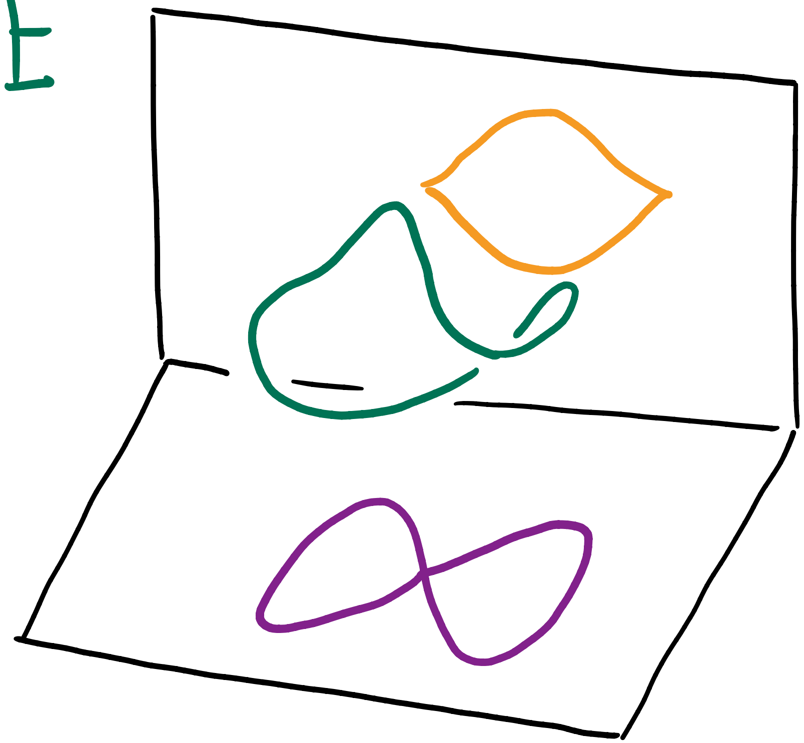
HOW TO UNTANGLE PLANE CURVES IN THE 3RD DIMENSION



HOW CONTACT TOPOLOGY PROVES CLASSICAL THEOREMS

VERA VÉRTESI

UNIVERSITY OF VIENNA

NOV. 10. 2021.

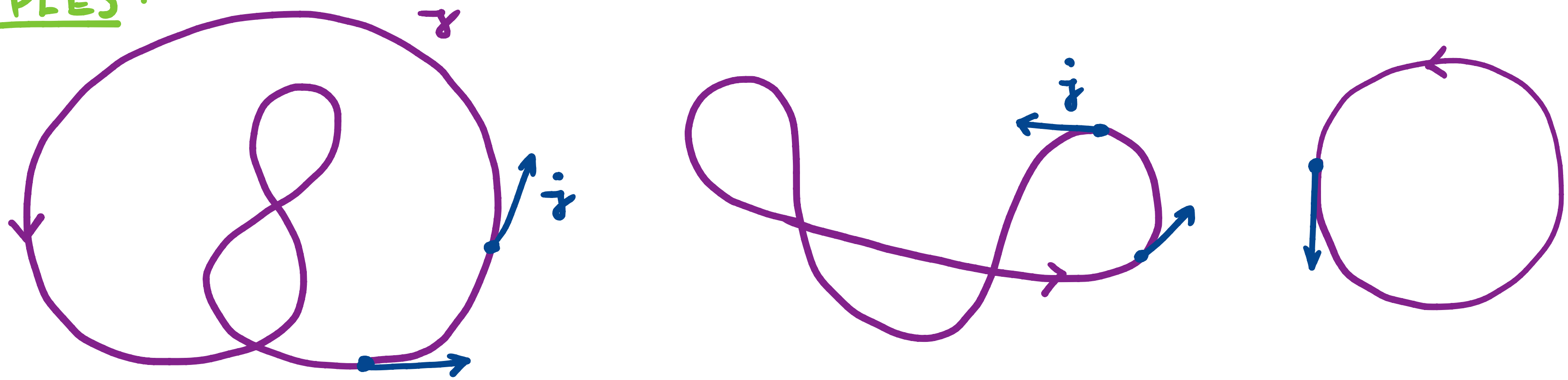
REGULAR CLOSED CURVES IN THE PLANE

$\gamma : S^1 \looparrowright \mathbb{R}^2$

- γ CONTINUOUS
- $\dot{\gamma}$ EXISTS
  NONZERO
  & CONTINUOUS

$\Big\}$ IMMERSION

EXAMPLES:



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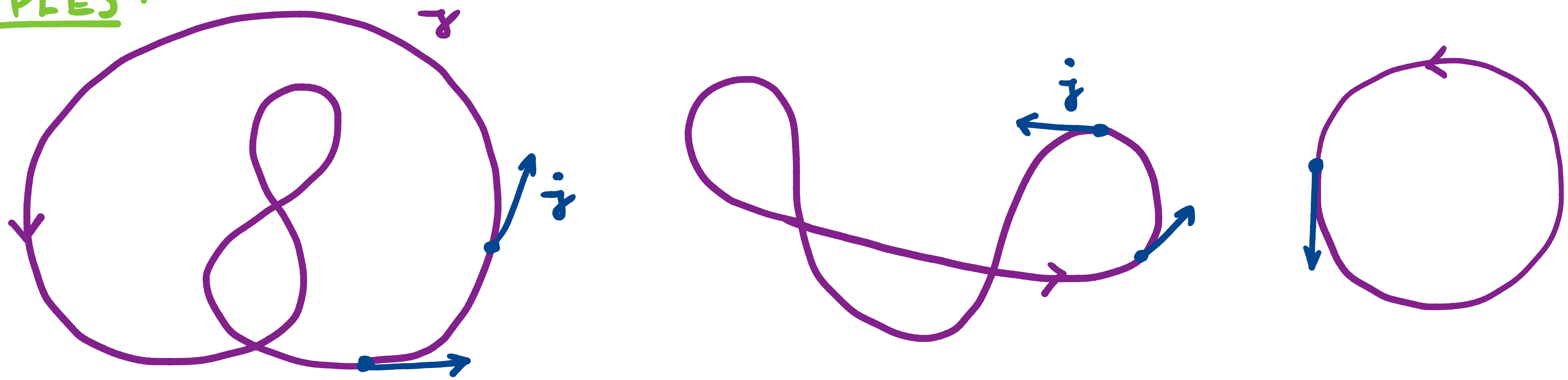

NON-EXAMPLES

γ NOT CONTINUOUS

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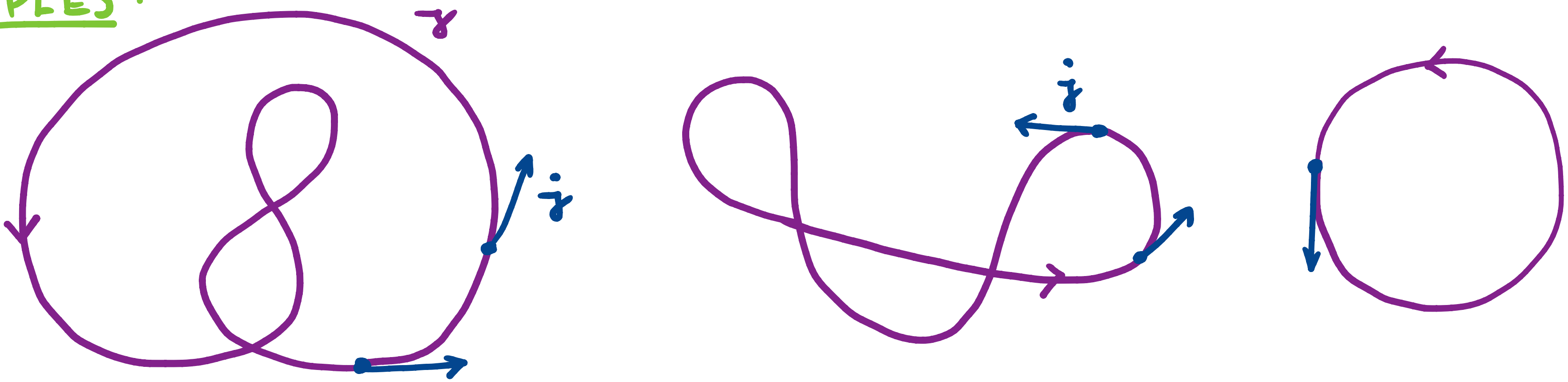
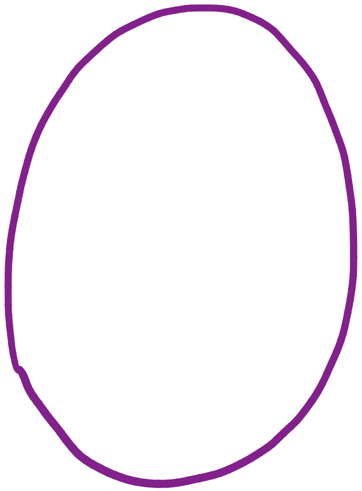


NON-EXAMPLES

γ NOT CONTINUOUS $\dot{\gamma}$ DOES NOT EXIST

REGULAR HOMOTOPY



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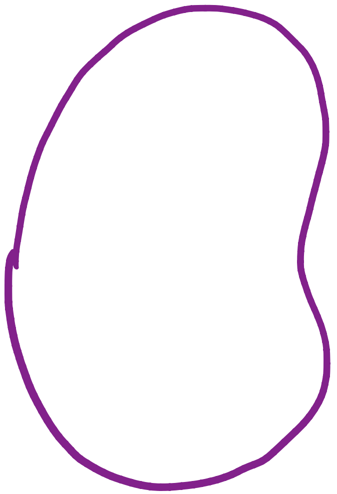

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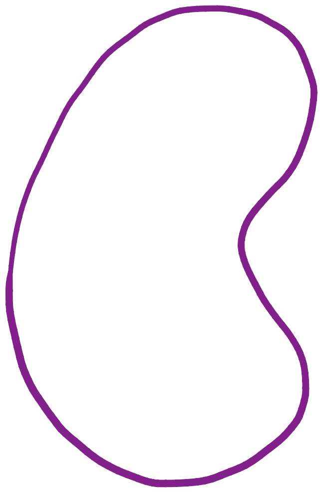

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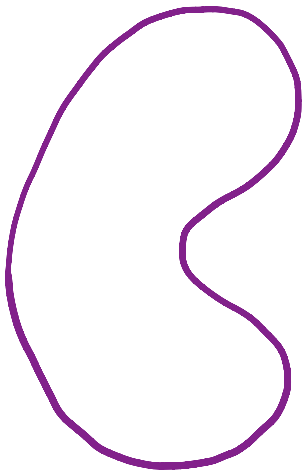

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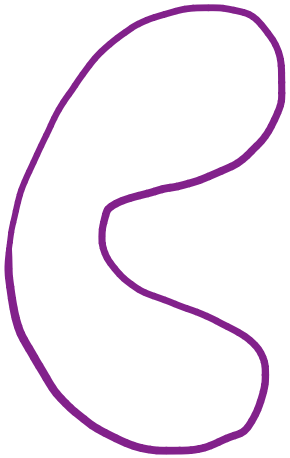

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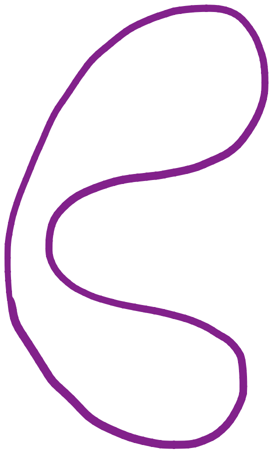

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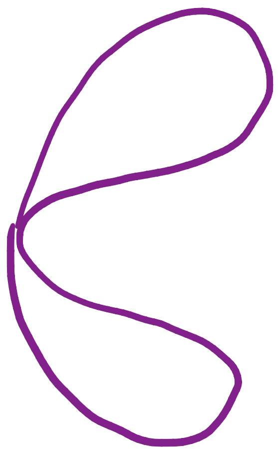

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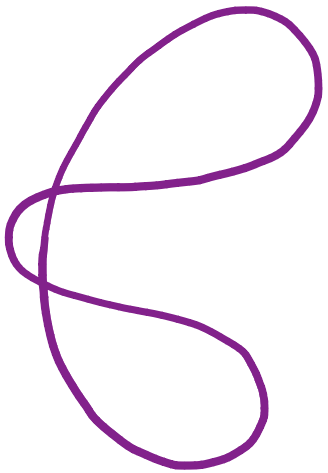

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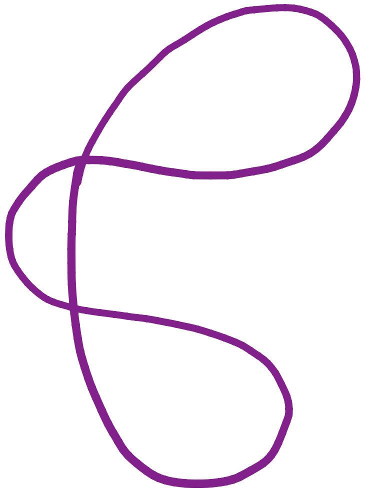

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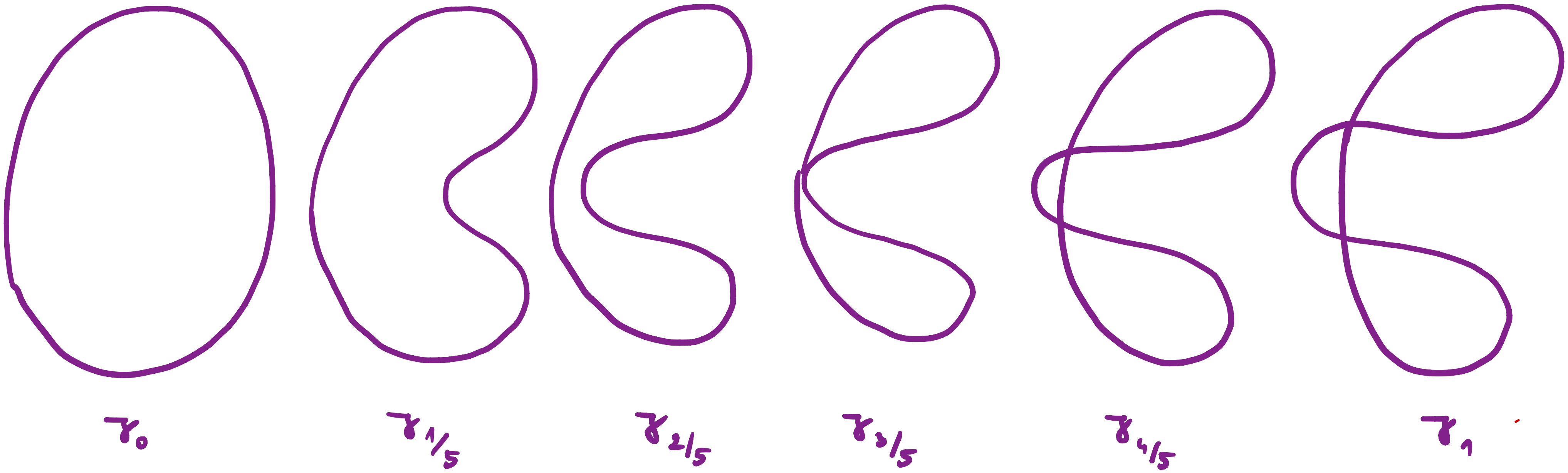

A MOVIE $\{\gamma_t\}$ OF REGULAR CLOSED CURVES:

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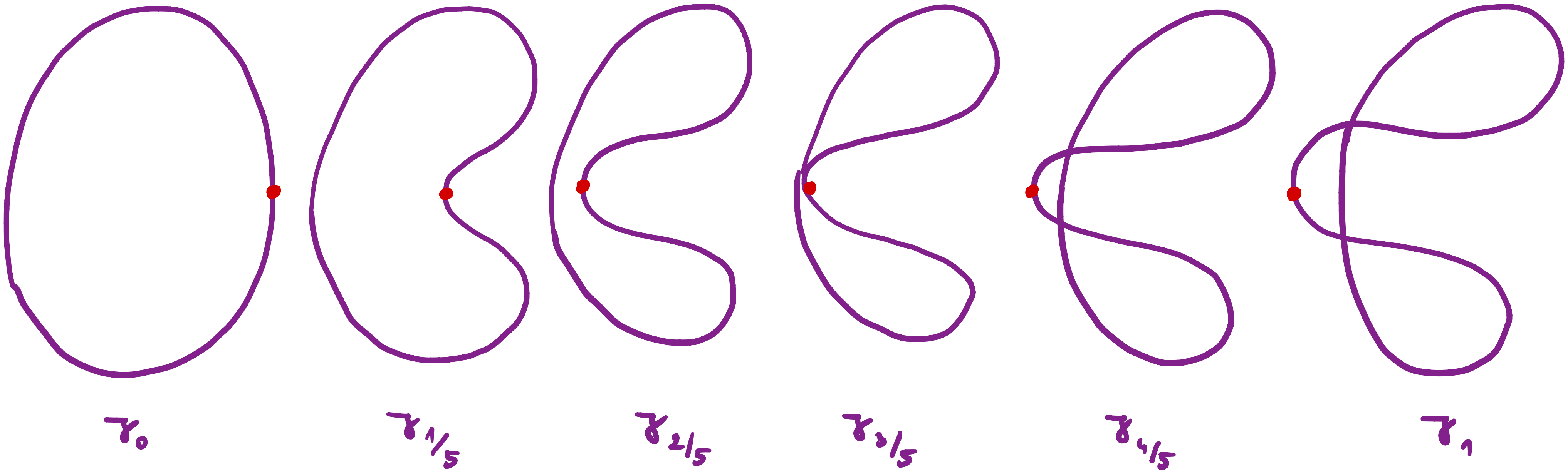

