOP



TRIVIAL BYPASS

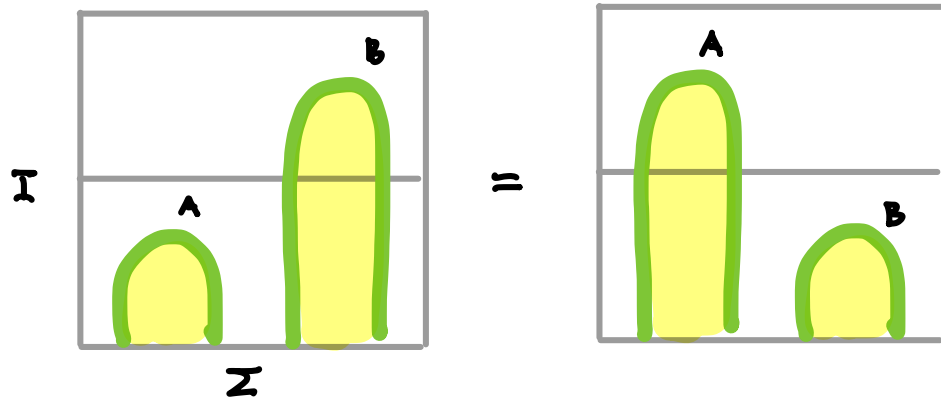


INTERLUDE - STACKING BYPASSES

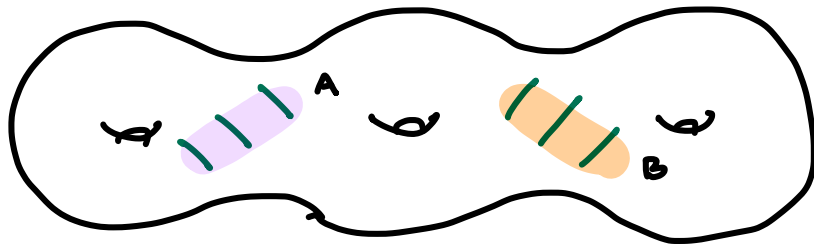


THE FOLLOWING MOVES DO NOT CHANGE $\mathfrak{Z}$

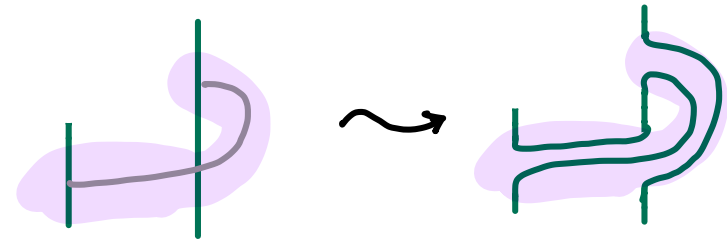
COMMUTATION



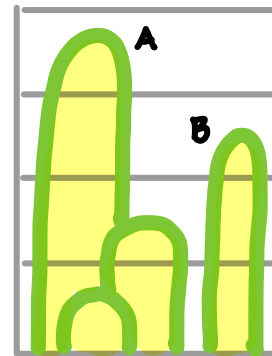
FROM TOP



TRIVIAL BYPASS



E.G.:

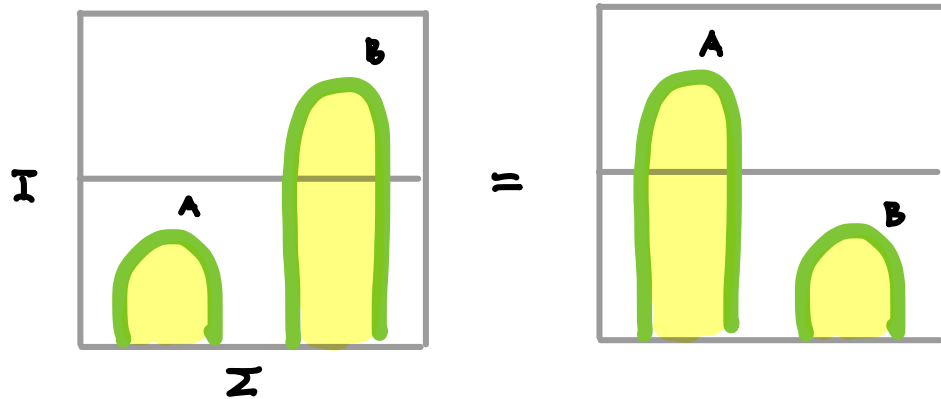


INTERLUDE - STACKING BYPASSES

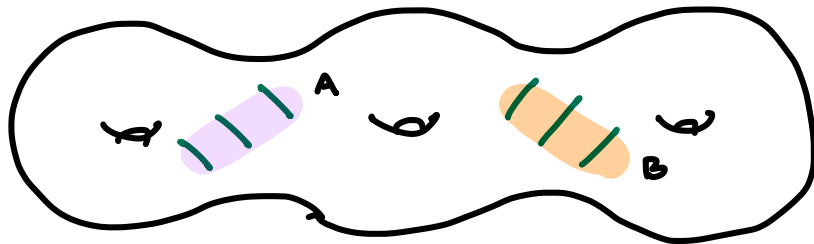


THE FOLLOWING MOVES DO NOT CHANGE $\mathfrak{Z}$

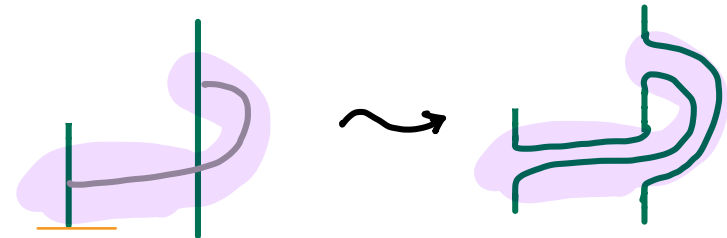
COMMUTATION



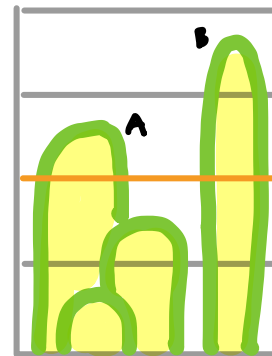
FROM TOP



TRIVIAL BYPASS



E.G.:

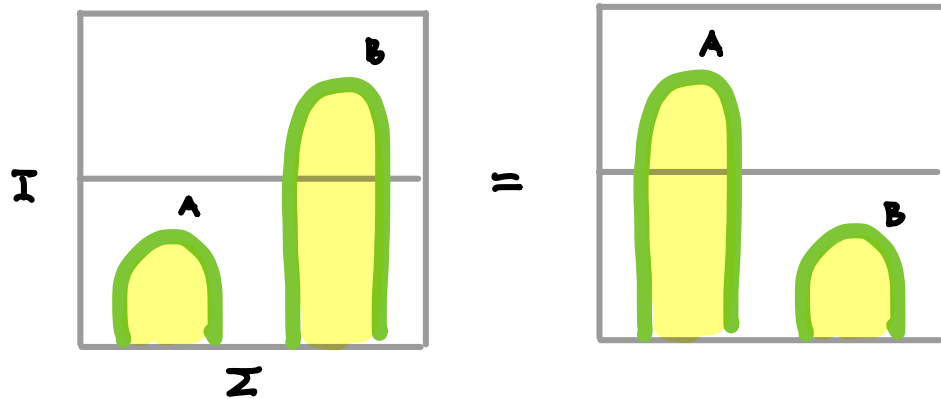


INTERLUDE - STACKING BYPASSES

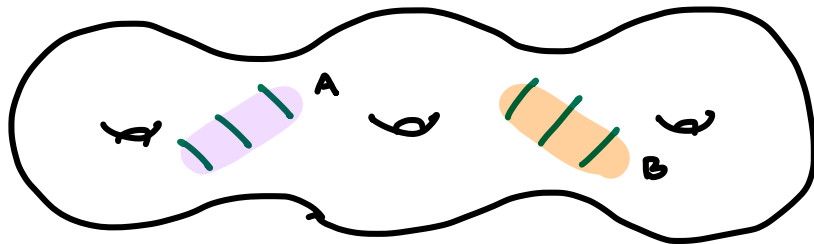


THE FOLLOWING MOVES DO NOT CHANGE $\mathfrak{Z}$

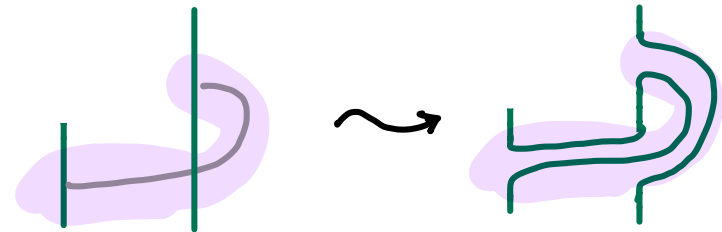
COMMUTATION



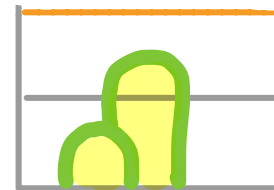
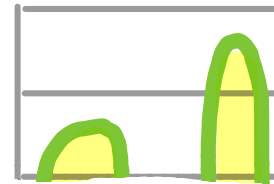
FROM TOP