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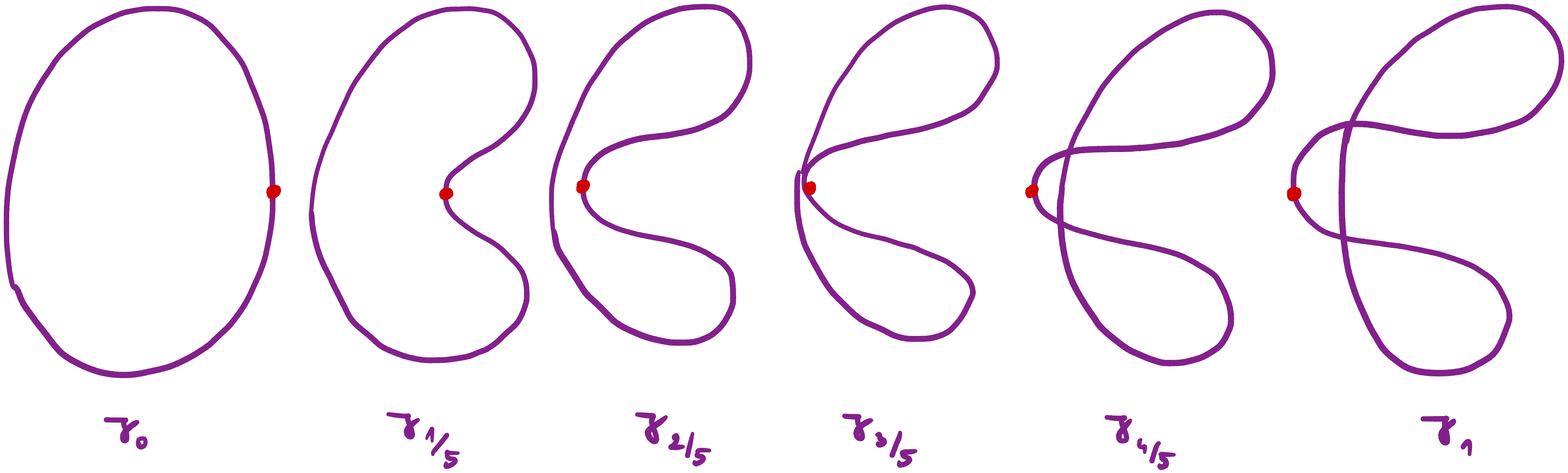

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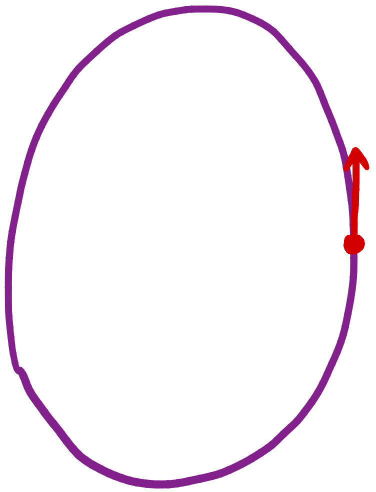

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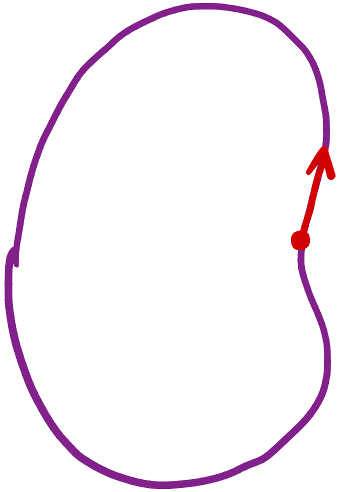

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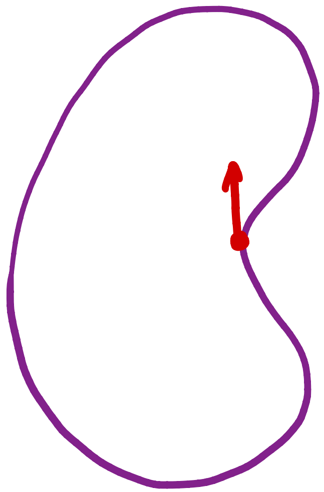

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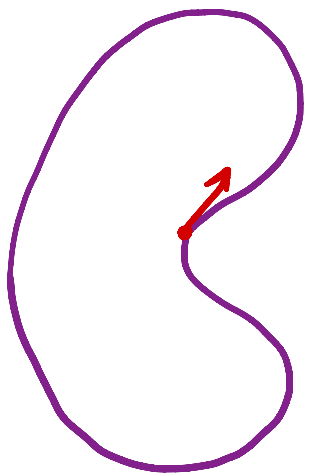

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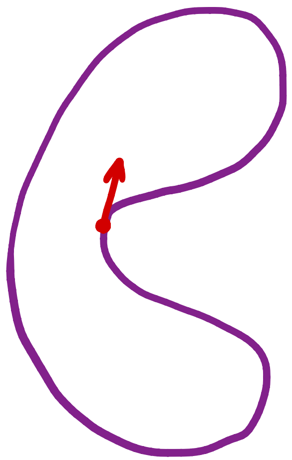

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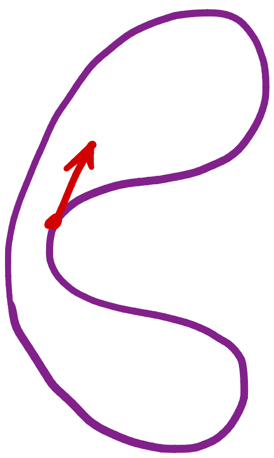

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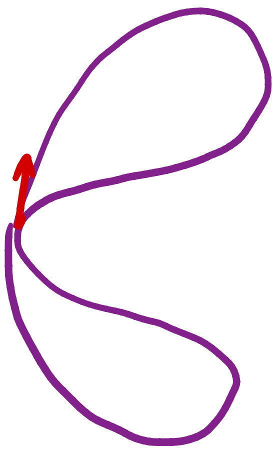

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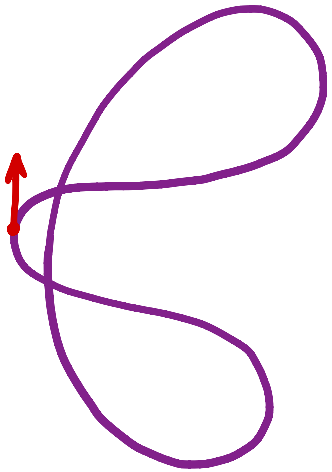

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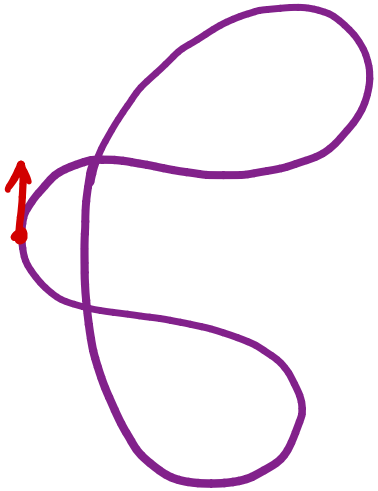

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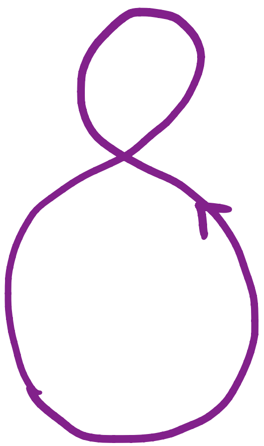

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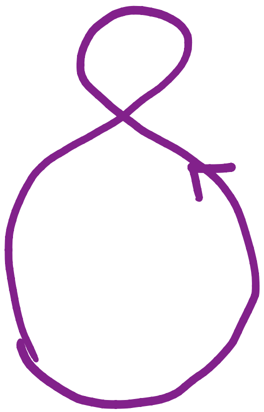

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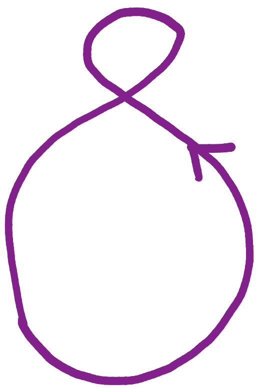

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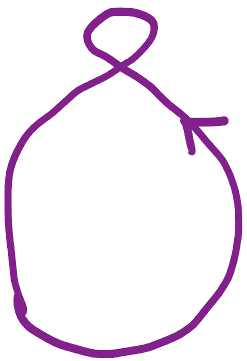

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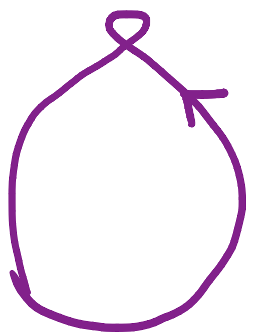

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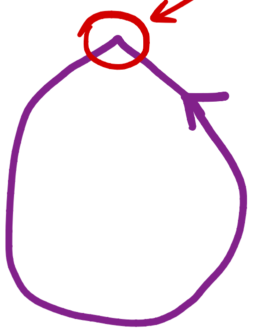

CUSP ⇒ NOT REGULAR !

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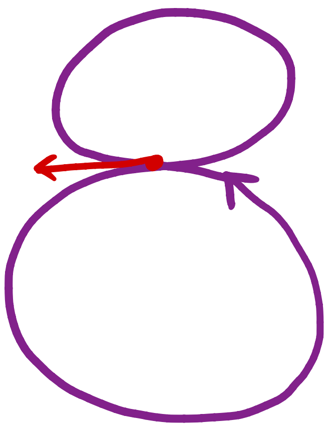

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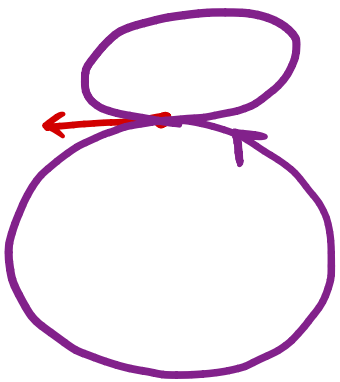

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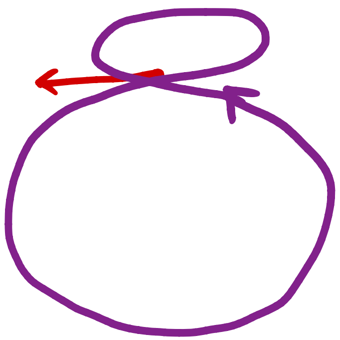

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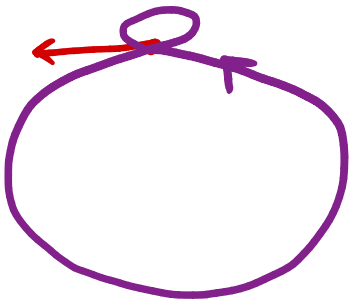

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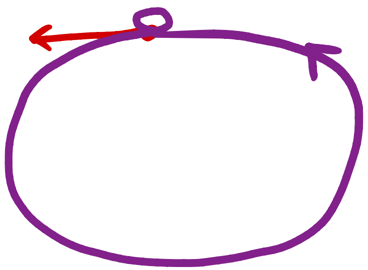

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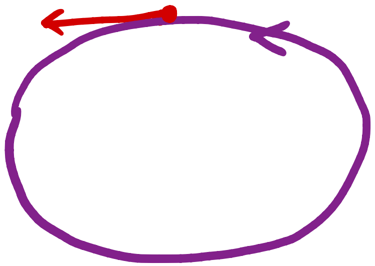

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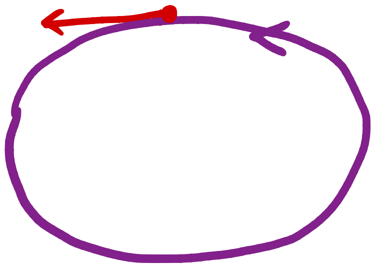

← STILL NOT A REGULAR HOMOTOPY

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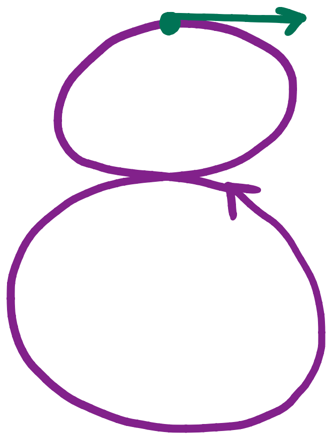

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← STILL NOT A REGULAR HOMOTOPY

WHY ?

KEEP TRACK OF ONE MORE TANGENT

A MOVIE $\{\gamma_t\}$ OF REGULAR CLOSED CURVES :

- γ_t IS CONTINUOUS
- $\dot{\gamma}_t$ EXISTS, NONZERO & CONTINUOUS
- THE PATH $I \to \mathbb{R}^2$, $t \mapsto \gamma_t(s)$ IS CONTINUOUS FOR EACH $s \in S^1$
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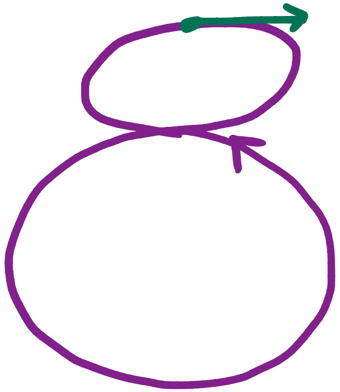
REGULAR HOMOTOPY – NON EXAMPLE



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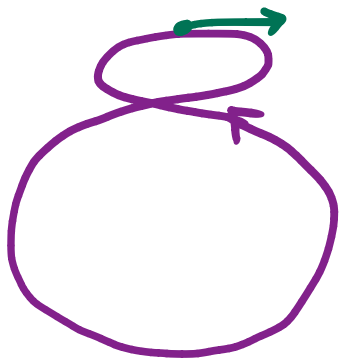

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REGULAR HOMOTOPY – NON EXAMPLE

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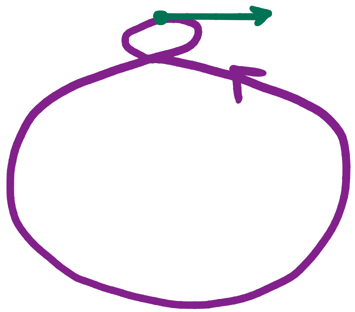

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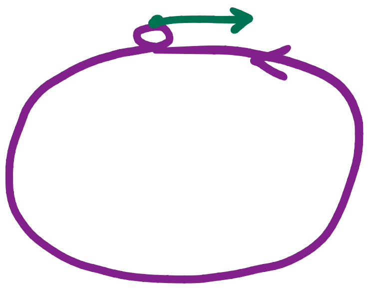

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REGULAR HOMOTOPY – NON EXAMPLE

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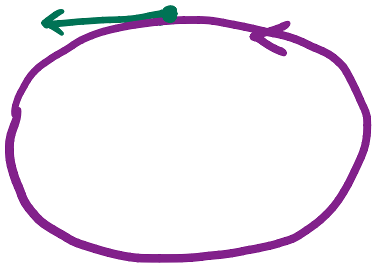

← STILL NOT A REGULAR HOMOTOPY

WHY?

$t \mapsto \dot{\gamma}_t(s)$ IS NOT CONTINUOUS!

A MOVIE $\{\gamma_t\}$ OF REGULAR CLOSED CURVES:

- γ_t IS CONTINUOUS
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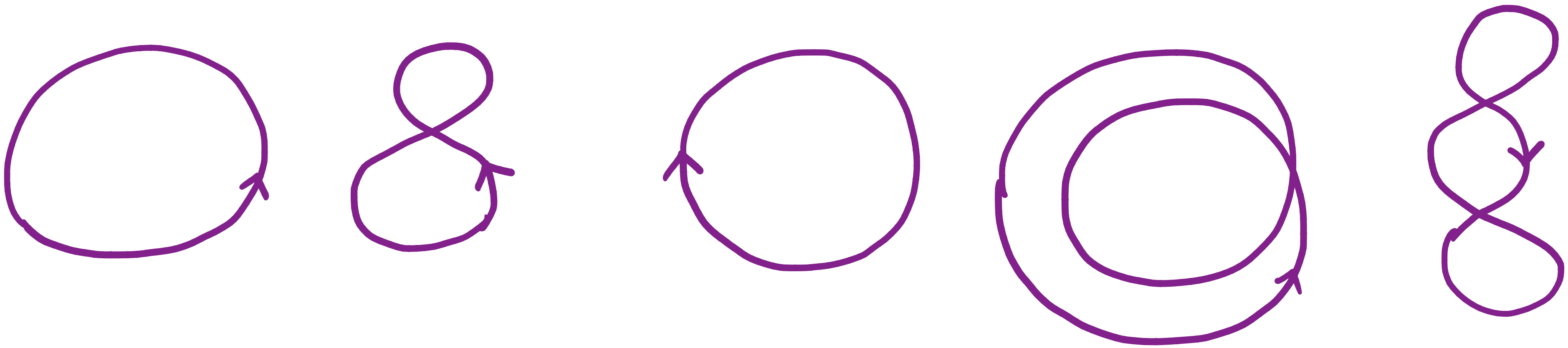
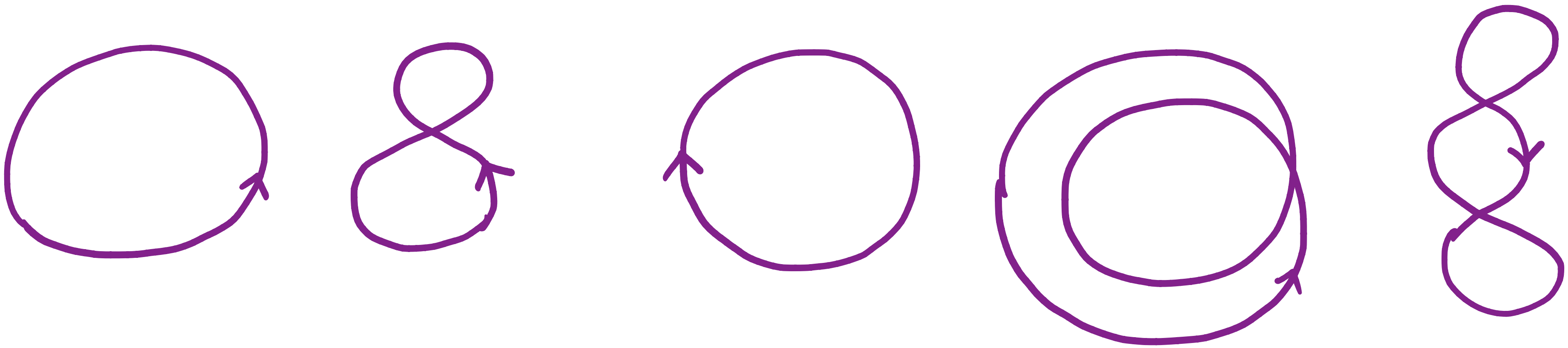
REGULAR HOMOTOPY

QUESTION: WHICH CURVES ARE REGULAR HOMOTOPIC?