TRIVIAL BYPASS



E.G.:

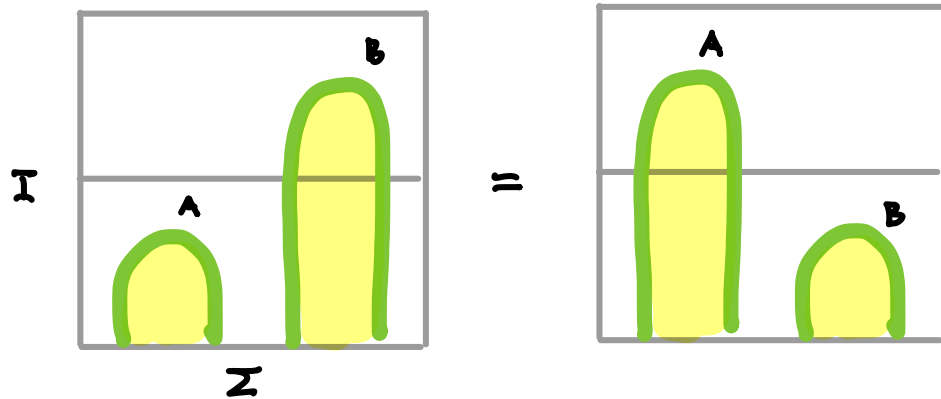


INTERLUDE - STACKING BYPASSES

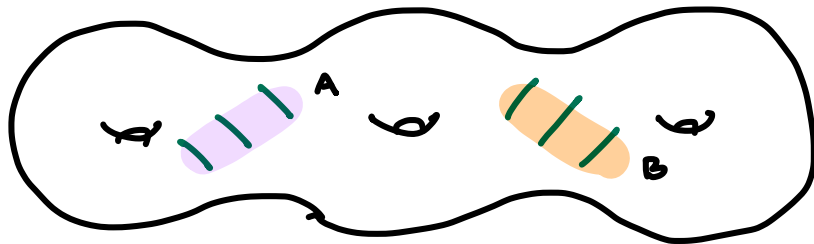


THE FOLLOWING MOVES DO NOT CHANGE $\mathfrak{Z}$

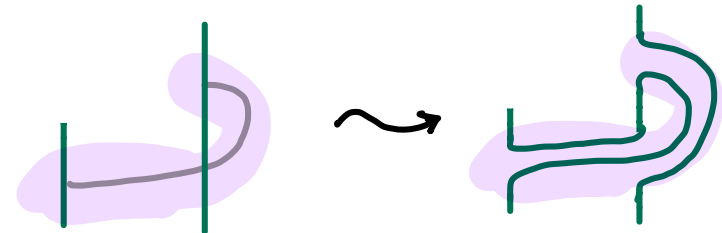
COMMUTATION



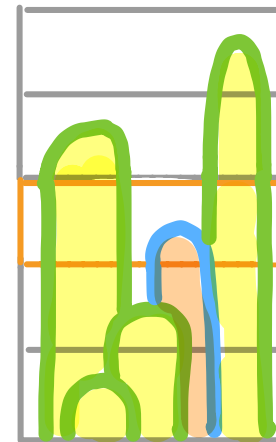
FROM TOP



TRIVIAL BYPASS



E.G.:



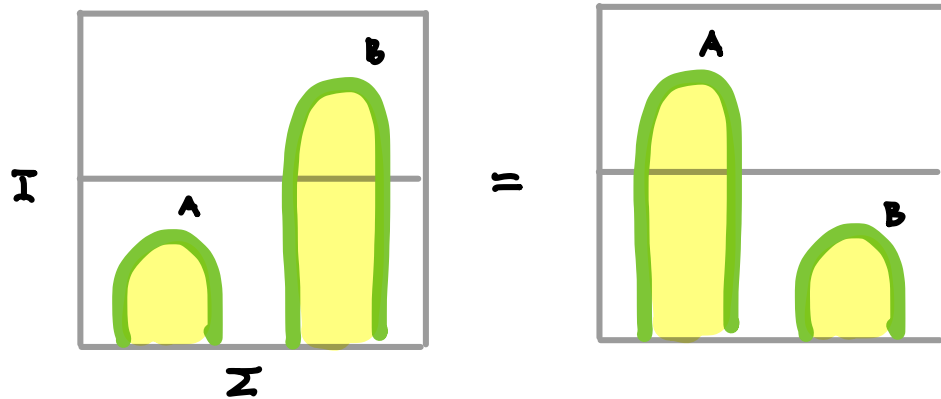
← TRIVIAL

INTERLUDE - STACKING BYPASSES

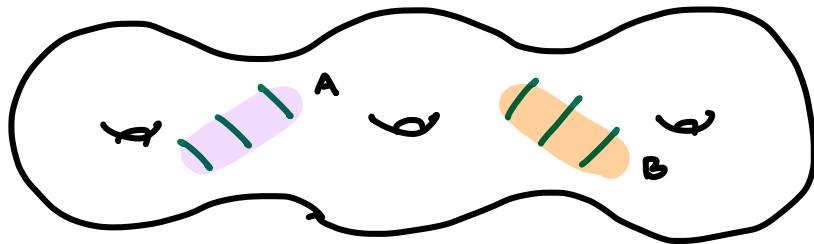


THE FOLLOWING MOVES DO NOT CHANGE $\mathfrak{Z}$

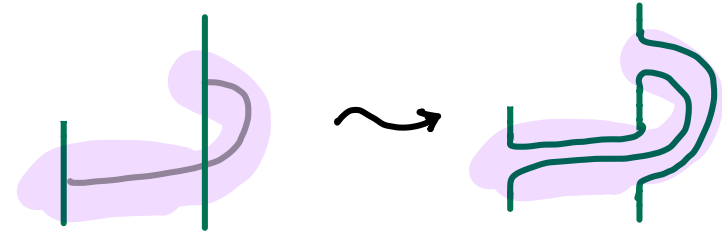
COMMUTATION



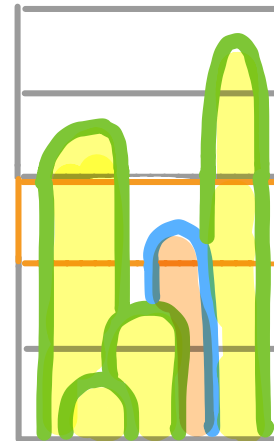
FROM TOP



TRIVIAL BYPASS



E.G.:



← TRIVIAL

...