REGULAR HOMOTOPY

QUESTION: WHICH CURVES ARE REGULAR HOMOTOPIC?



ANSWER: FIND A PROPERTY THAT DOESN'T CHANGE DURING REGULAR HOMOTOPY.

WHITNEY - GRAUSTEIN THEOREM (1937):

γ_0 IS REGULARLY HOMOTOPIC TO γ_1

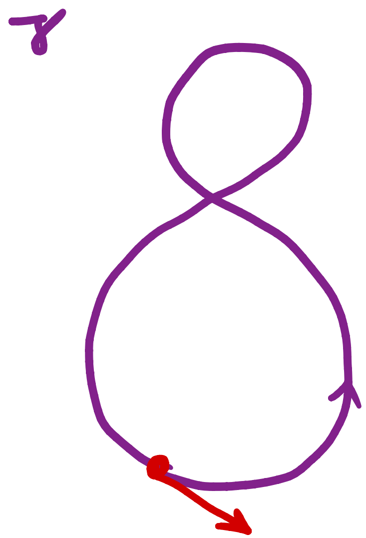
$$\Updownarrow$$

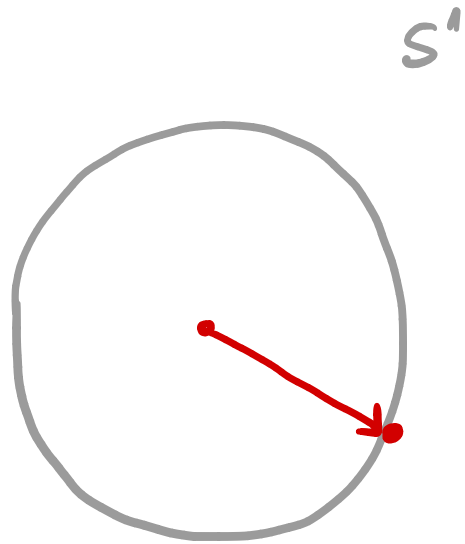
$$W(\gamma_0) = W(\gamma_1)$$

WINDING NUMBER

$\gamma : S^1 \longrightarrow \mathbb{R}^2$ REGULAR PLANE CURVE $(\Rightarrow \dot{\gamma} \neq 0)$

DEFINE: $\dfrac{\dot{\gamma}}{|\dot{\gamma}|} : S^1 \longrightarrow S^1$





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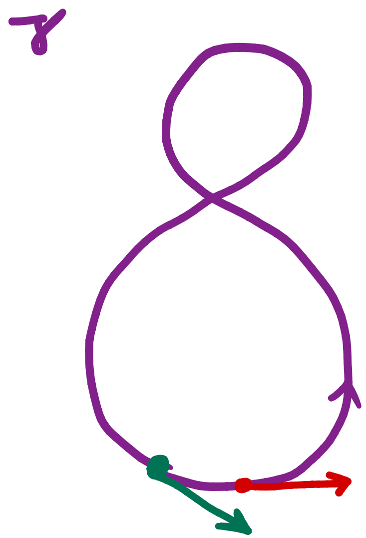

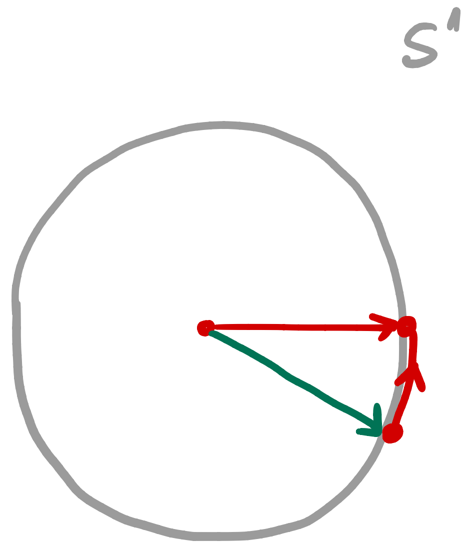

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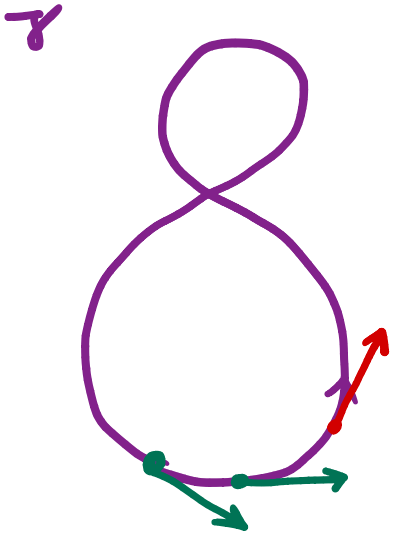

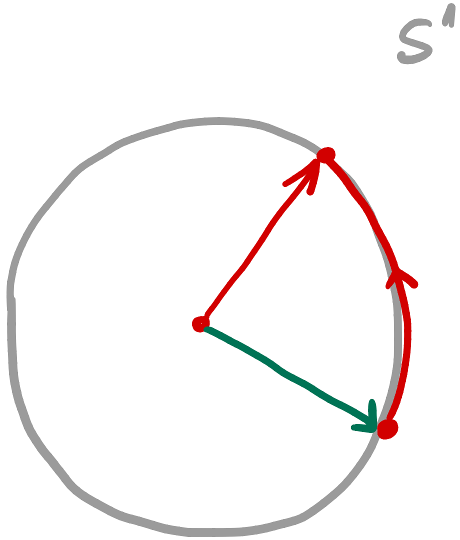

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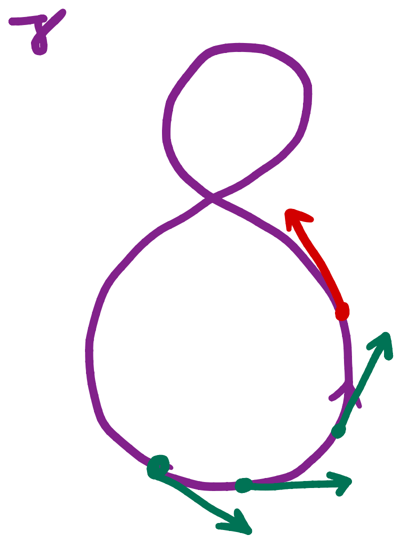

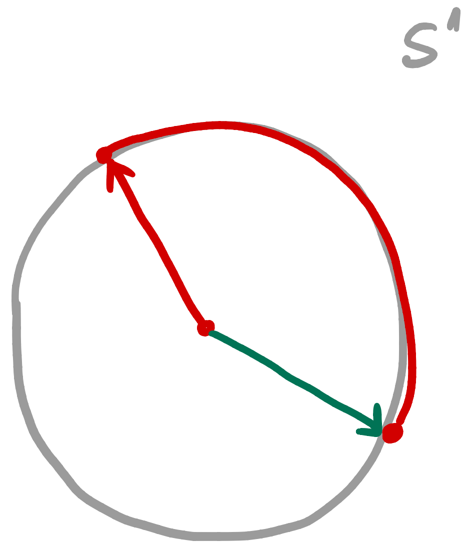

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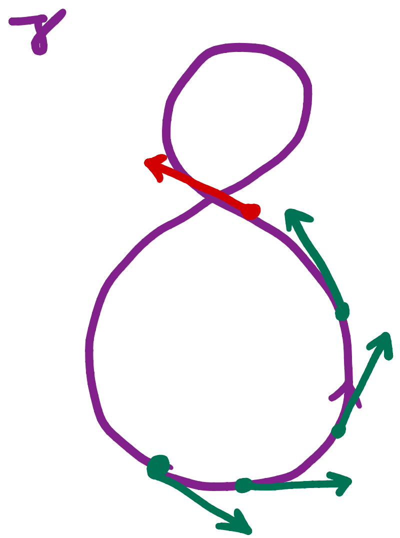

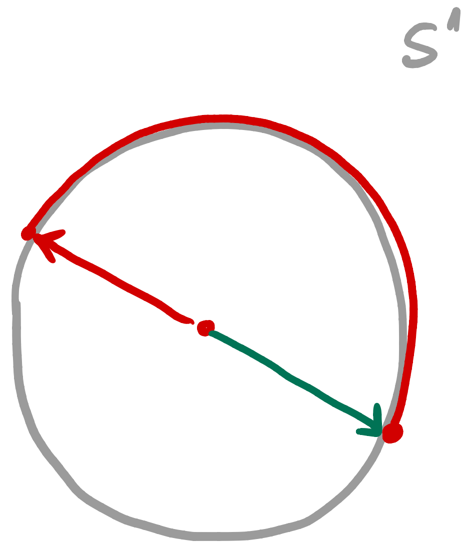

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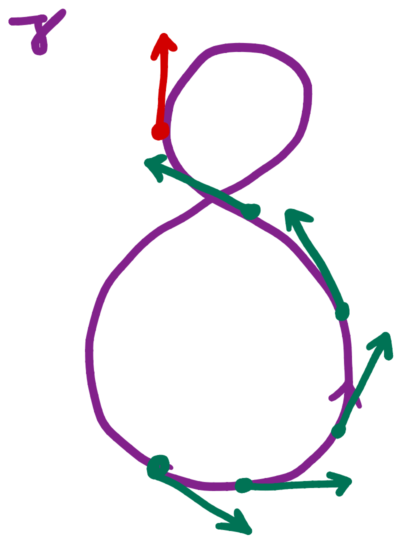

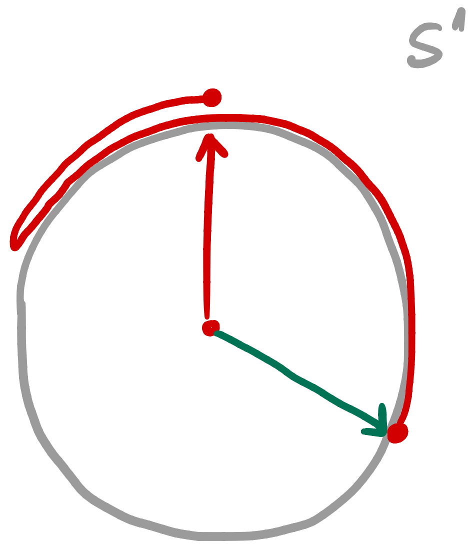

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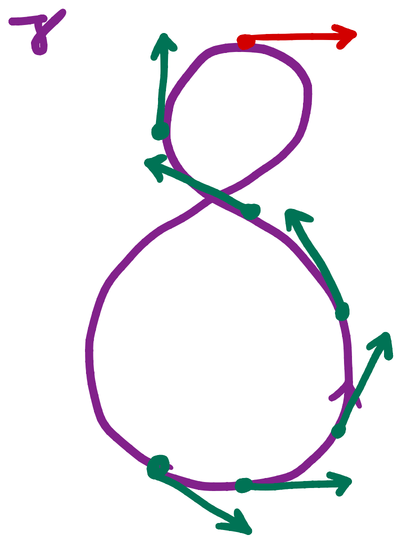

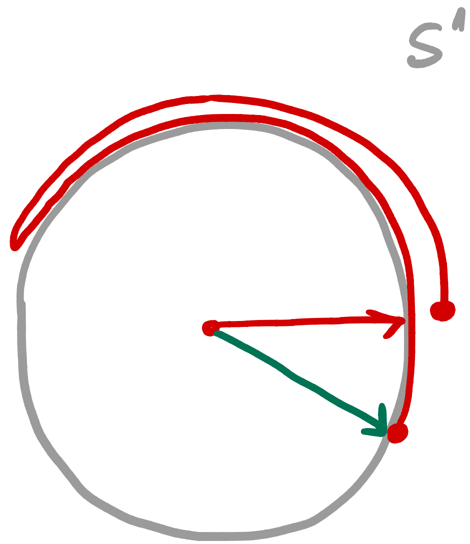

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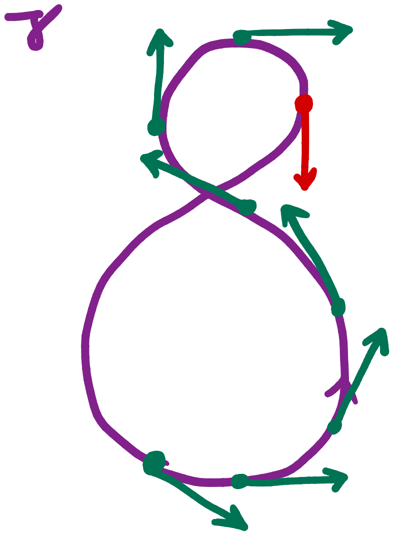

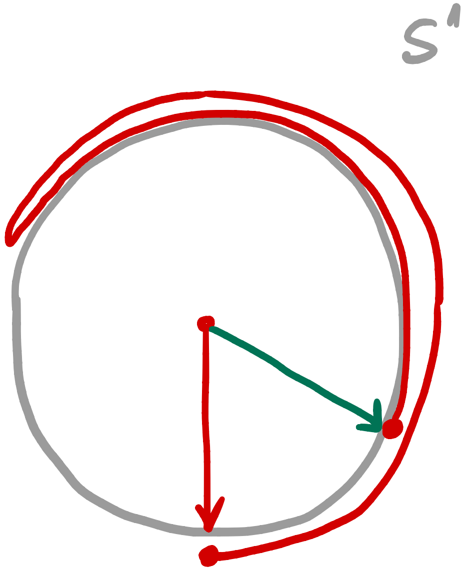

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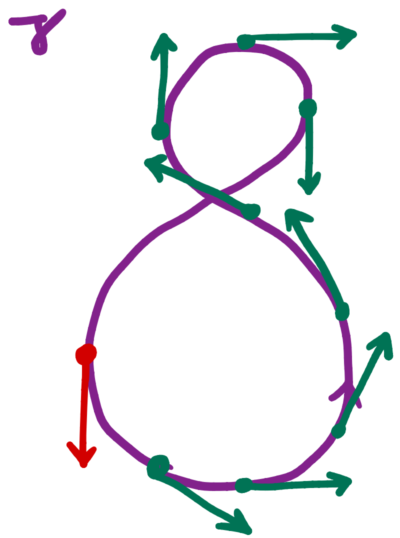

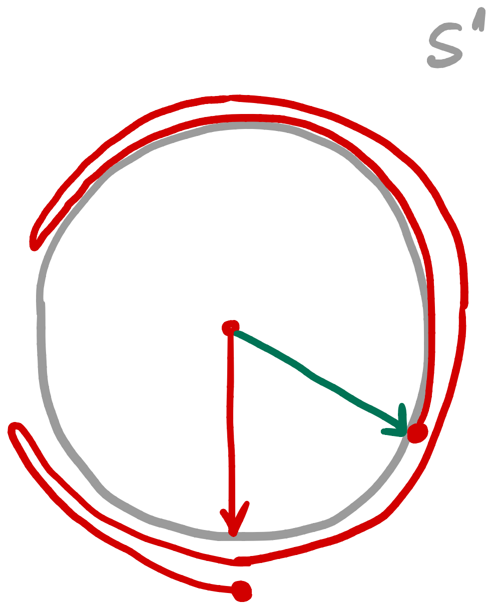

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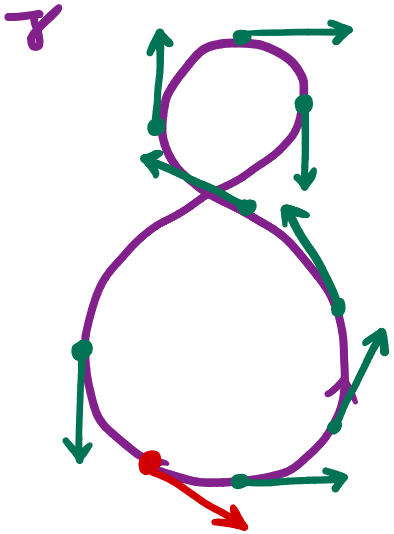

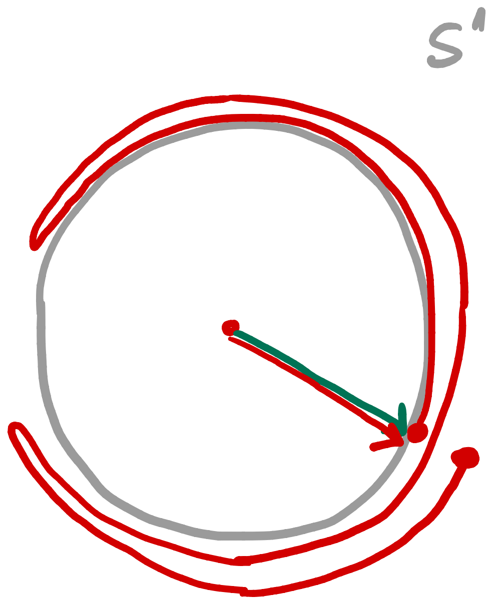

$W = 0$

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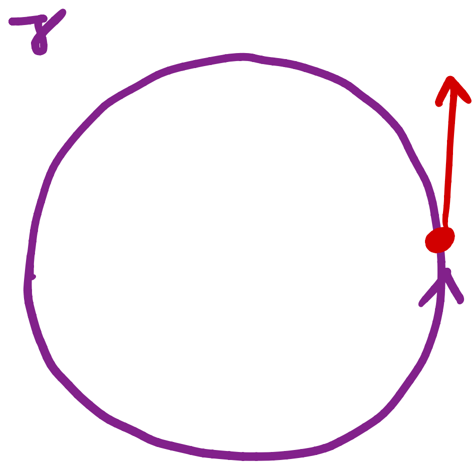