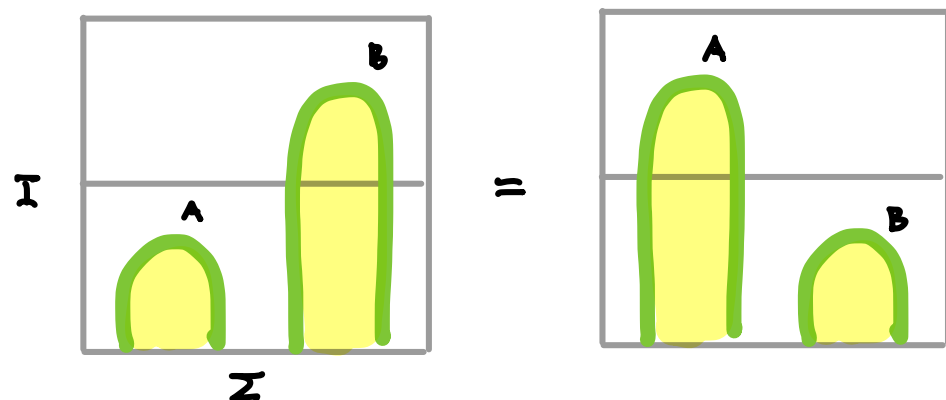
STILL GET SAME $\mathfrak{Z}$

INTERLUDE - STACKING BYPASSES

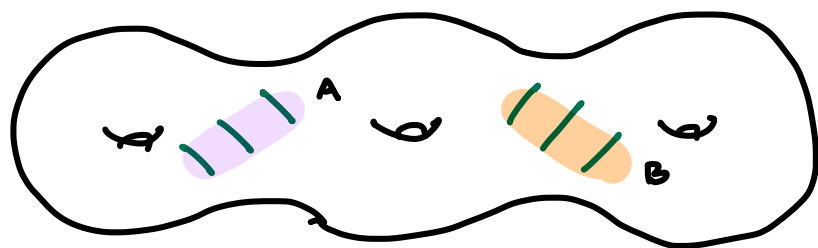


THE FOLLOWING MOVES DO NOT CHANGE $\mathfrak{Z}$

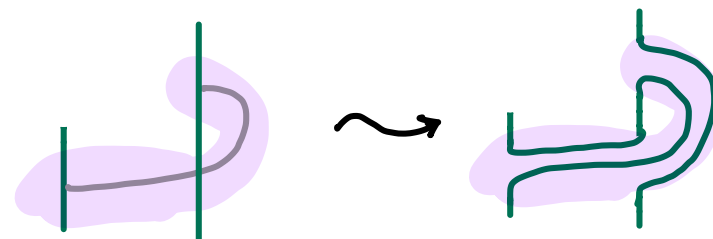
COMMUTATION



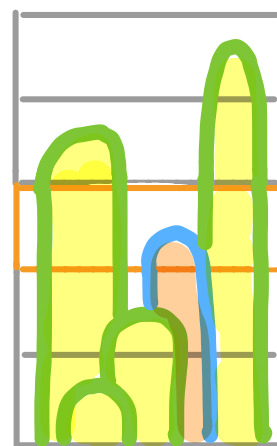
FROM TOP



TRIVIAL BYPASS



E.G.:



← TRIVIAL

...

STILL GET SAME $\mathfrak{Z}$

THM (TIAN, BREEN-HONDA-HUANG) ON $\Sigma \times I$ ANY TWO DECOMPOSITION OF $\mathfrak{Z}$ INTO STACKS OF BYPASS SLICES ARE RELATED VIA

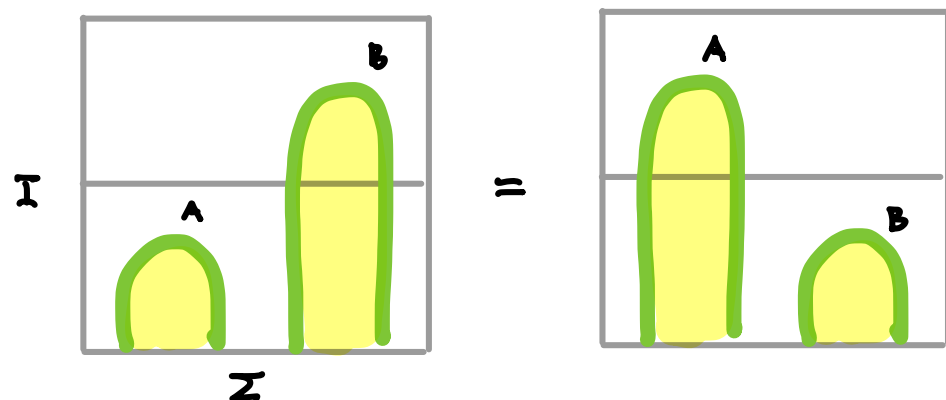
- ADDING A TRIVIAL BYPASS
- COMMUTING DISJOINT BYPASSES

INTERLUDE - STACKING BYPASSES

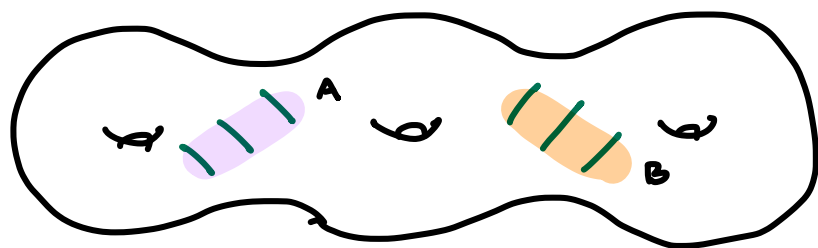


THE FOLLOWING MOVES DO NOT CHANGE $\mathfrak{z}$

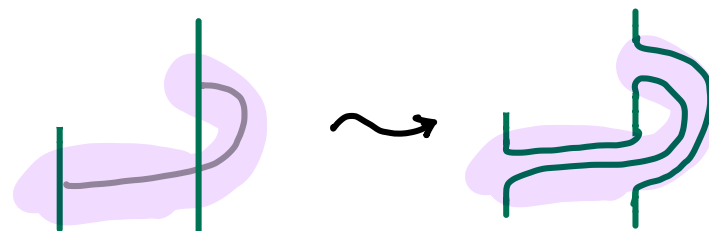
COMMUTATION



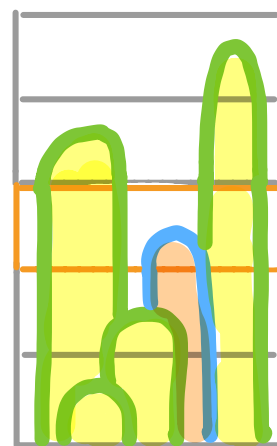
FROM TOP



TRIVIAL BYPASS



E.G.:



← TRIVIAL

...

STILL GET SAME $\mathfrak{z}$

THM (TIAN, BREEN-HONDA-HUANG) ON $\mathbb{Z} \times I$ ANY TWO DECOMPOSITION OF $\mathfrak{z}$ INTO STACKS OF BYPASS SLICES ARE RELATED VIA

- ADDING A TRIVIAL BYPASS
- COMMUTING DISJOINT BYPASSES

BYPASS MOVES

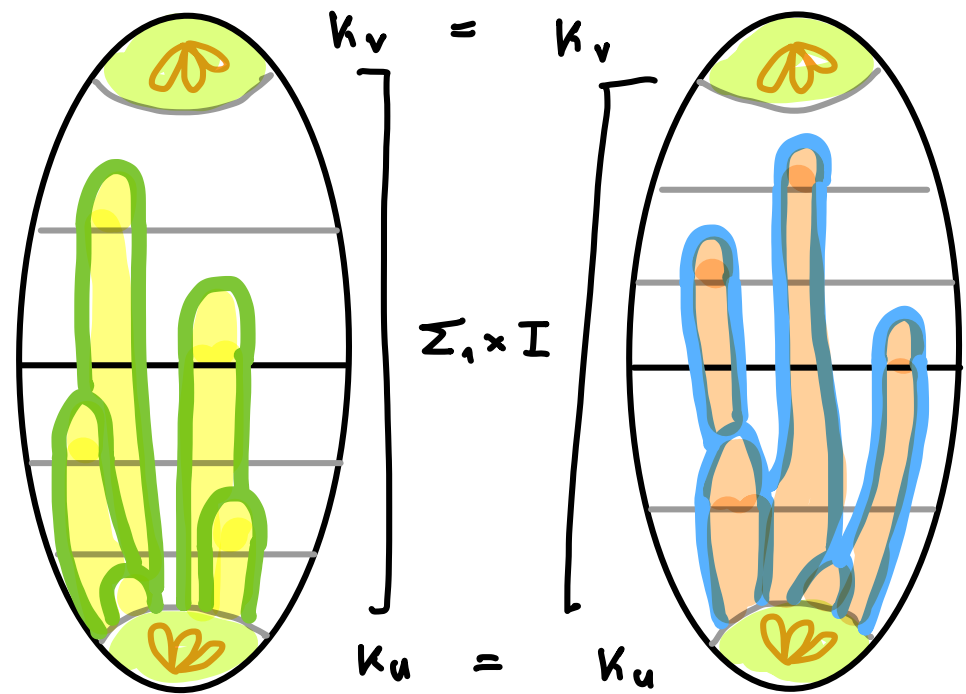
STEP 3

$$\mathcal{H}_2 = \hat{\mathcal{H}}(U_k, V_k, \mathcal{B}) \quad \mathcal{H}'_2 = \hat{\mathcal{H}}(U_k, V_k, \mathcal{B}')$$