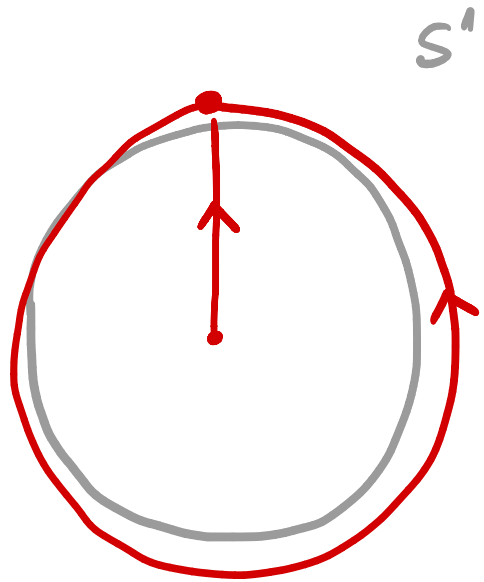

$W = 1$

WINDING NUMBER

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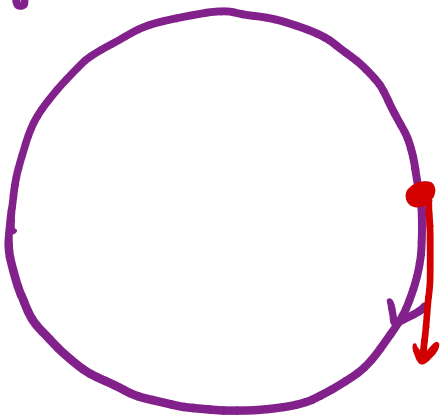
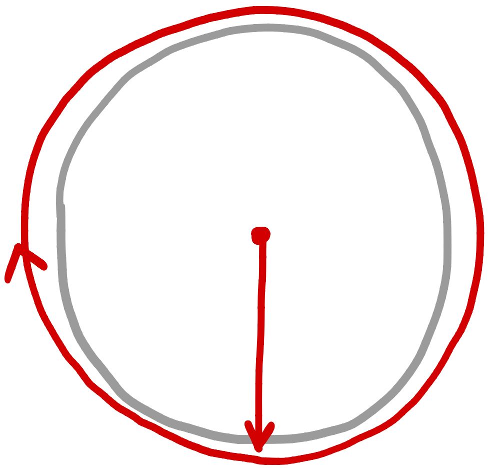
DEFINE: $\dfrac{\dot\gamma}{|\dot\gamma|} : S^1 \longrightarrow S^1$

γ



S^1



$W = -1$

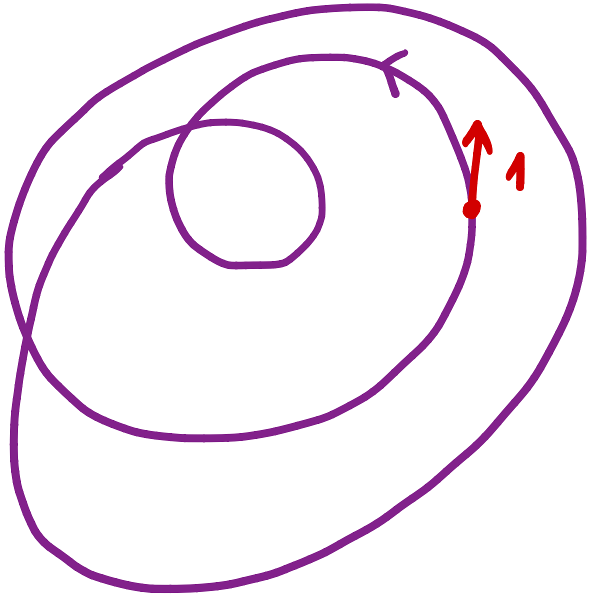
$$\boxed{W(\gamma) = \mathrm{DEG}\left(\frac{\dot\gamma}{|\dot\gamma|}\right)}$$

WINDING NUMBER - COMPUTATION

$\gamma : S^1 \longrightarrow \mathbb{R}^2$ REGULAR PLANE CURVE

CHOOSE A DIRECTION (E.G. : $\uparrow$)

- LET'S CHECK HOW MANY TIMES $\dot{\gamma} \parallel \uparrow$



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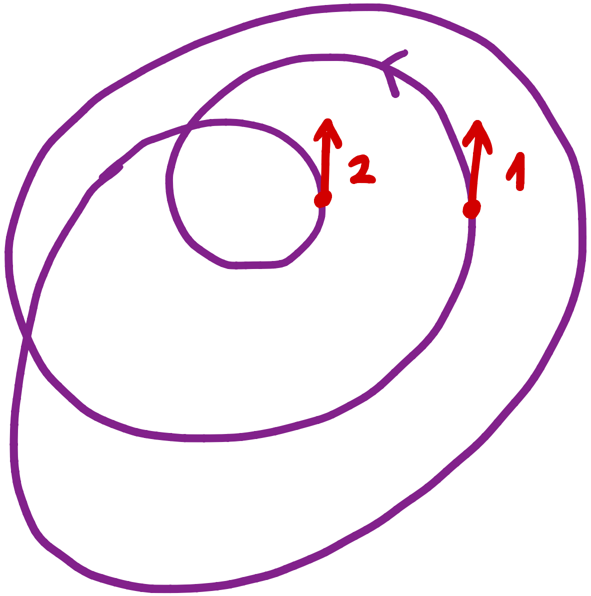

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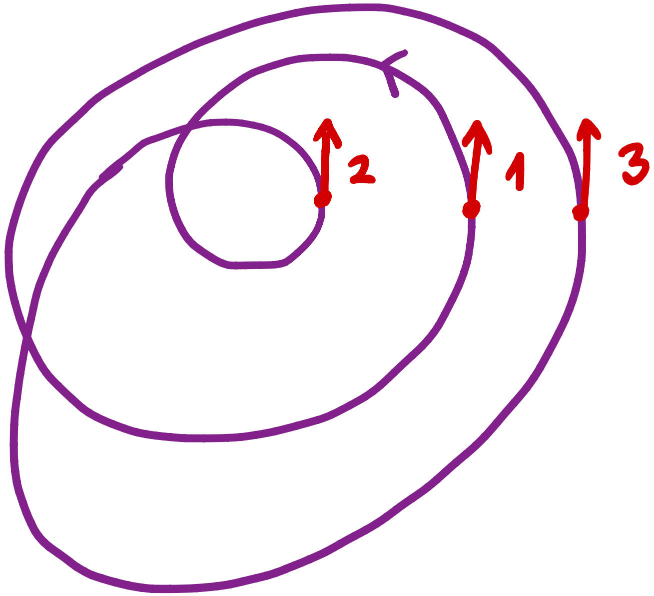

& $W(\gamma) = 3$

WINDING NUMBER - COMPUTATION

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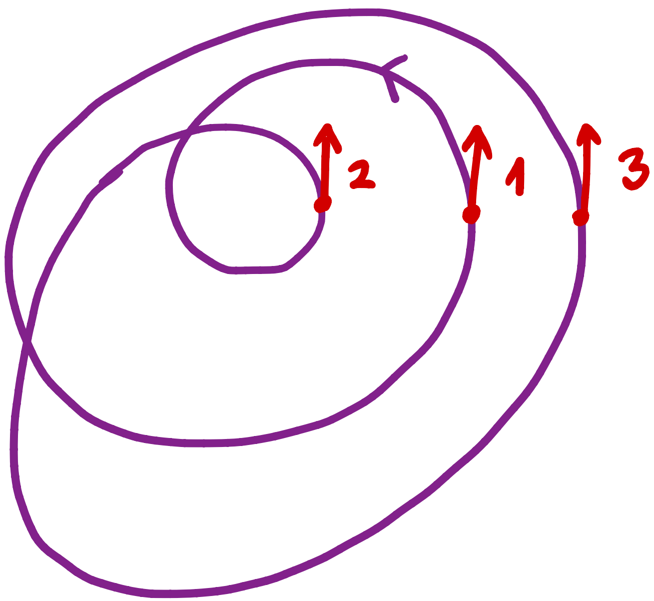

& $W(\gamma) = 3$

THEOREM: $W(\gamma) = \#\{\dot{\gamma} \parallel \uparrow\}$

WINDING NUMBER - COMPUTATION

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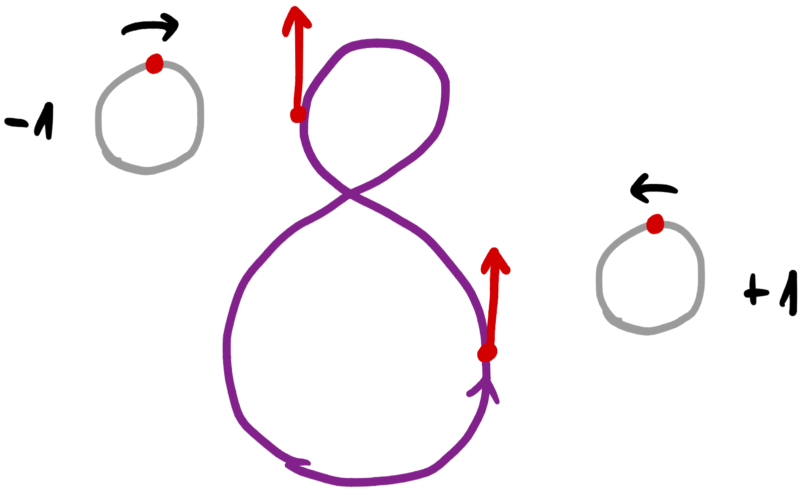
CHOOSE A DIRECTION (E.G.: $\uparrow$)

• LET'S CHECK HOW MANY TIMES $\dot{\gamma} \parallel \uparrow$



& $w(\gamma) = 3$

THEOREM: $w(\gamma) = \#\{\gamma \parallel \uparrow\}$

BUT! WE HAVE TO COUNT W/ SIGN

WINDING NUMBER - FOR COMPLEX CURVES

FOR $\gamma : S^1 \to \mathbb{C}$, $S^1 \subset \mathbb{C}$

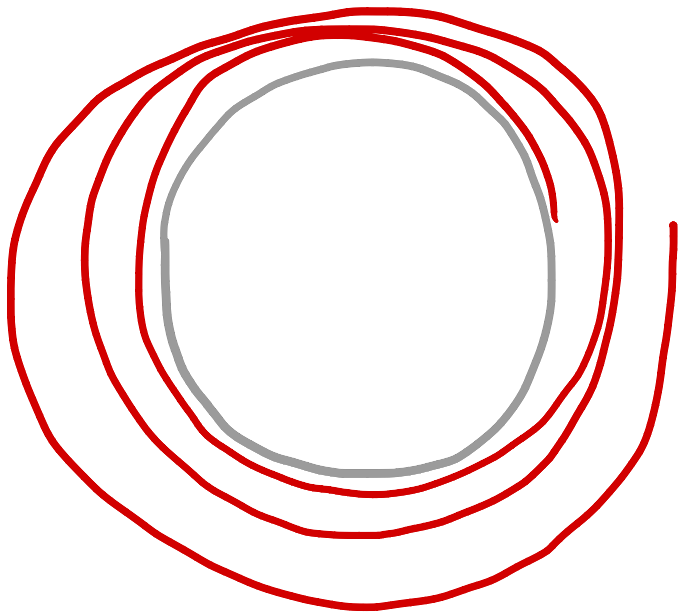
CONSIDER $\dot{\gamma} : S^1 \to \mathbb{C} \setminus \{0\}$, $S^1 \subset \mathbb{C}$

$$\boxed{W(\gamma) = \mathrm{IND}(\dot{\gamma}, 0) = \frac{1}{2\pi i} \oint_{\dot{\gamma}} \frac{dz}{z}}$$

EXAMPLE: $\gamma_n : S^1 \to S^1$

$z \mapsto z^n$

$\underline{W(\gamma_n) = n}$



WINDING NUMBER

$\gamma : S^1 \longrightarrow \mathbb{R}^2$ REGULAR PLANE CURVE

DEFINE: $\dfrac{\dot\gamma}{|\dot\gamma|} : S^1 \longrightarrow S^1$

$$\boxed{W(\gamma) = \deg\left(\frac{\dot\gamma}{|\dot\gamma|}\right)}$$

$$\{\text{REGULAR PLANE CURVES}\} \xrightarrow{W} \mathbb{Z}$$

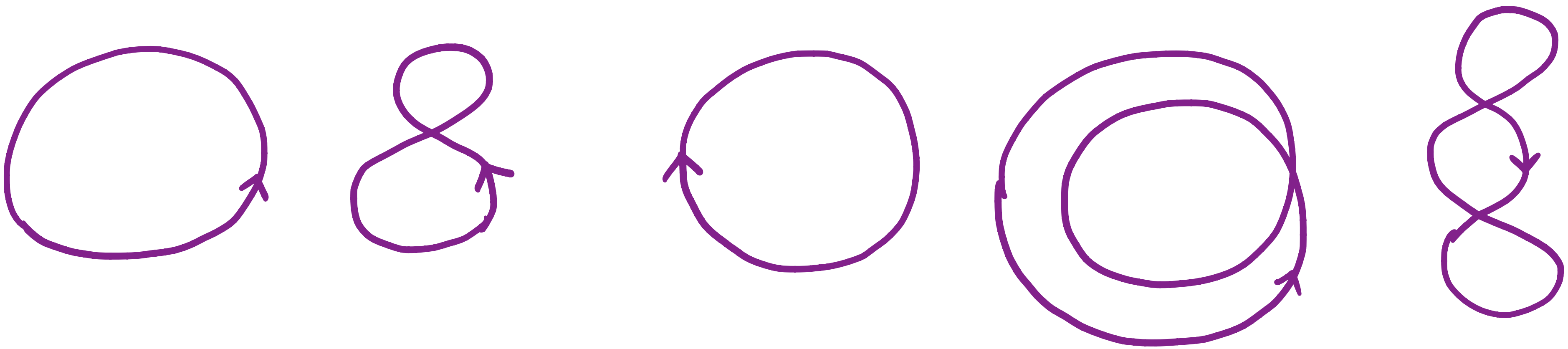
DISCREET

REGULAR HOMOTOPIC CURVES HAVE EQUAL WINDING #S

$$\Rightarrow \boxed{\{\text{REGULAR PLANE CURVES}\} \big/ \text{REGULAR HOMOTOPY} \xrightarrow{W} \mathbb{Z}}$$

WINDING NUMBER

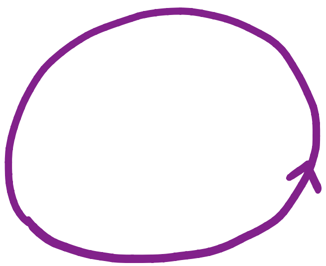
QUESTION: WHICH CURVES ARE REGULAR HOMOTOPIC ?



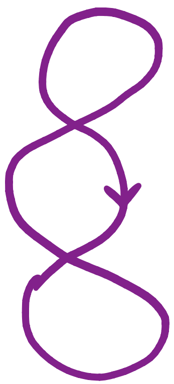
W = 1 0 -1 2 1

$$\{\text{REGULAR PLANE CURVES}\} \big/ \text{REGULAR HOMOTOPY} \xrightarrow{W} \mathbb{Z}$$

$\Rightarrow$ ONLY



&

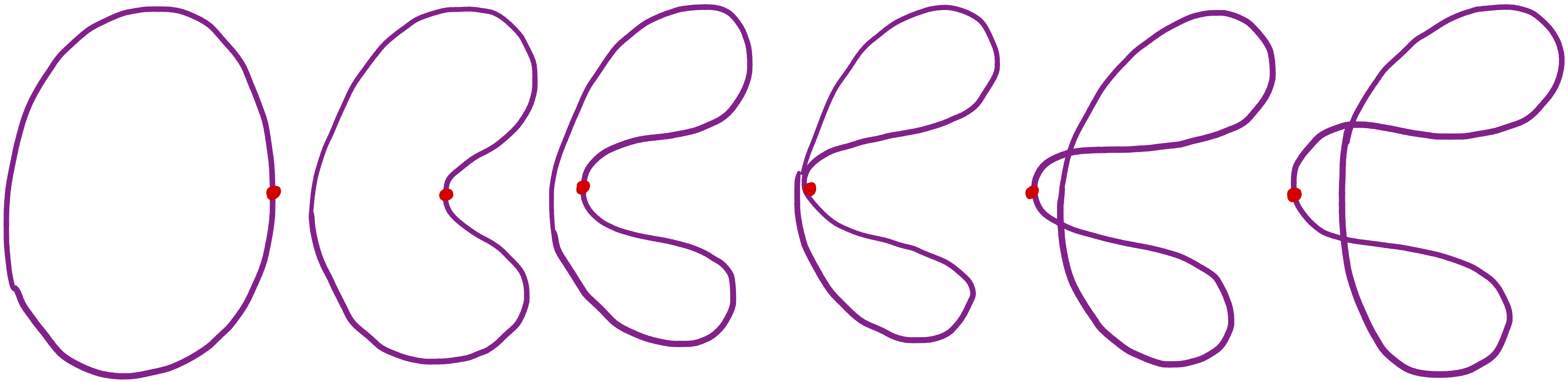


CAN BE REGULAR HOMOTOPIC

ARE THEY ?

WHITNEY - GRAUSTEIN THEOREM

YES:



IN GENERAL:

WHITNEY - GRAUSTEIN THEOREM:

$$\{\text{REGULAR PLANE CURVES}\} \Big/ \text{REGULAR HOMOTOPY} \xrightarrow{W} \mathbb{Z}$$

IS BIJECTIVE.

WHITNEY-GRAUSTEIN THEOREM - PROOF

WHITNEY-GRAUSTEIN THEOREM:

$$\{\text{REGULAR PLANE CURVES}\} \Big/ \text{REGULAR HOMOTOPY} \xrightarrow{W} \mathbb{Z}$$

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PROOF

- W IS WELL-DEFINED ✓

WHITNEY-GRAUSTEIN THEOREM - PROOF

WHITNEY - GRAUSTEIN THEOREM:

$$\{\text{REGULAR PLANE CURVES}\} \big/ \text{REGULAR HOMOTOPY} \xrightarrow{W} \mathbb{Z}$$

IS BIJECTIVE.

PROOF

- W IS WELL-DEFINED ✓
- SURJECTIVE : $W(\gamma_n) = n$ $\qquad \gamma_n : z \mapsto z^n$ ✓

WHITNEY-GRAUSTEIN THEOREM – PROOF

WHITNEY-GRAUSTEIN THEOREM:

$$\{\text{REGULAR PLANE CURVES}\} / \text{REGULAR HOMOTOPY} \xrightarrow{W} \mathbb{Z}$$

IS BIJECTIVE.

PROOF

- W IS WELL-DEFINED ✓
- SURJECTIVE: $W(\gamma_n) = n$ $\qquad \gamma_n : z \mapsto z^n$
- INJECTIVE:

  PROVED BY WHITNEY IN 1937

  GOAL: PRESENT A 3D-PROOF OF GEIGES-ELIASHBERG FROM 2007 USING CONTACT TOPOLOGY

STRATEGY OF PROOF

WANT TO SHOW: $W(\gamma_0) = W(\gamma_1) \Rightarrow \gamma_0$ REG. HOMOTOPIC TO γ_1

INSTEAD: WE WILL FIND STANDARD FORMS γ_n

SO THAT $W(\gamma) = n \Rightarrow \gamma$ REG. HOMOTOPIC TO γ_n

HOW DO WE LOOK FOR THE STANDARD FORM?

STRATEGY OF PROOF

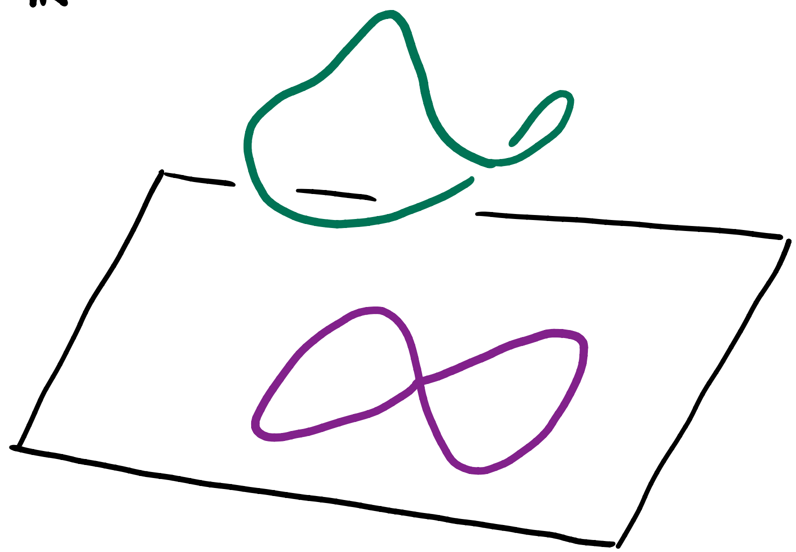
WANT TO SHOW: $W(\gamma_0) = W(\gamma_1) \Rightarrow \gamma_0$ REG. HOMOTOPIC TO γ_1

INSTEAD: WE WILL FIND STANDARD FORMS γ_n

SO THAT $W(\gamma) = i \Rightarrow \gamma$ REG. HOMOTOPIC TO γ_n

HOW DO WE LOOK FOR THE STANDARD FORM?

- WANT TO THINK OF γ AS A PROJECTION OF A KNOT IN $\mathbb{R}^3$



STRATEGY OF PROOF

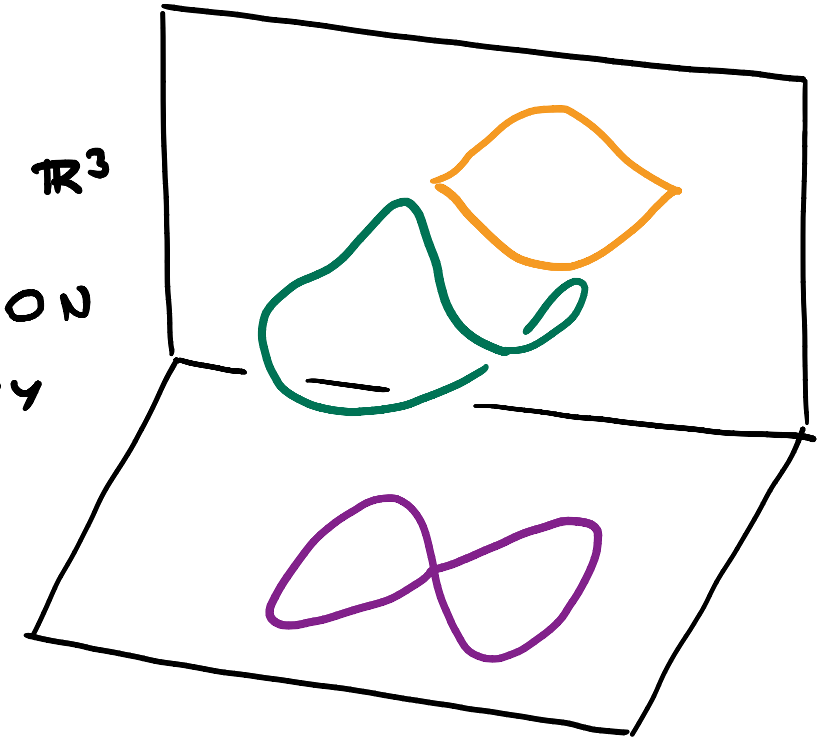
WANT TO SHOW: $W(\gamma_0) = W(\gamma_1) \Longrightarrow \gamma_0$ REG. HOMOTOPIC TO γ_1

INSTEAD: WE WILL FIND STANDARD FORMS γ_n

SO THAT $W(\gamma) = i \Rightarrow \gamma$ REG. HOMOTOPIC TO γ_n

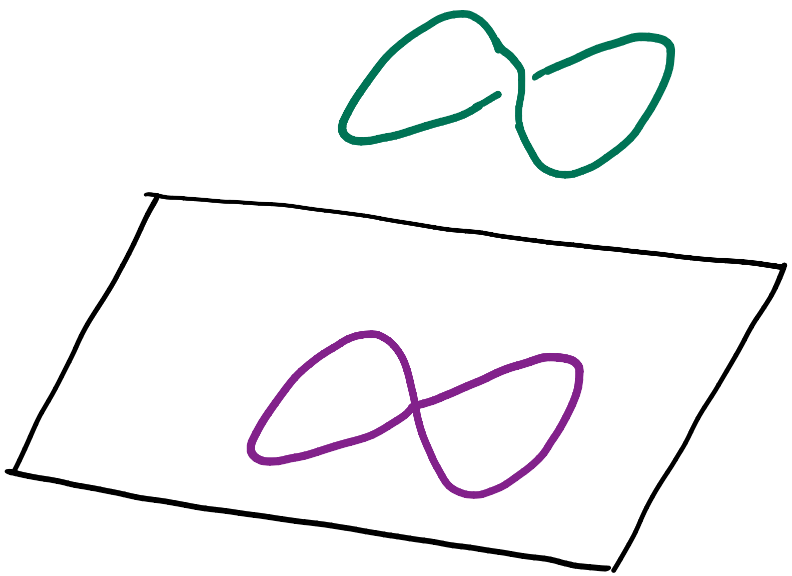
HOW DO WE LOOK FOR THE STANDARD FORM?

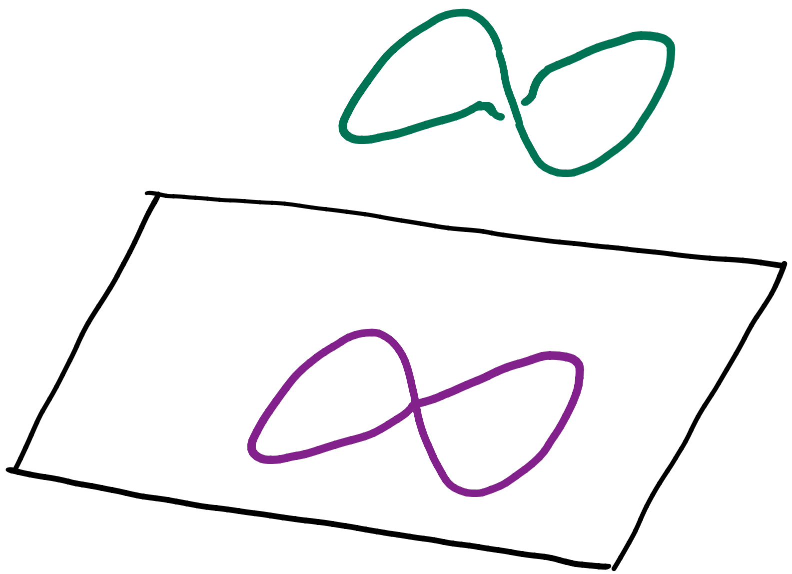
- WANT TO THINK OF γ AS A PROJECTION OF A KNOT IN $\mathbb{R}^3$
- THEN USE ANOTHER PROJECTION TO REDUCE REGULAR HOMOTOPY TO SOMETHING THAT IS COMBINATORIALLY CHECKABLE



LIFTING THE CURVES TO $\mathbb{R}^3$

- A PRIORI WE HAVE MULTIPLE WAYS OF LIFTING γ





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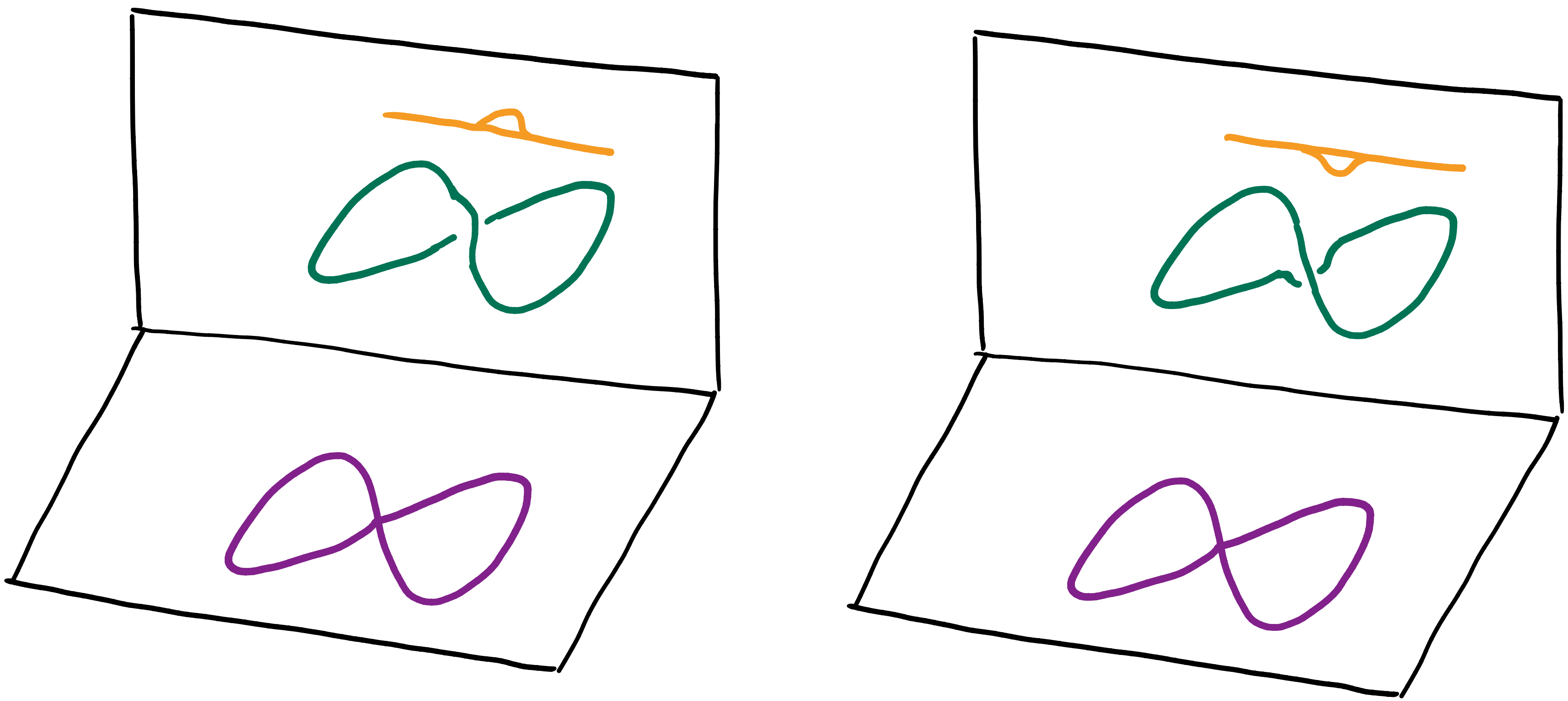


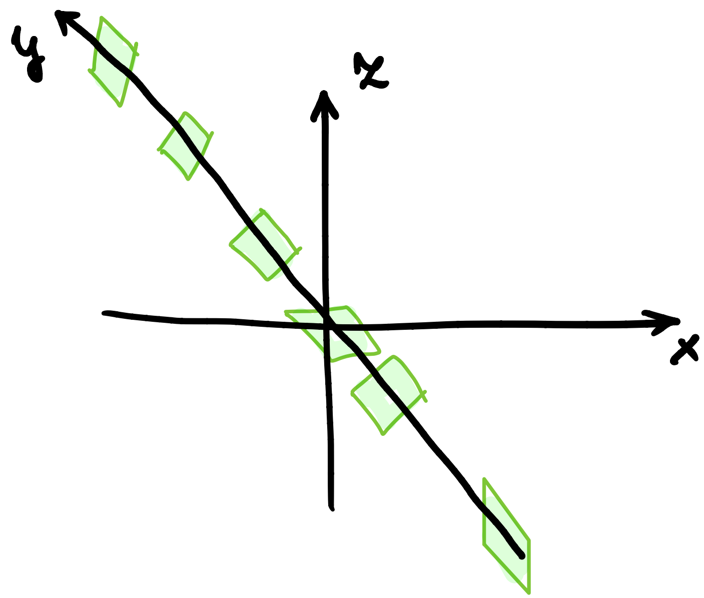
- BUT THEN THE PROJECTION IS BORING OR CAN BE ALMOST ANYTHING

$\Rightarrow$ WE HAVE TO PUT RESTRICTIONS ON THE LIFT TO MAKE IT UNIQUE

STANDARD CONTACT STRUCTURE ON $\mathbb{R}^3$

2-PLANE DISTRIBUTION (= 2-PLANE AT EVERY POINT)

$$\xi = \ker(dz - y\,dx) = \mathrm{span}\left\langle \frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\rangle$$



THE LIFTS WILL NEED TO BE TANGENT TO ξ

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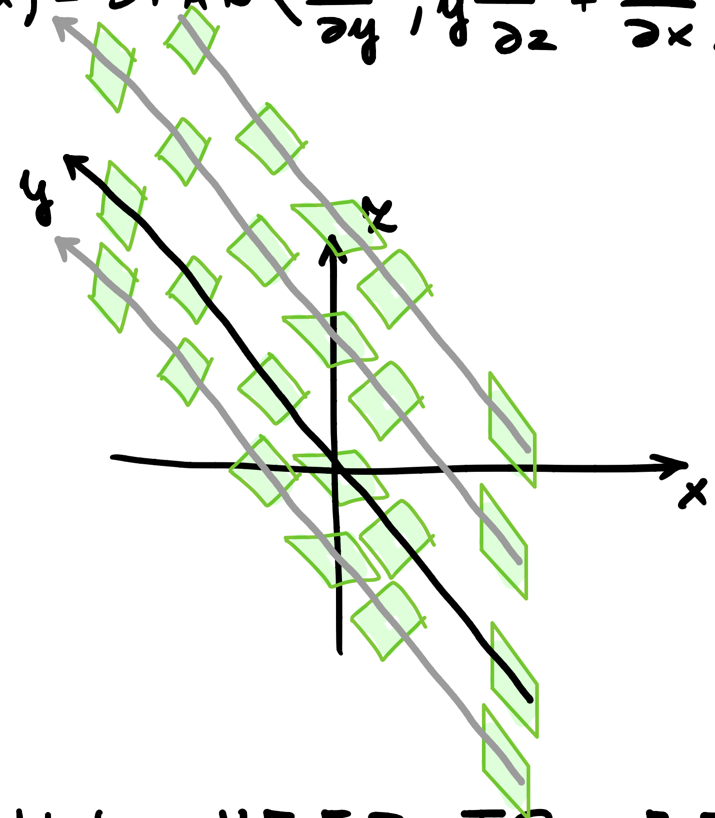

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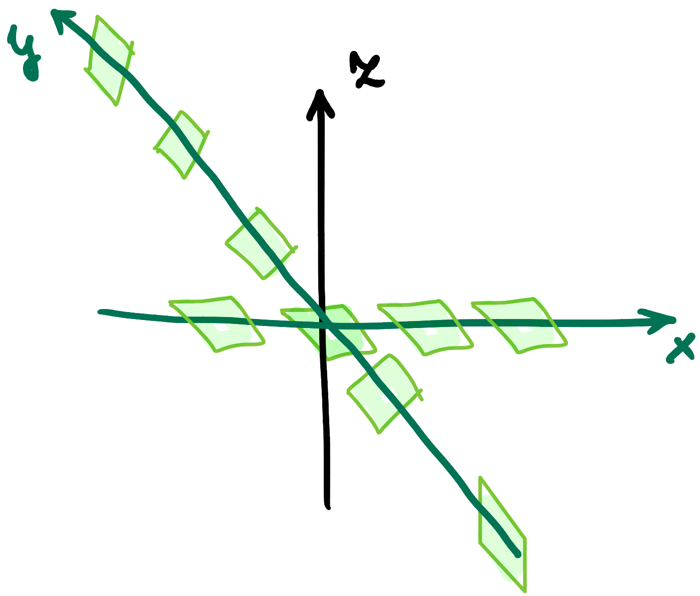
LEGENDRIAN KNOTS

$L: S^1 \hookrightarrow \mathbb{R}^3$ IS **LEGENDRIAN** IF $\dot{L} \subseteq \xi = \ker(dz - y\,dx)$