WHERE $\mathcal{B}$ & $\mathcal{B}'$ ARE TWO
BYPASS - DECOMPOSITIONS
OF $\mathbb{S}^1 \setminus M - (N(U_k) \cup N(V_k))$

$\xrightarrow[\text{B-H-H}]{\text{TIAN}}$ $\mathcal{B}$ & $\mathcal{B}'$ ARE RELATED
VIA BYPASS MOVES

$$\mathcal{B} = \mathcal{B}_0 \rightsquigarrow \mathcal{B}_1 \rightsquigarrow \dots \rightsquigarrow \mathcal{B}_\ell = \mathcal{B}'$$



THM (L-V): BYPASS MOVES ON $\Sigma_1 \times I$ CORRESPOND TO CONTACT
STAB & DESTAB OF $\hat{\mathcal{H}}$

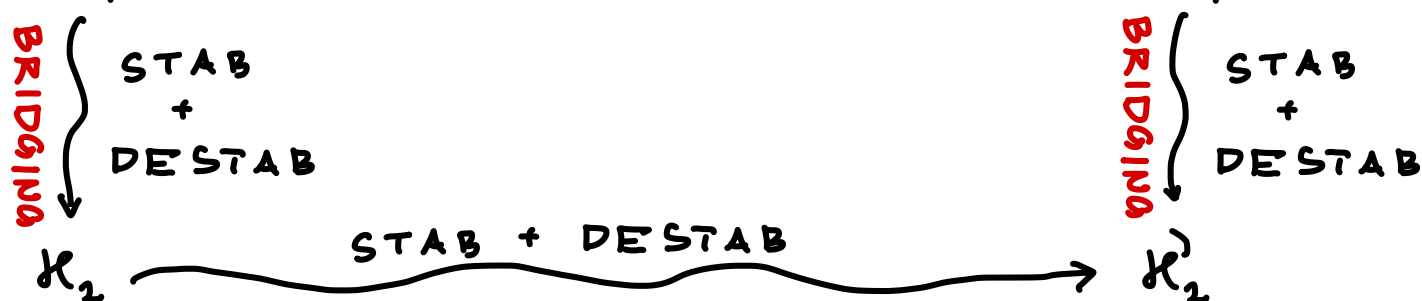
$$\underbrace{\hat{\mathcal{H}}(U_k, V_k, \mathcal{B})}_{\mathcal{H}_2} \xrightarrow{\text{STAB+DESTAB}} \hat{\mathcal{H}}(U_k, V_k, \mathcal{B}_1) \rightsquigarrow \dots \xrightarrow{\text{STAB+DESTAB}} \hat{\mathcal{H}}(U_k, V_k, \mathcal{B}')_{\mathcal{H}'_2}$$

SUMMARY

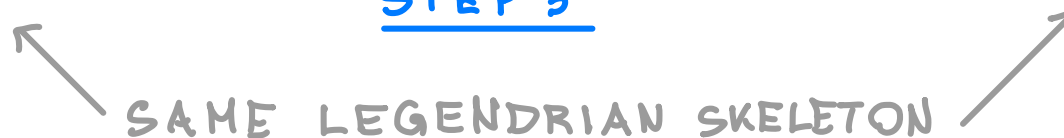
STEP 1



STEP 2



STEP 3

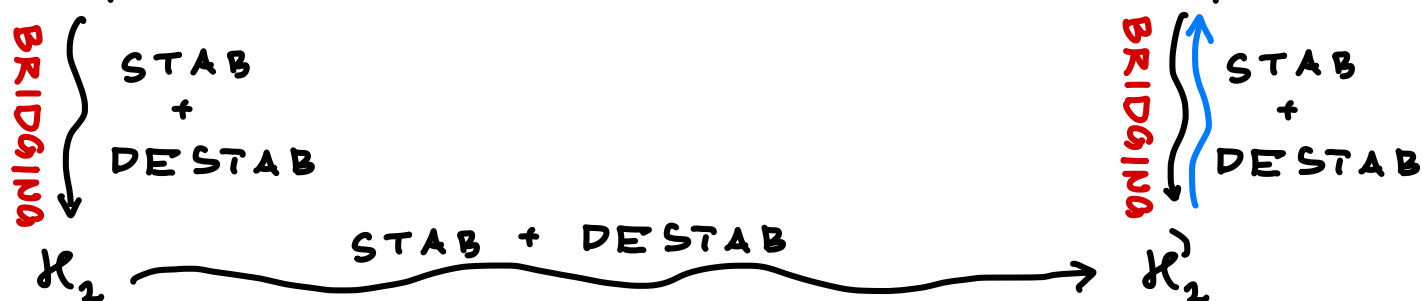


SUMMARY

STEP 1



STEP 2



STEP 3

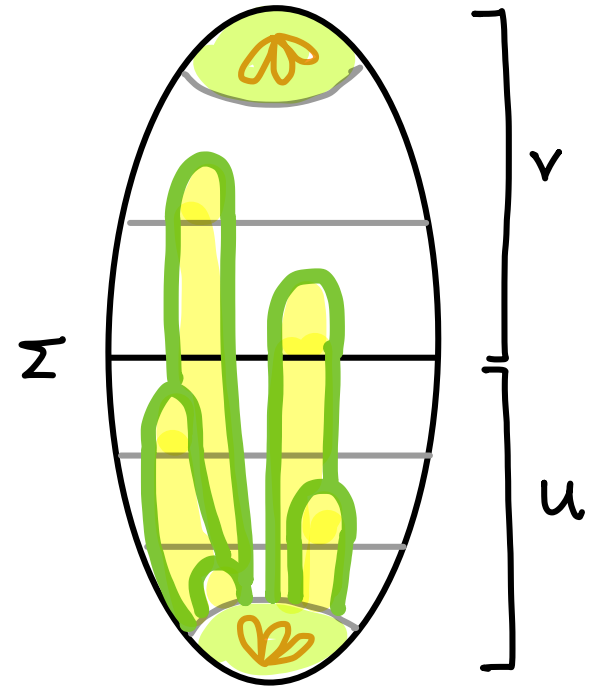
$\Rightarrow \mathcal{H} \text{ \& } \mathcal{H}' \text{ ARE RELATED VIA A SEQUENCE OF CONTACT STAB. \& DESTAB}$



KEY POINT

IN STEP 2 WE NEEDED:

THM (L-V): $\mathcal{H}$ CTCT HD $\implies$ THE BRIDGE $\hat{\mathcal{H}}$
OF $\mathcal{H}$ IS CONNECTED TO $\mathcal{H}$ VIA CTCT
STAB'S & DESTAB'S



KEY POINT

IN STEP 2 WE NEEDED:

THM (L-V): $\mathcal{K}$ CTCT HD $\implies$ THE BRIDGE $\hat{\mathcal{K}}$ OF $\mathcal{K}$ IS CONNECTED TO $\mathcal{K}$ VIA CTCT STAB'S & DESTAB'S

AN ELEMENTARY STEP IN THE PROOF

PROP (L-V): $\mathcal{K}(M=U \cup V)$ CTCT HD, B BYPASS ONTO $Z < V$

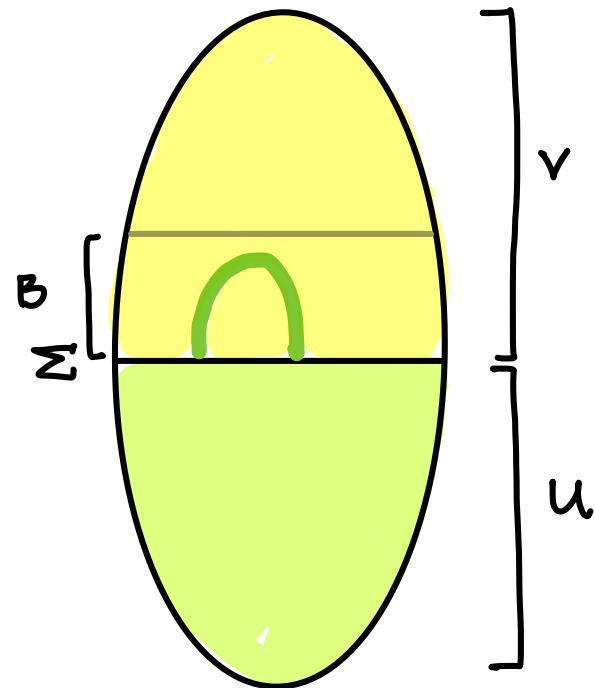
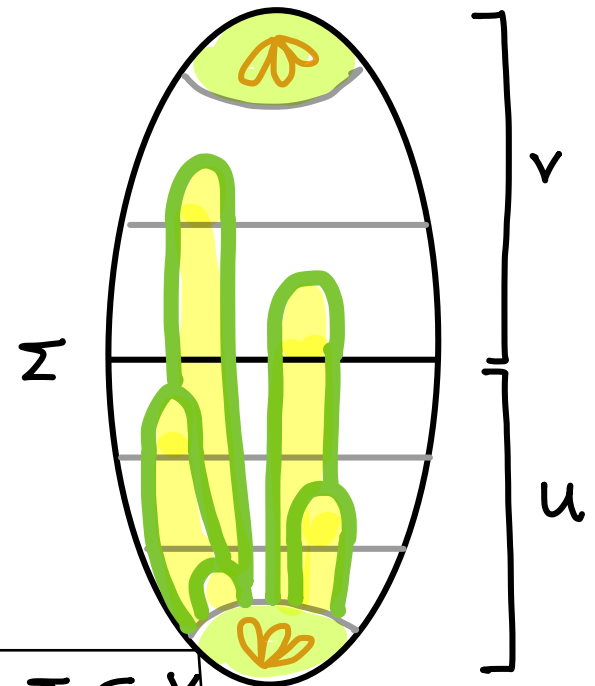
THEN FOR $\bullet U' = U \cup 1-h$

$\bullet V' = \overline{M - U'}$

THE CTCT HD $\mathcal{K}'(M=U' \cup V')$

IS CONNECTED TO $\mathcal{K}$ VIA CTCT STAB'S & DESTAB'S

PROP $\implies$ THM:



KEY POINT

IN STEP 2 WE NEEDED:

THM (L-V): $\mathcal{K}$ CTCT HD $\implies$ THE BRIDGE $\hat{\mathcal{K}}$ OF $\mathcal{K}$ IS CONNECTED TO $\mathcal{K}$ VIA CTCT STAB'S & DESTAB'S

AN ELEMENTARY STEP IN THE PROOF

PROP (L-V): $\mathcal{K}(M=U \cup V)$ CTCT HD, B BYPASS ONTO $Z < V$

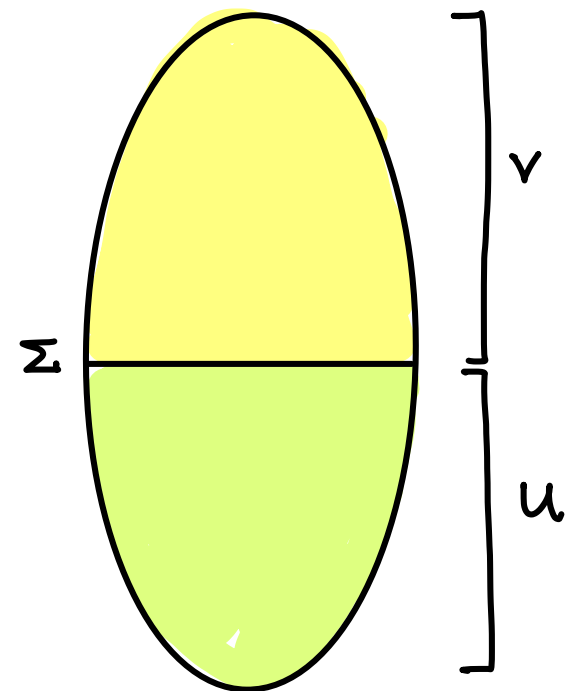
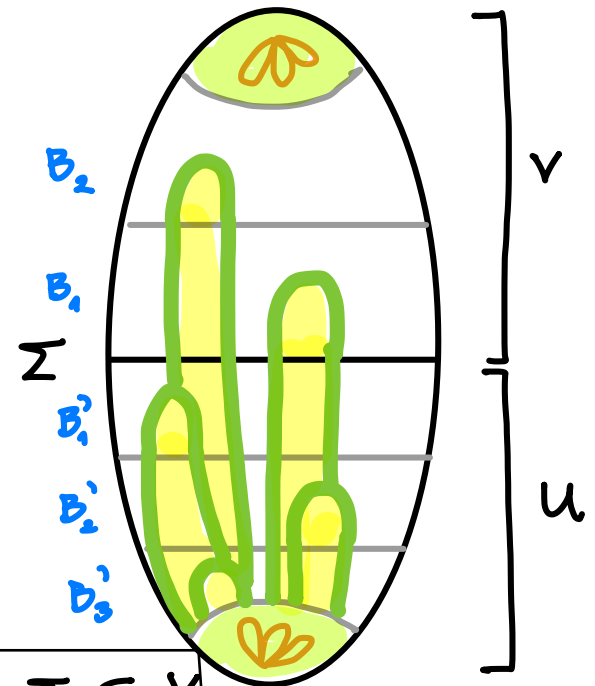
THEN FOR $\bullet U' = U \cup 1-h$

$\bullet V' = \overline{M - U'}$

THE CTCT HD $\mathcal{K}'(M=U' \cup V')$

IS CONNECTED TO $\mathcal{K}$ VIA CTCT STAB'S & DESTAB'S

PROP $\implies$ THM: START FROM $\mathcal{K}$



KEY POINT

IN STEP 2 WE NEEDED:

THM (L-V): $\mathcal{K}$ CTCT HD $\implies$ THE BRIDGE $\hat{\mathcal{K}}$ OF $\mathcal{K}$ IS CONNECTED TO $\mathcal{K}$ VIA CTCT STAB'S & DESTAB'S

AN ELEMENTARY STEP IN THE PROOF

PROP (L-V): $\mathcal{K}(M=U \cup V)$ CTCT HD, B BYPASS ONTO $Z < V$

THEN FOR • $U' = U \cup 1-h$

• $V' = \overline{M - U'}$

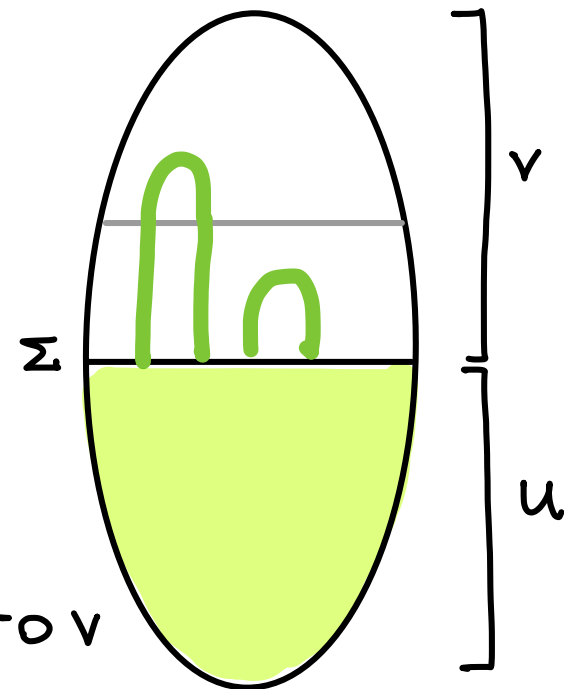
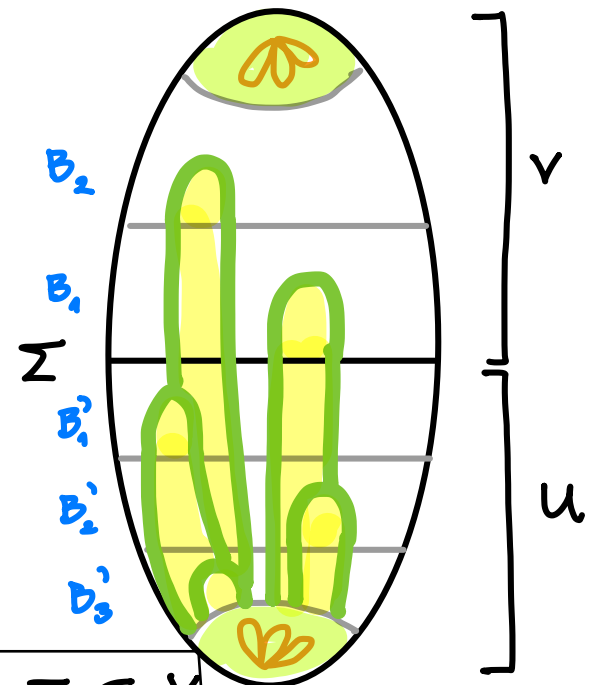
THE CTCT HD $\mathcal{K}'(M=U' \cup V')$