EXAMPLE: X-AXIS & Y-AXIS ARE BOTH LEGENDRIANS



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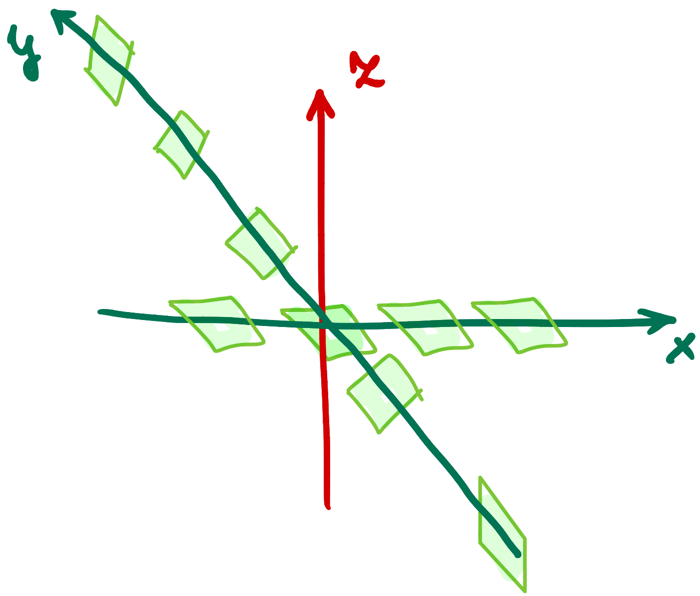

Z-AXIS IS NOT A LEGENDRIAN

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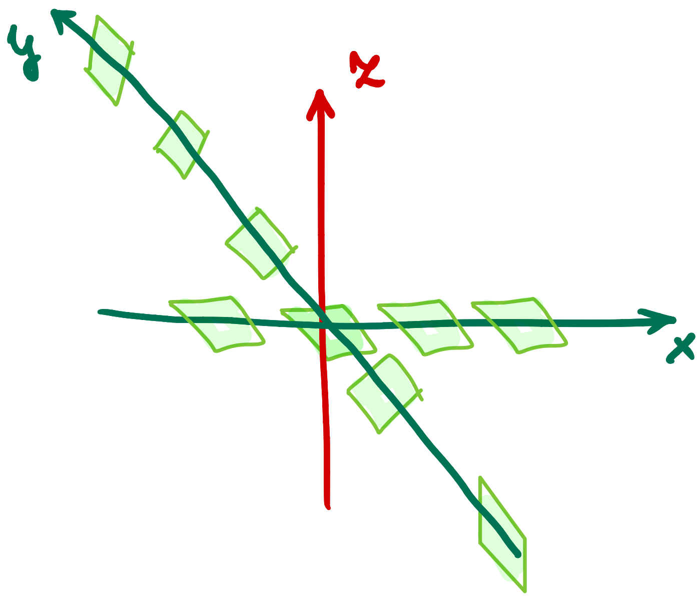


Z-AXIS IS NOT A LEGENDRIAN

$L(s) = (x(s), y(s), z(s))$ IS LEGENDRIAN $\Leftrightarrow \dot{L}(s) \in \xi_{L(s)}$

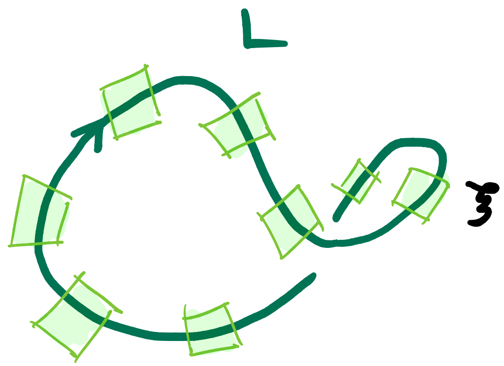
$\Leftrightarrow \boxed{\dot{z}(s) - y(s) \cdot \dot{x}(s) = 0}$

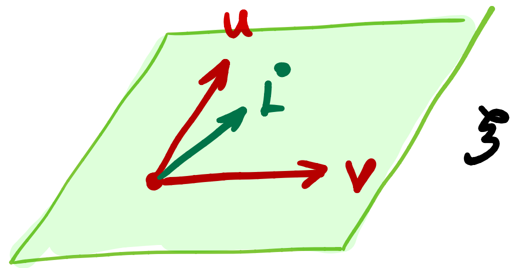
ROTATION NUMBER

L LEGENDRIAN KNOT

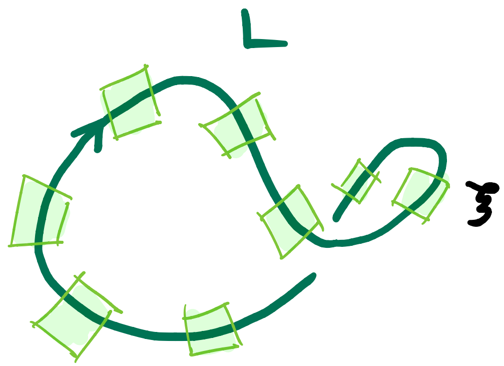
BASIS → $u = \frac{\partial}{\partial y}$, $v = y\frac{\partial}{\partial z} + \frac{\partial}{\partial x}$

$$\dot{L} : S^1 \longrightarrow \xi = \text{SPAN}\left\langle \frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\rangle$$





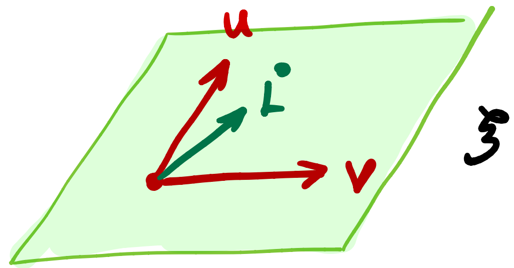
ROTATION NUMBER



L LEGENDRIAN KNOT

BASIS $\rightarrow$ u v

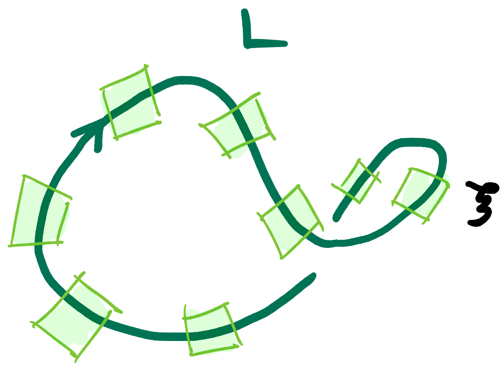
$$\dot{L} : S^1 \longrightarrow \xi = \mathrm{SPAN}\left\langle \overset{u}{\overset{\parallel}{\frac{\partial}{\partial y}}} , \overset{v}{\overset{\parallel}{y\frac{\partial}{\partial z} + \frac{\partial}{\partial x}}} \right\rangle$$



- WRITE $\dot{L}$ IN THE BASIS:

$$\dot{L} = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}$$

ROTATION NUMBER



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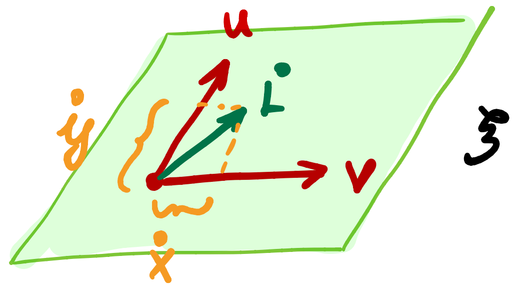

- WRITE $\dot{L}$ IN THE BASIS:

$$\dot{L} = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} \underset{\uparrow}{=} \dot{x}\left(y\frac{\partial}{\partial z} + \frac{\partial}{\partial x}\right) + \dot{y}\frac{\partial}{\partial y} = \dot{x}\,\underline{v} + \dot{y}\,\underline{u}$$

L LEGENDRIAN

SO THE COORDINATES OF $\dot{L}$ ARE $(\dot{x}, \dot{y})$

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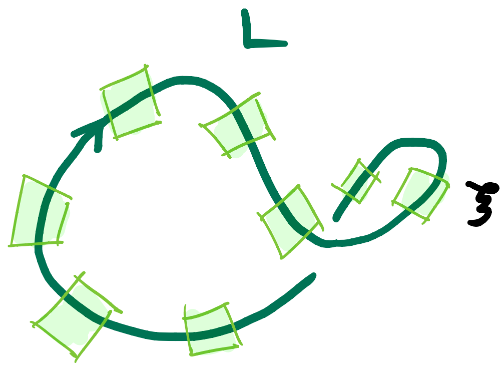

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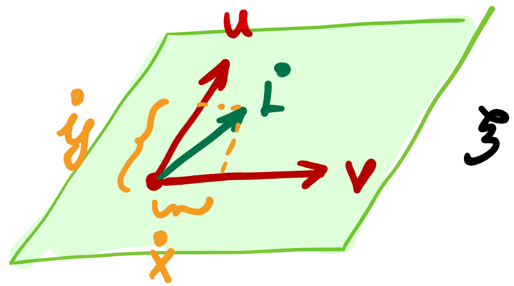

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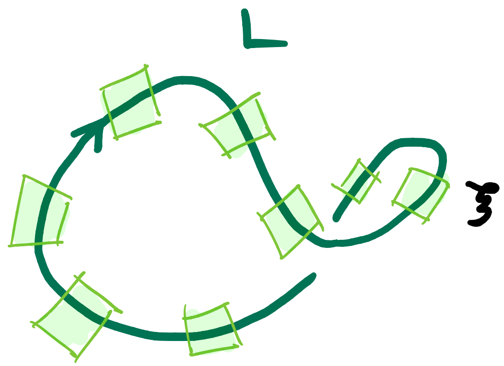
ROTATION NUMBER



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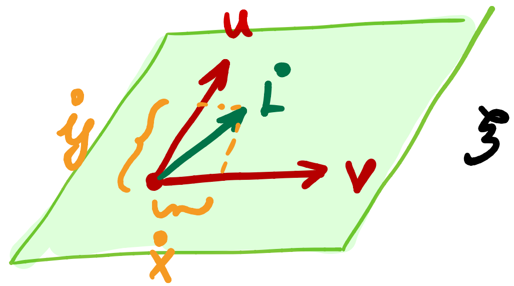

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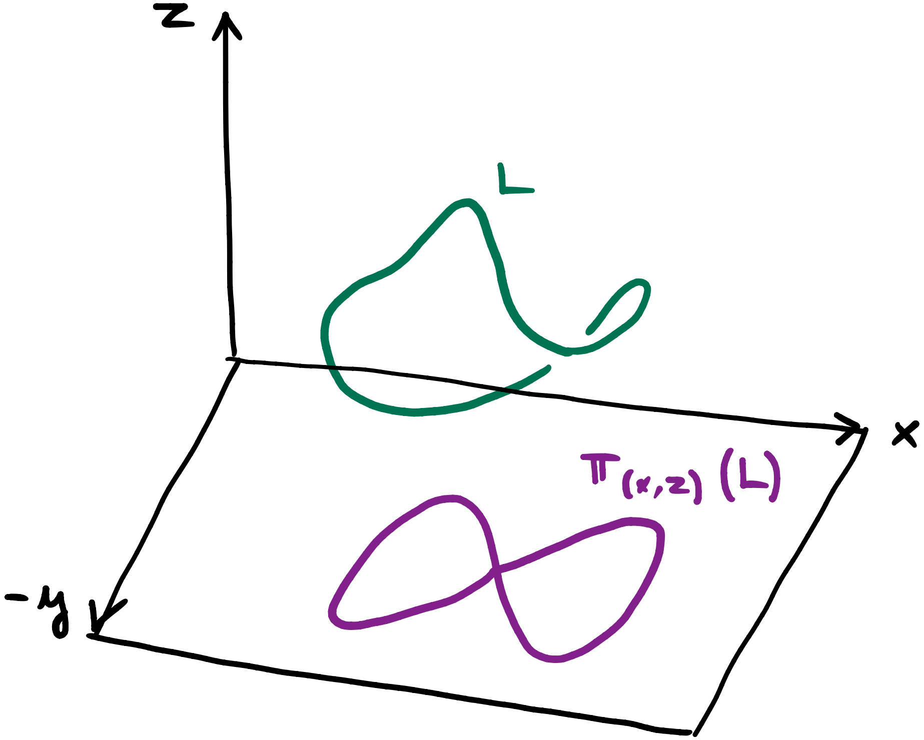
THIS IS THE WINDING # OF L'S PROJECTION TO THE (x, y)-PLANE

LAGRANGIAN PROJECTION

L LEGENDRIAN KNOT $\Leftrightarrow$ $\dot{z} - y\dot{x} = 0$



$$\pi_{(x,z)} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto (x,y)$$

$$\gamma(s) = \pi_{(x,y)}(L) = (x(s), y(s))$$

- **CLAIM:** $\gamma = \pi_{(x,y)}(L)$ IS AN IMMERSION

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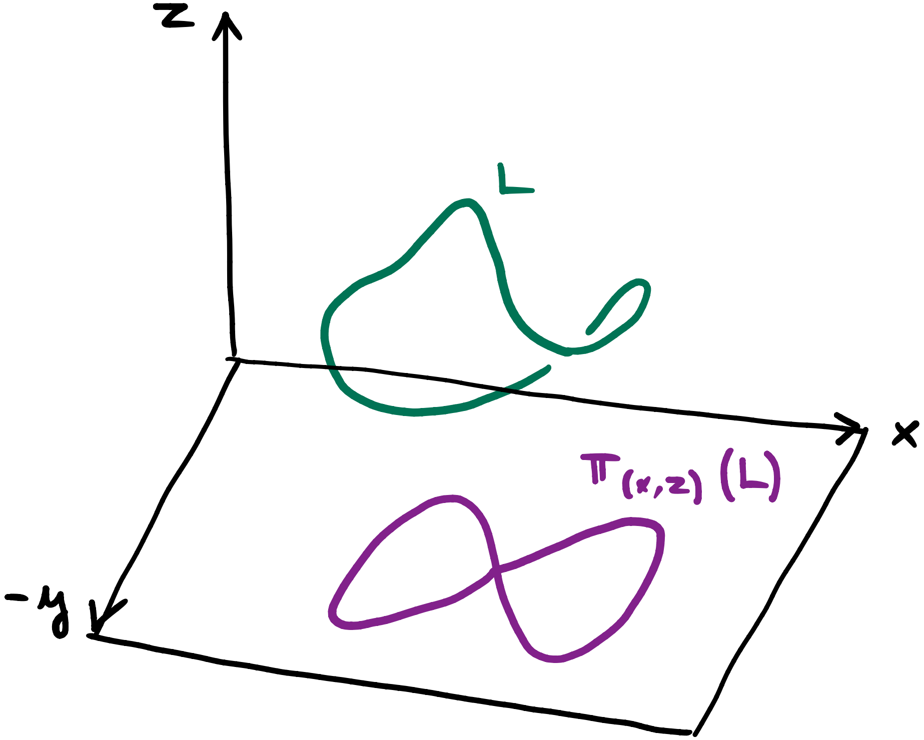
L LEGENDRIAN KNOT $\iff$ $\dot z - y\dot x = 0$



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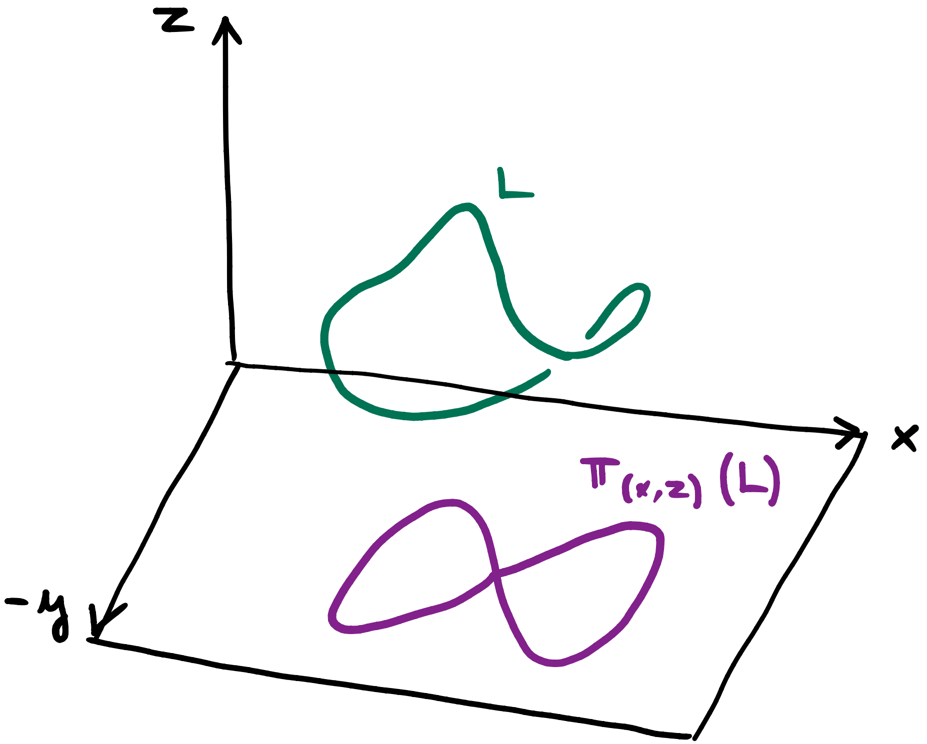

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BUT $\frac{\partial}{\partial z} \notin \xi$ $\quad \left(\xi = \left\langle \frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\rangle\right)$