IS CONNECTED TO $\mathcal{K}$ VIA CTCT STAB'S & DESTAB'S

PROP $\implies$ THM: START FROM $\mathcal{K}$

• ATTACH THE 1-h OF B_1 & B_2 TO U

• TURN UPSIDE DOWN & ATTACH THE 2-h OF B_2' TO V



KEY POINT

IN STEP 2 WE NEEDED:

THM (L-V): $\mathcal{K}$ CTCT HD $\implies$ THE BRIDGE $\hat{\mathcal{K}}$ OF $\mathcal{K}$ IS CONNECTED TO $\mathcal{K}$ VIA CTCT STAB'S & DESTAB'S

AN ELEMENTARY STEP IN THE PROOF

PROP (L-V): $\mathcal{K}(M=U \cup V)$ CTCT HD, B BYPASS ONTO $Z < V$

THEN FOR • $U' = U \cup 1-h$

• $V' = \overline{M - U'}$

THE CTCT HD $\mathcal{K}'(M=U' \cup V')$

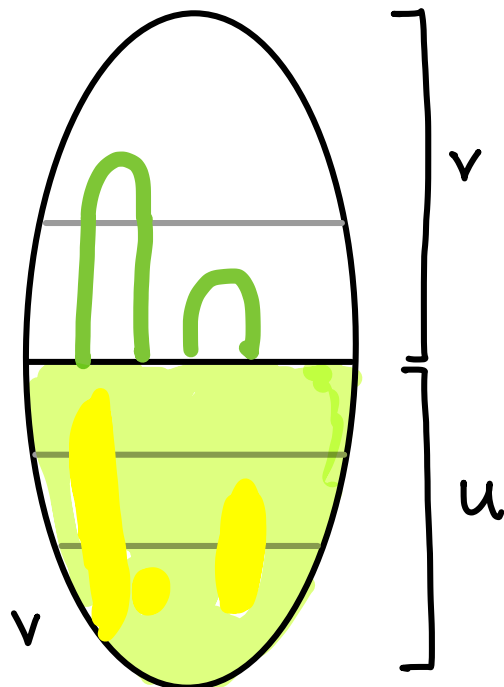
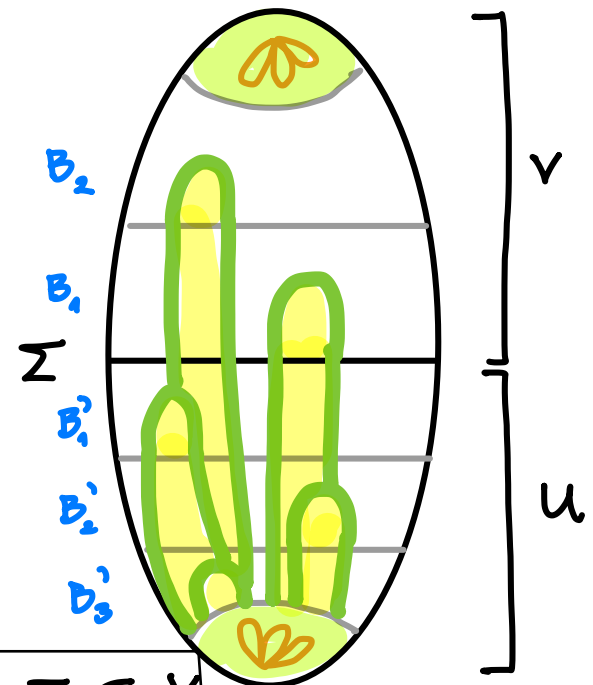
IS CONNECTED TO $\mathcal{K}$ VIA CTCT STAB'S & DESTAB'S

PROP $\implies$ THM: START FROM $\mathcal{K}$

• ATTACH THE 1-h OF B_1 & B_2 TO U

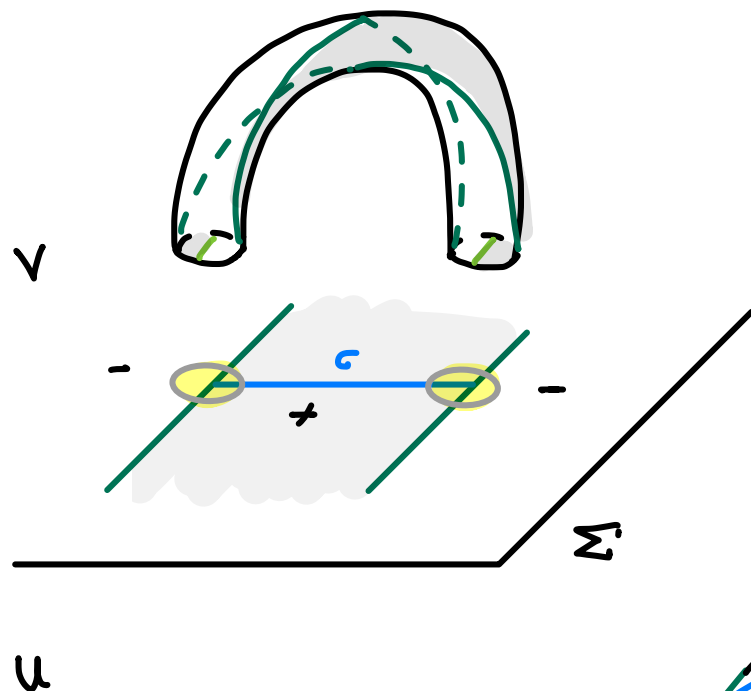
• TURN UPSIDE DOWN & ATTACH THE 2-h OF B_2' TO V

$\rightsquigarrow$ GET $\mathcal{K}'$ ■

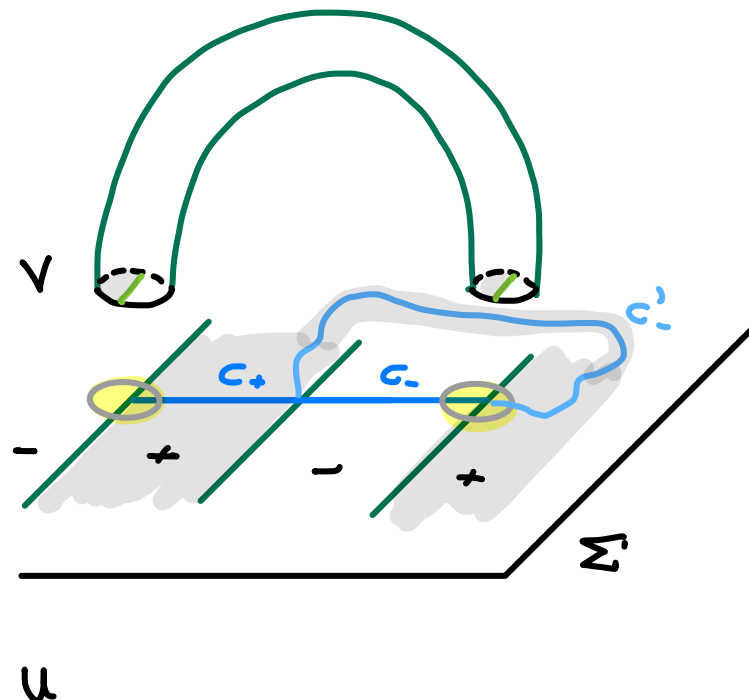


ADDING BYPASS 1-HANDLE

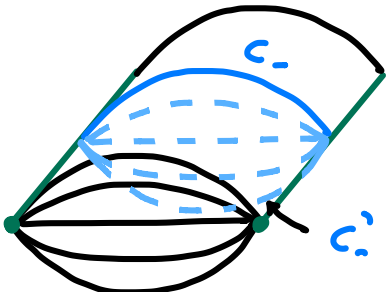
STABILISATION



BYPASS 1-HANDLE



RECALL: $U \cong \Sigma \times I / \sim$:

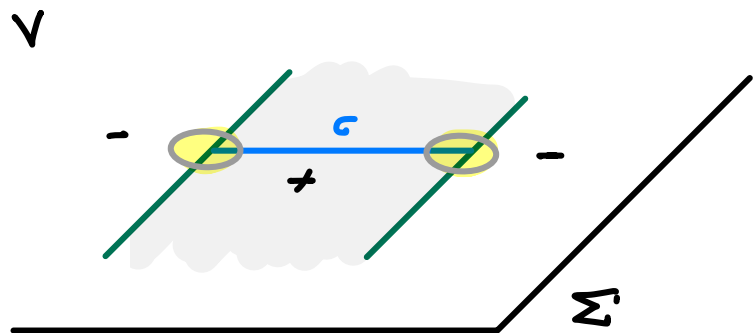


$c_+ \subset \Sigma_+$

SO THE BYPASS 1-HANDLE ADDITION ALONG $c_+ \cup c_- = \text{STAB ALONG}$
 $\parallel$
 STABILISATION ALONG $c_+ \cup c_-$

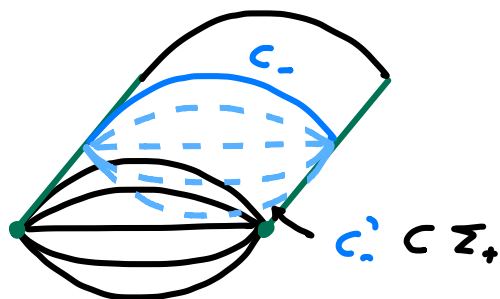
ADDING BYPASS 1-HANDLE

STABILISATION



U

RECALL: $U \cong \Sigma \times I / \sim$

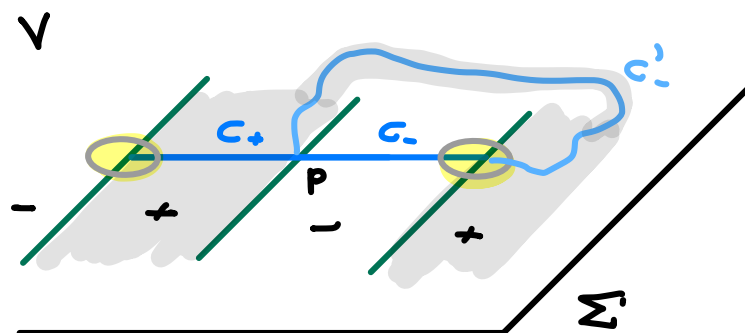


SO THE BYPASS 1-HANDLE ADDITION ALONG $C_+ \cup C_- = \text{STAB ALONG}$

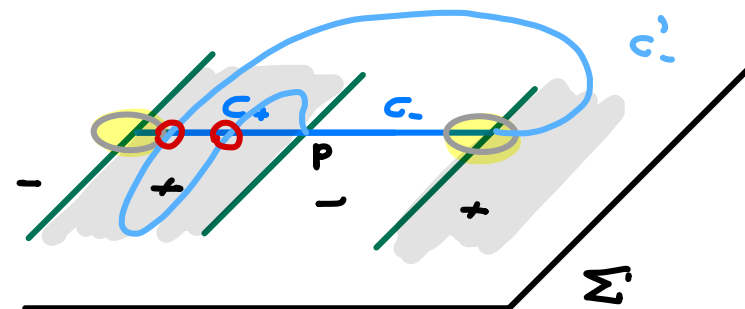
STABILISATION ALONG $C_+ \cup C_-$

ONLY WORKS IF $C_- \cap C_+ = p$

BYPASS 1-HANDLE

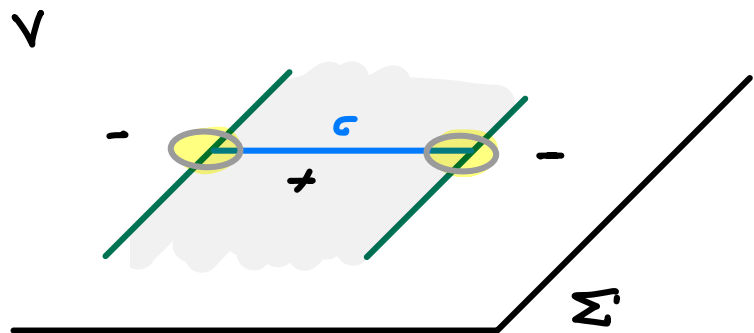


U



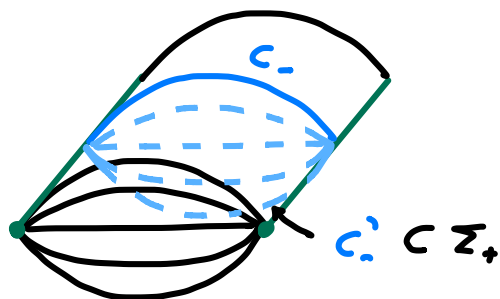
ADDING BYPASS 1-HANDLE

STABILISATION



U

RECALL: $U \cong \Sigma \times I / \sim$



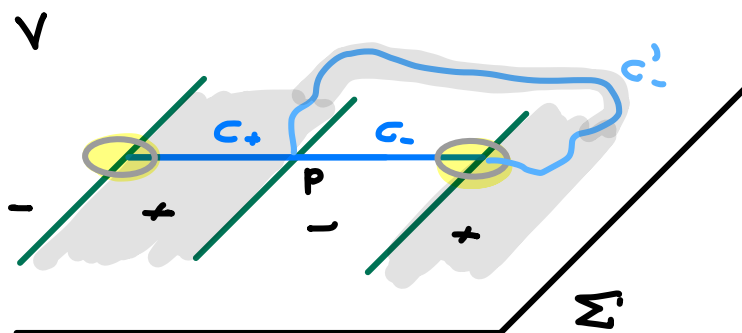
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STABILISATION ALONG $C_+ \cup C_-$

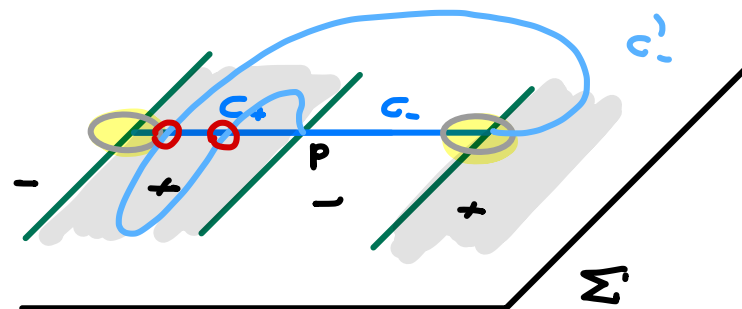
ONLY WORKS IF

$$C'_- \cap C_+ = p$$

BYPASS 1-HANDLE



U



WE SHOW: AFTER MORE (DE)STAB WE CAN MAKE SURE $C'_- \cap C_+ = p$

ADDING BYPASS 1-HANDLE

WE GET

PROP (L-V): $\mathcal{K}(M=U \cup V)$ CTCT HD, B BYPASS ONTO $Z \subset V$

THEN FOR $\bullet U' = U \cup 1-h$

$\bullet V' = \overline{M - U'}$

THE CTCT HD $\mathcal{K}'(M=U' \cup V')$ IS CONNECTED TO $\mathcal{K}$
VIA CTCT STAB'S & DESTAB'S

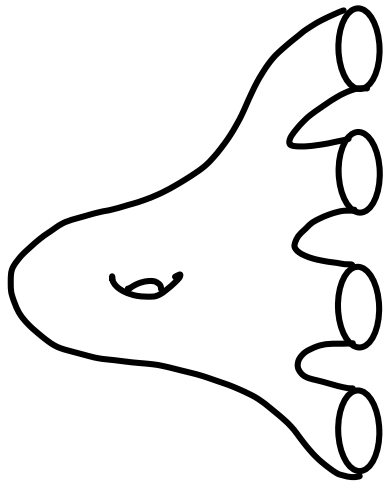
COR (L-V): $\mathcal{K}(M=U \cup V)$ CTCT HD, B BYPASS ONTO $Z \subset V$

IF $U' = U \cup B$ IS TIGHT, THEN FOR $V' = \overline{M - U}$

THE CTCT HD $\mathcal{K}'(M=U' \cup V')$ IS CONNECTED TO $\mathcal{K}$
VIA CTCT STAB'S & DESTAB'S

THIS GIVES A SIMPLER PROOF FOR GIROUX CORRESPONDENCE
FOR TIGHT CONTACT STRUCTURES

FURTHER DIRECTIONS



WHAT'S NEXT?