SO $\dot\gamma(s) \neq \underline{0}$ ✓

LAGRANGIAN PROJECTION

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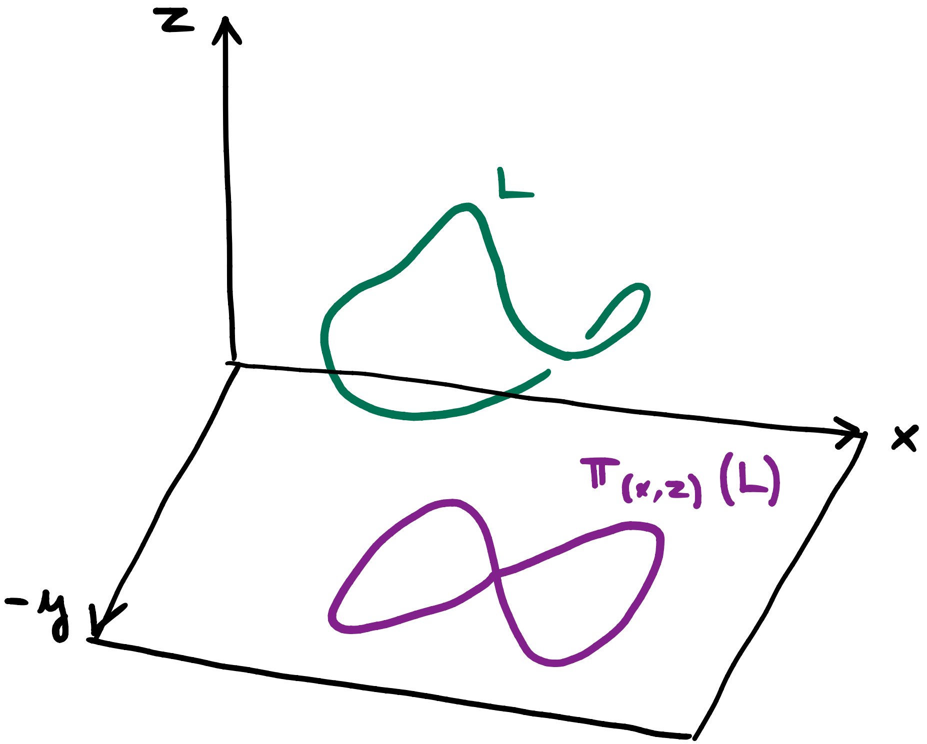

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- **CLAIM:** $\gamma = \pi_{(x,z)}(L)$ IS AN IMMERSION
- WE CAN RECOVER THE z-COORDINATE

$$\boxed{z(s_0) = z(0) + \int_0^{s_0} y(s)\,\dot{x}(s)\,ds} \Leftarrow \dot{z} = y \cdot \dot{x}$$

RECOGNISING LAGRANGIAN PROJECTIONS

$$\boxed{z(s_0) = z(0) + \int_0^{s_0} y(s)\,\dot{x}(s)\,ds}$$

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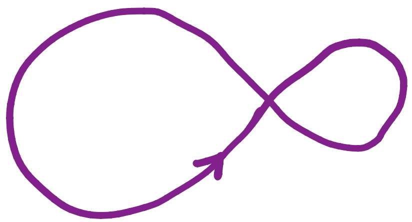

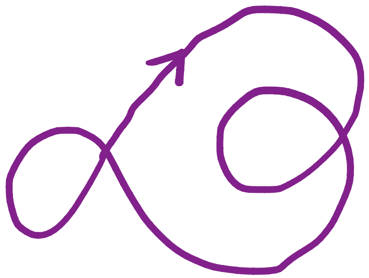

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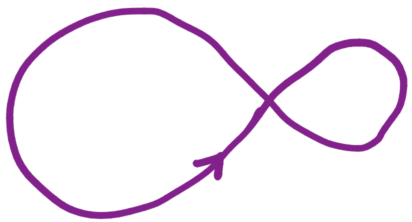

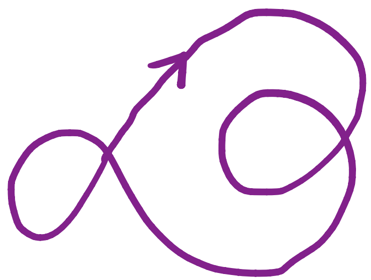

NEED: $z(0) = z(2\pi)$ I.E. $\oint y(s)\,\dot{x}(s)\,ds = 0$

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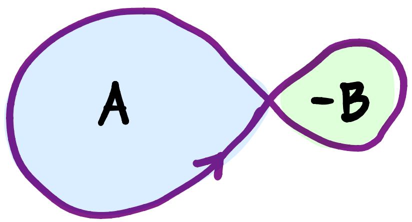

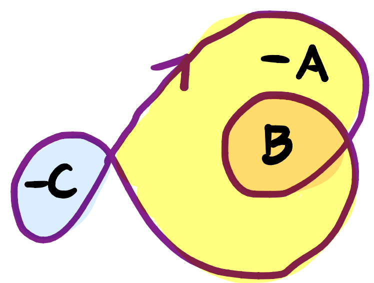

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SIGNED AREA

SO: WE'D NEED: $A - B = 0$ $\quad -A + B - C = 0$

RECOGNISING LAGRANGIAN PROJECTIONS

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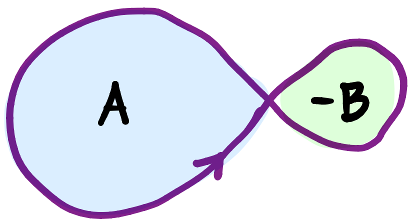

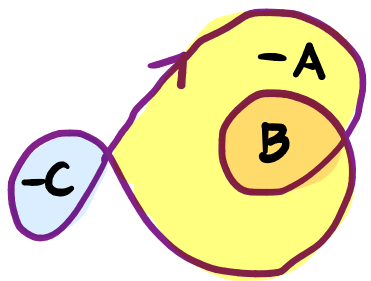

NEED: $z(0) = z(2\pi)$ I.E. $\oint y(s)\,\dot{x}(s)\,ds = 0$

SIGNED AREA

SO: WE'D NEED: $A - B \neq 0$ $\quad -A + B - C \neq 0$

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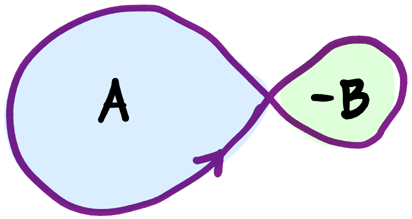

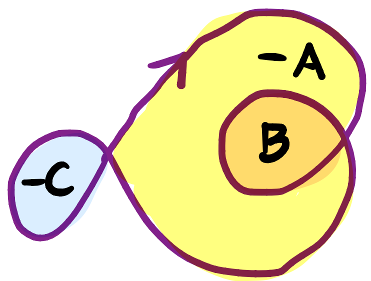

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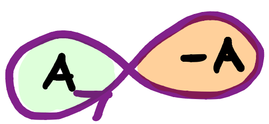
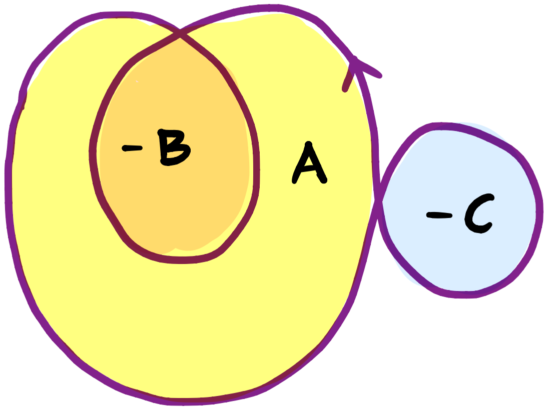
THIS WOULD LIFT:

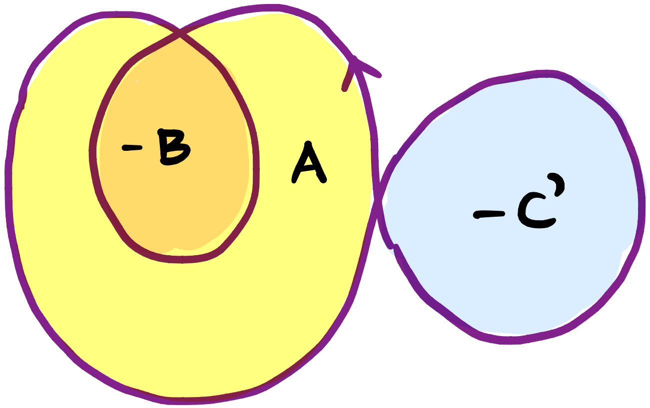


LAGRANGIAN PROJECTIONS VS. REGULAR PLANECURVES

- $\gamma : S^1 \to \mathbb{R}^2$ REGULAR PLANE CURVE
- BY REGULAR HOMOTOPY WE CAN ARRANGE TO HAVE $\oint y(s)\,\dot{x}(s)\,ds = 0$



INCREASE C TO HAVE $\rightsquigarrow$ $A - B - C' = 0$



$\Rightarrow \gamma = \Pi_{(x,y)}(L)$ FOR SOME LEGENDRIAN

LAGRANGIAN PROJECTIONS VS. REGULAR PLANE CURVES

- $\gamma : S^1 \to \mathbb{R}^2$ REGULAR PLANE CURVE
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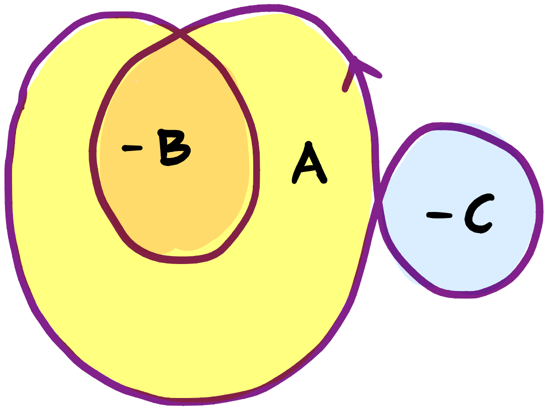

INCREASE C TO HAVE

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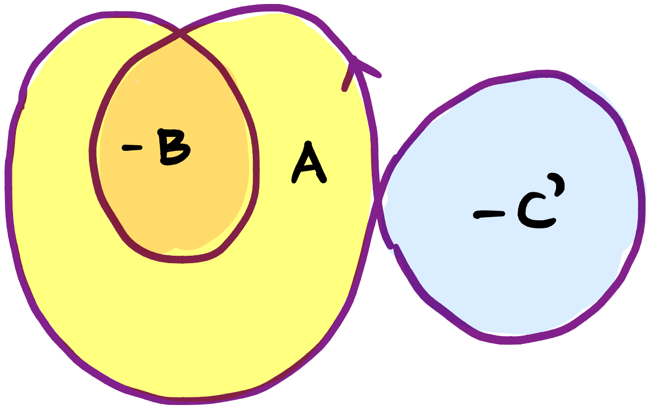
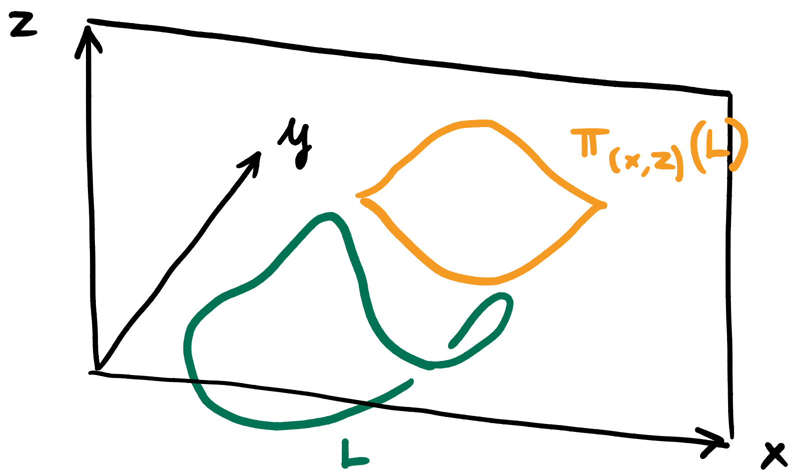


$\Rightarrow \gamma = \Pi_{(x,y)}(L)$ FOR SOME LEGENDRIAN

RECALL : $\boxed{W(\gamma) = \mathrm{ROT}(L)}$

FRONT PROJECTION

L LEGENDRIAN KNOT $\Leftrightarrow$ $\dot{z} - y\dot{x} = 0$



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$$(x,y,z) \longmapsto (x,z)$$

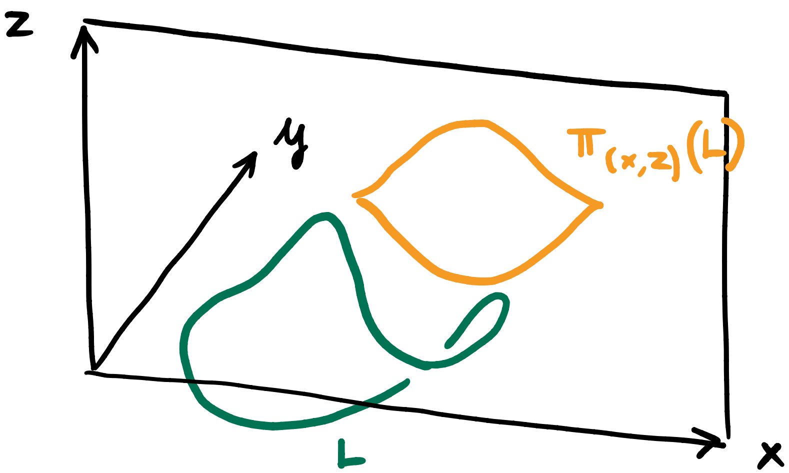
$$F(s) = \pi_{(x,z)}(L) = (x(s), z(s))$$

WE CAN RECOVER THE y-COORDINATE

$$\boxed{y = \frac{\partial z}{\partial x}} \quad \leftarrow \text{ SLOPE OF } \pi_{(x,z)}(L)$$

FRONT PROJECTION

L LEGENDRIAN KNOT $\Leftrightarrow$ $\dot{z} - y\dot{x} = 0$



$$\pi_{(x,z)} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto (x,z)$$

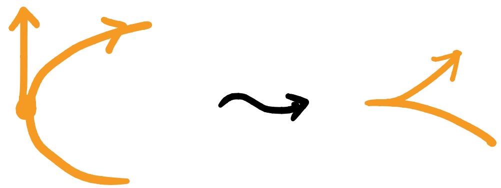
$$F(s) = \pi_{(x,z)}(L) = (x(s), z(s))$$

WE CAN RECOVER THE y-COORDINATE

$$\boxed{y = \frac{\partial z}{\partial x}} \leftarrow \text{SLOPE OF } \pi_{(x,z)}(L)$$

$\Rightarrow$ FRONT PROJECTIONS

- HAVE NO VERTICAL TANGENCIES



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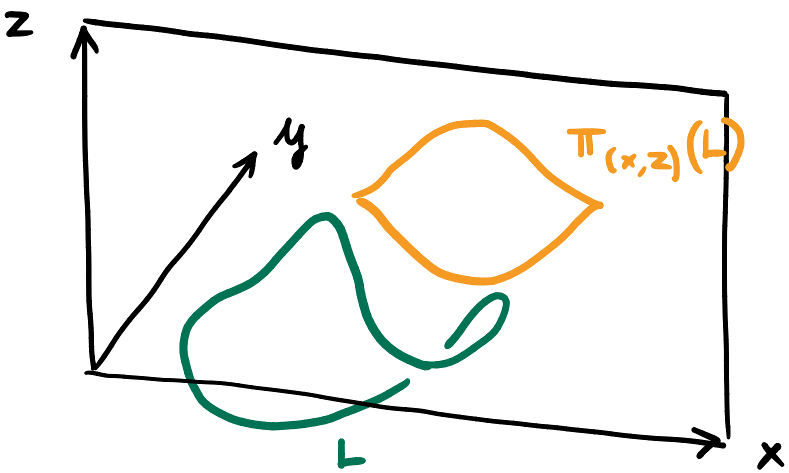

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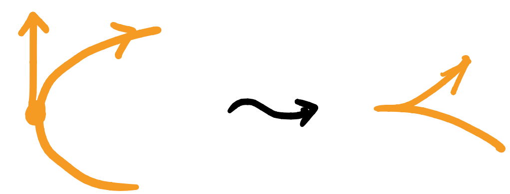
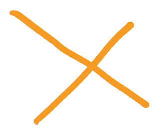
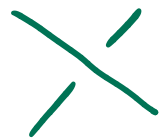
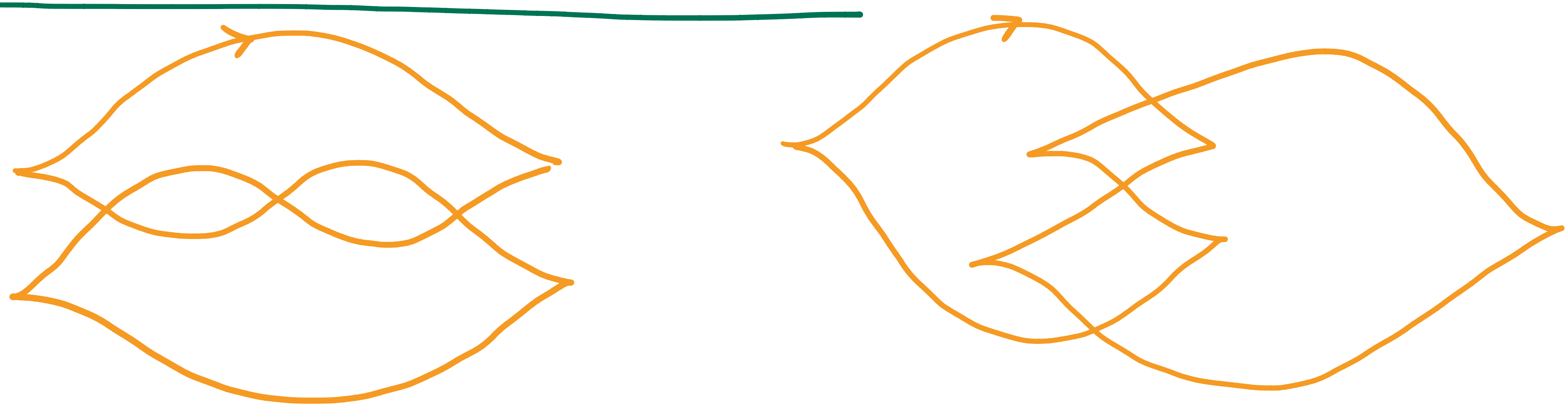
- HAVE NO VERTICAL TANGENCIES 

- INTERSECTIONS OF $\pi_{(x,z)}(L)$:  ON L : 