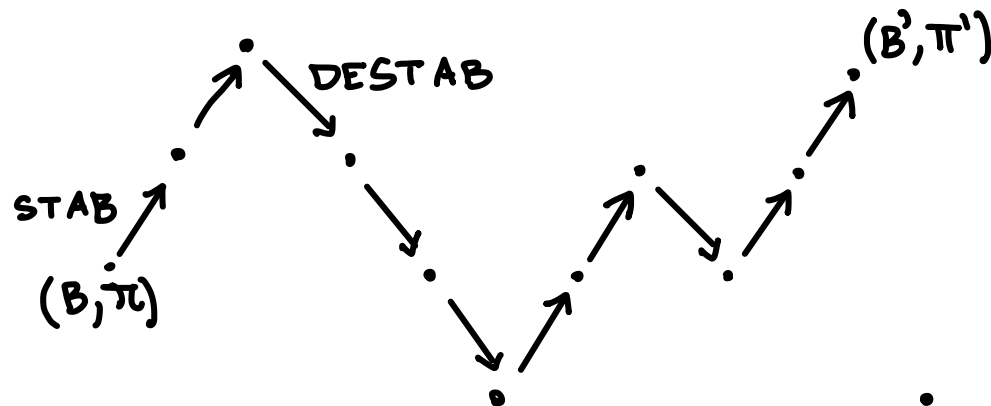
FEELS LIKE END OF A LONG STORY

BUT AS ALWAYS THERE IS A LOT TO DO:

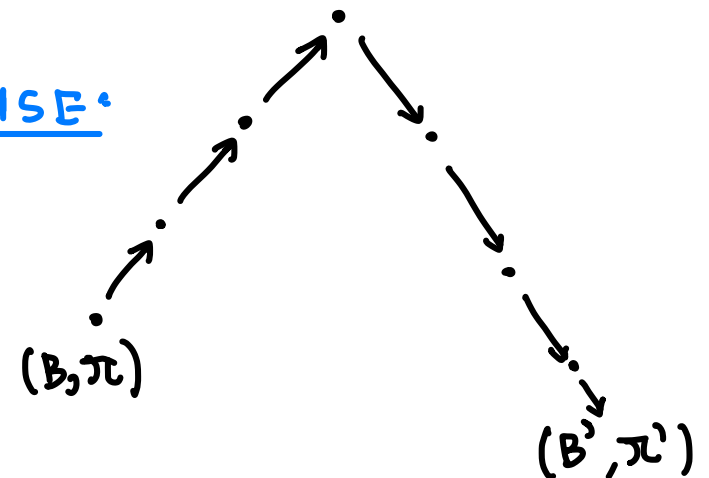
- (HOPEFULLY) MINOR ISSUE:

COMMON STABILISATION VS. SEQUENCE OF (DE)STABILISATIONS

WHAT WE PROVED:



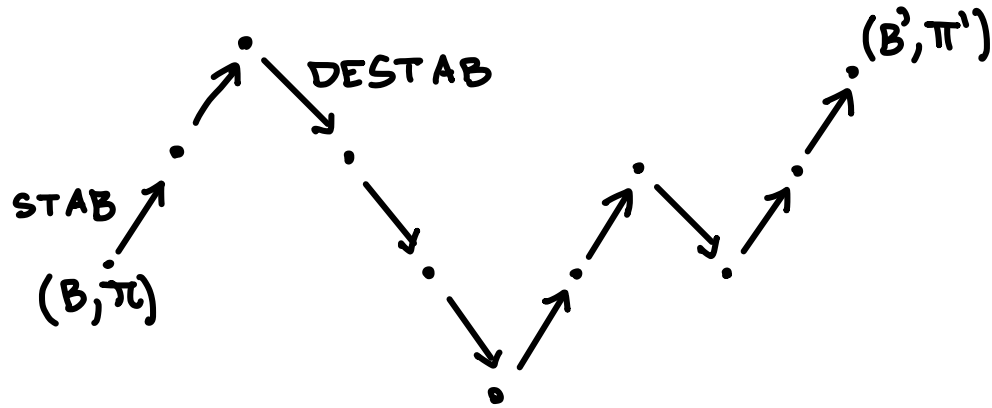
WHAT SOME OF THE APPLICATIONS USE:



ARE THEY EQUIVALENT?

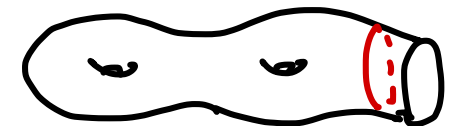
WHAT'S NEXT?

• HOW MANY STABILISATIONS DO WE NEED?



• ETNYRE: DOES EVERY CONTACT STRUCTURE HAVE A GENUS 1 OPEN BOOK?

- OVERTWISTED CONTACT STRUCTURES HAVE PLANAR (= GENUS 0) OPEN BOOKS (ETNYRE)
- THERE ARE CONTACT STRUCTURES THAT DO NOT HAVE PLANAR OPEN BOOKS (ETNYRE)
- POSSIBLE COUNTEREXAMPLE (MASSOT)



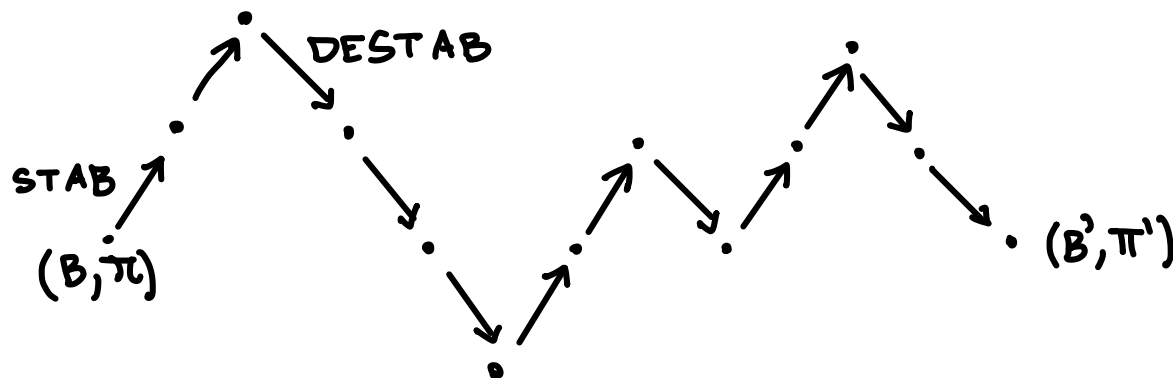
WHAT'S NEXT?

THM (WAND): LEGENDRIAN SURGERY PRESERVES TIGHTNESS

- THE PROOF RELIES ON AN EQUIVALENT CHARACTERISATION OF TIGHTNESS IN TERMS OF OPEN BOOKS
- THIS CHARACTERISATION IS GIVEN IN A SEQUENCE OF COMBINATORIAL DEFINITIONS THAT TAKE UP MULTIPLE PAGES

IS THERE A SIMPLER PROOF USING CONTACT HEEGAARD DECOMPOSITIONS ?

- MOVES BETWEEN OPEN BOOKS OF THE SAME GENUS / EULER CHARACTERISTIC



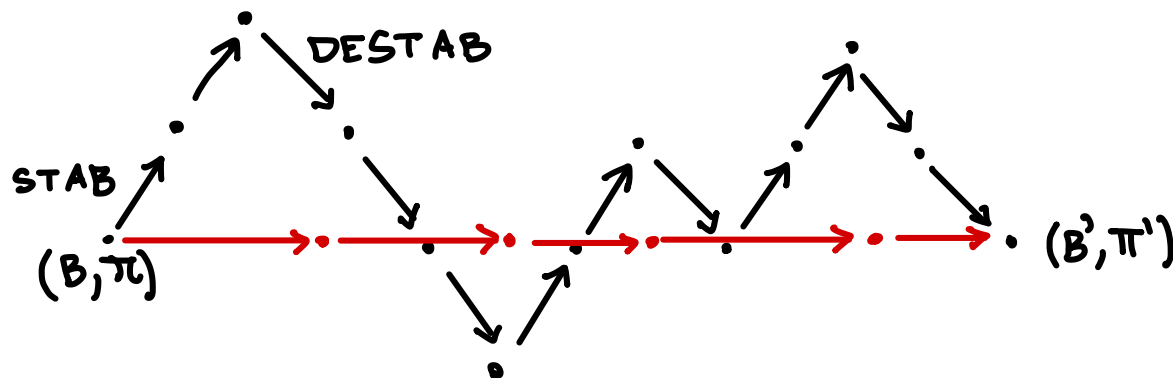
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THANKS FOR
YOUR
ATTENTION!