SOME FRONT PROJECTIONS



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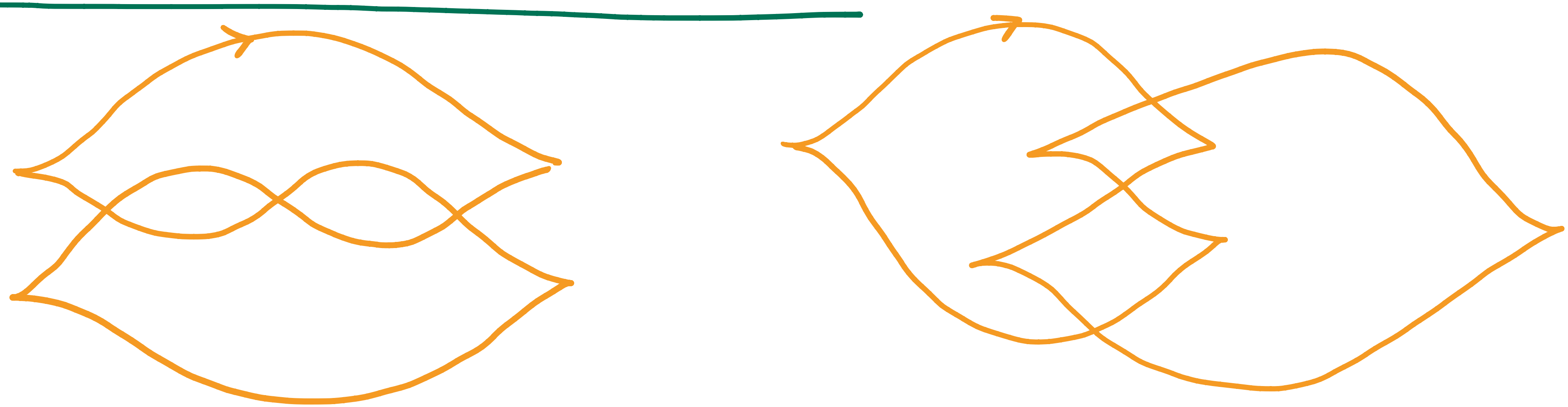
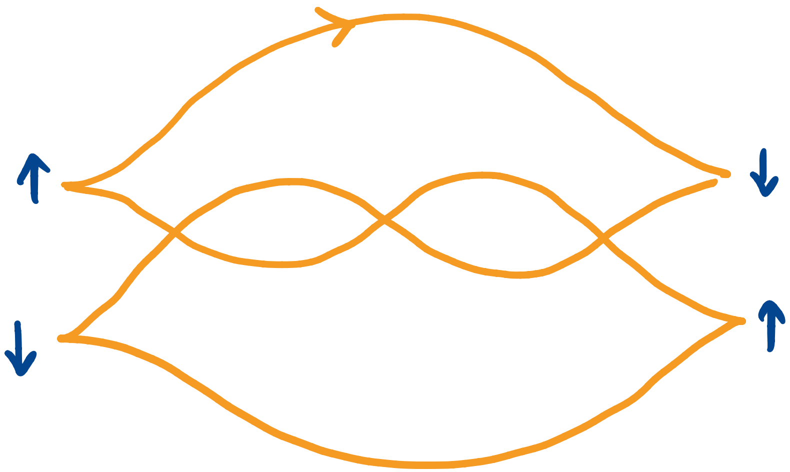


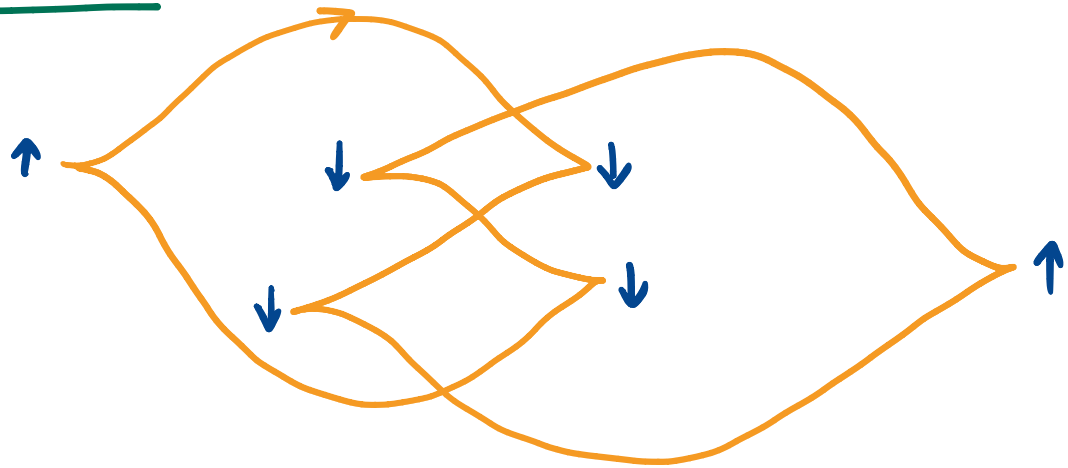
- ANY CUSPED CURVE CAN BE UNIQUELY LIFTED BY THE RULE $\boxed{y = \frac{\partial z}{\partial x}}$
- AND ROT(L) CAN BE COMBINATORIALLY COMPUTED

CLAIM: $\boxed{ROT(L) = \frac{1}{2}\left(\#\downarrow - \#\uparrow\right)}$

Some Front Projections



$$\text{ROT} = \tfrac{1}{2}(2-2) = 0$$



$$\text{ROT} = \tfrac{1}{2}(4-2) = 1$$

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ROTATION NUMBER IN THE FRONT PROJECTION

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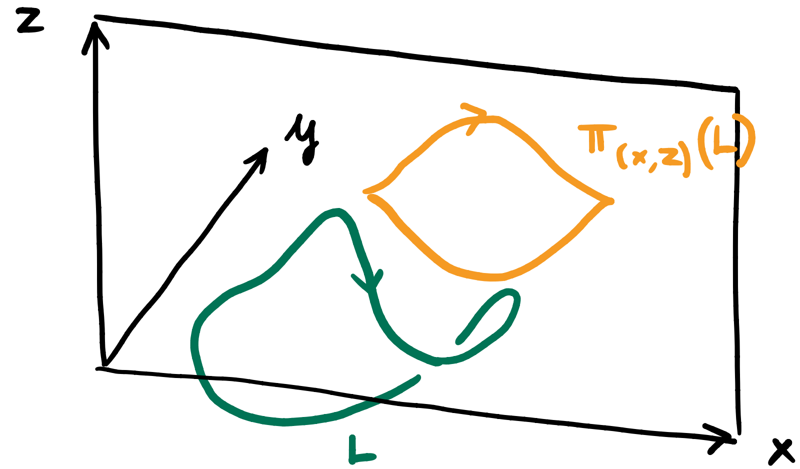
IDEA OF PROOF:

$$\dot{L} = \dot{y}\frac{\partial}{\partial y} + \dot{x}\left(y\frac{\partial}{\partial z} + \frac{\partial}{\partial x}\right) \quad \&$$

$$\mathrm{ROT}(L) \overset{def}{=} \mathrm{DEG}\left(\frac{(\dot{x},\dot{y})}{|(\dot{x},\dot{y})|}\right)$$



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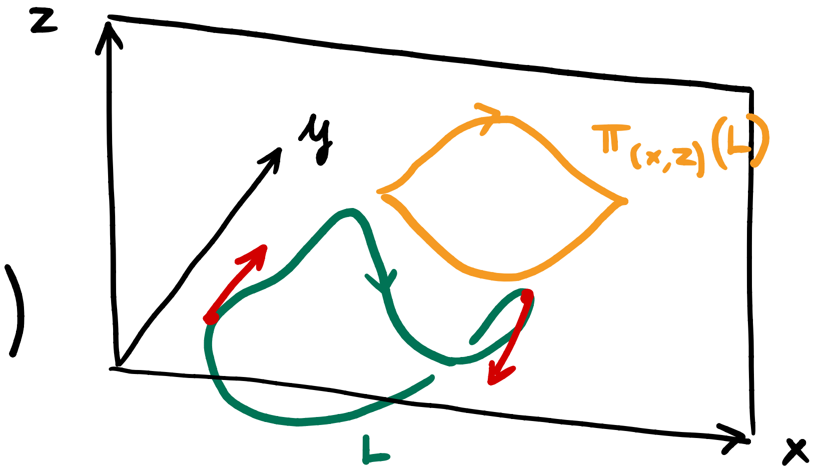

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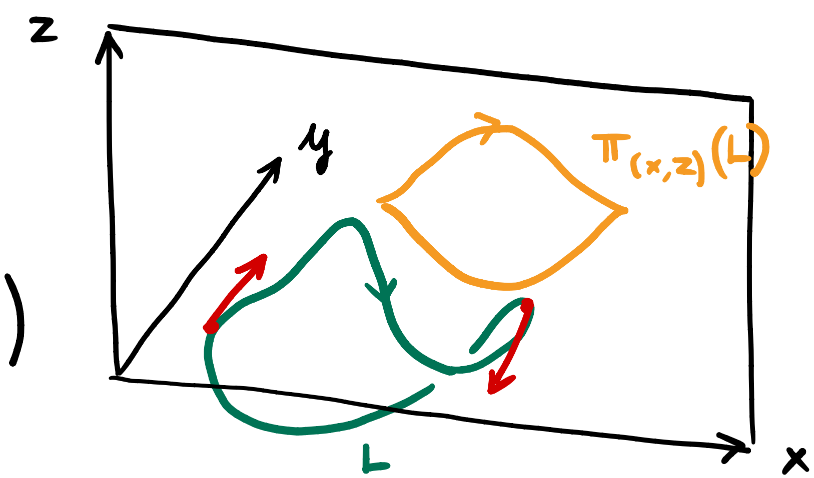

$$\dot{L} \parallel \frac{\partial}{\partial y} \iff \pi_{(x,z)}(L) \text{ HAS A CUSP}$$

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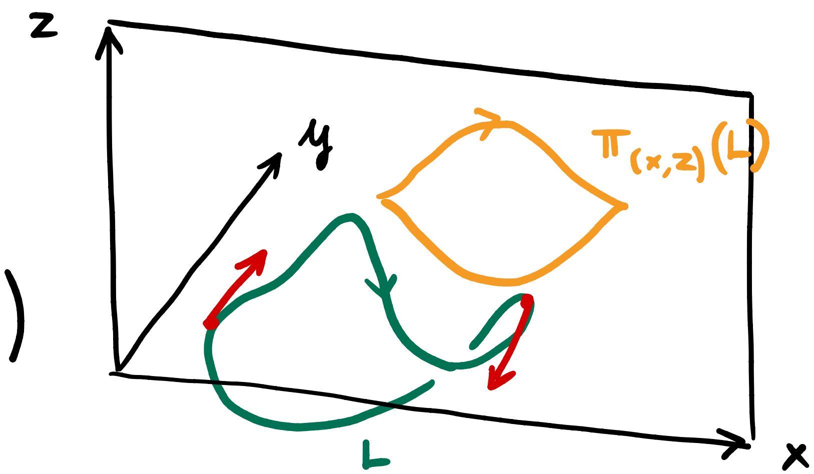

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$$\dot{L} \parallel \frac{\partial}{\partial y} \iff \pi_{(x,z)}(L) \text{ HAS A CUSP}$$

& THE SIGNS ARE  -1

 $+1$ ✓

$\square$

MOVES ON THE FRONT PROJECTION

- ANY CUSPED CURVE CAN BE UNIQUELY LIFTED
- SO A MOVIE (= 1-PARAMETER FAMILY) $\{F_t\}$ OF THESE CUSPED CURVES LIFTS UP TO A 1-PARAMETER FAMILY OF (IMMERSED) LEGENDRIANS $\{L_t\}$

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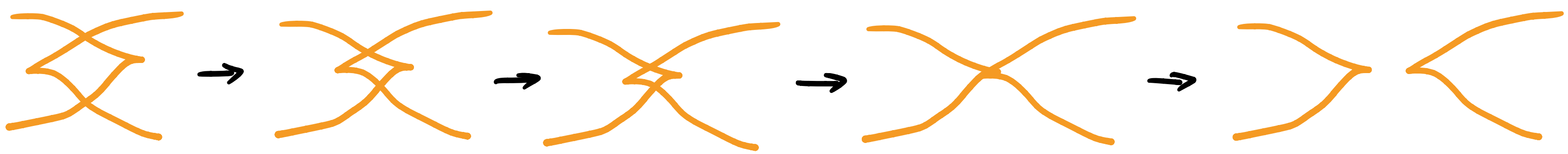

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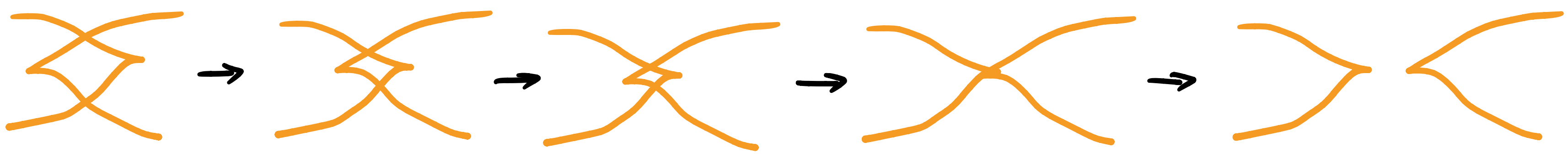
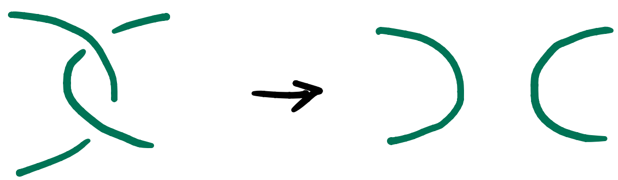


IN THE LIFT



NOT AN ISOTOPY

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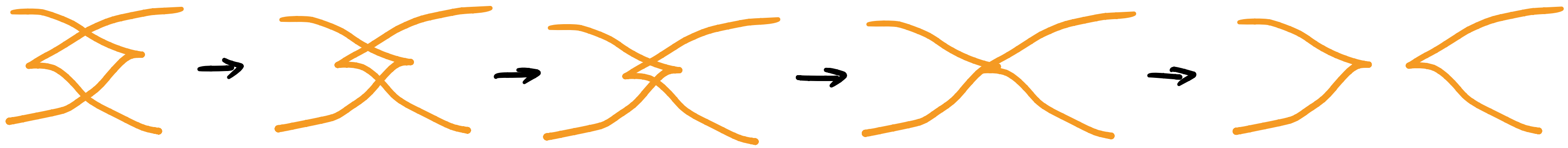

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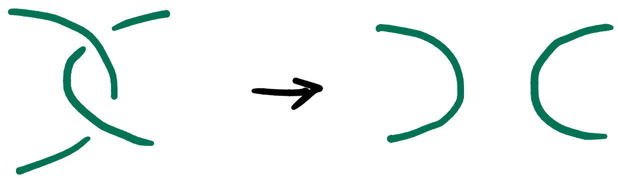
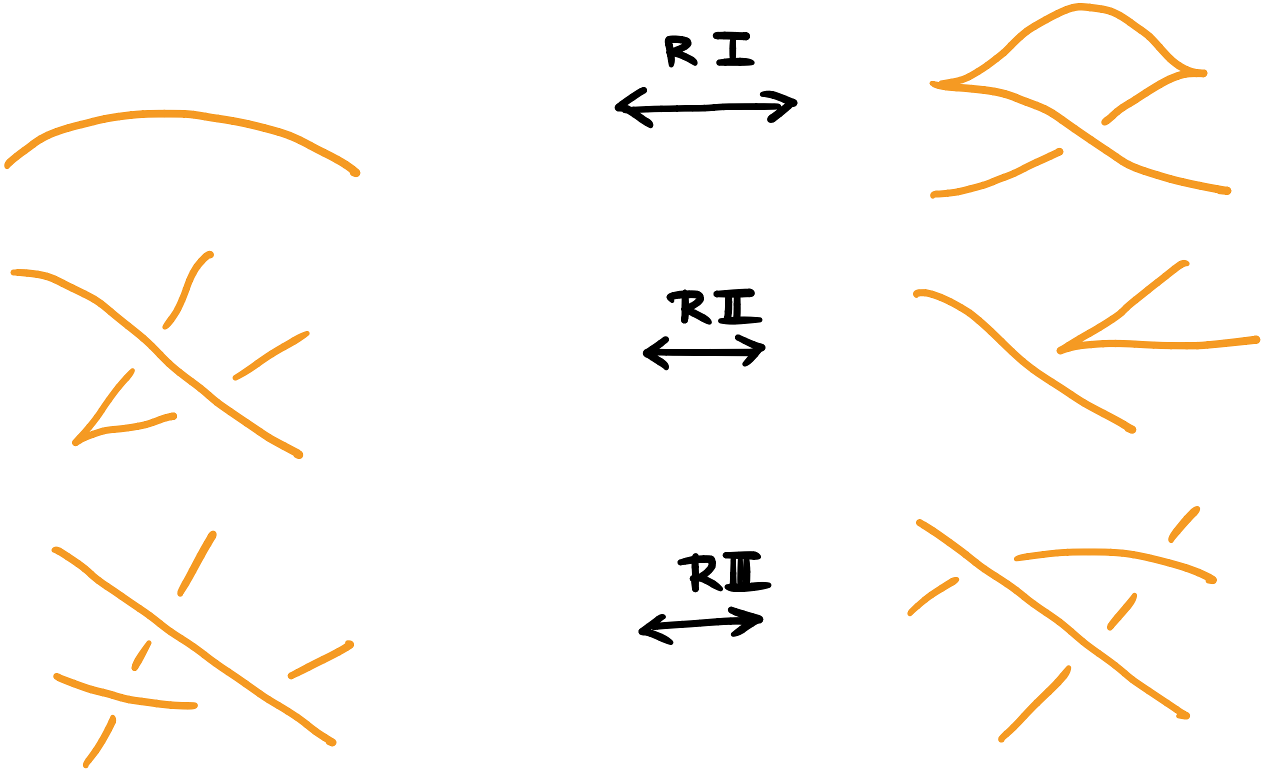


NOT AN ISOTOPY

- $ROT(L) = \frac{1}{2}(\#\downarrow - \#\uparrow)$ DOES NOT CHANGE DURING THESE MOVES

LEGENDRIAN REIDEMEISTER MOVES

THE FOLLOWING LOCAL MOVES ARE COMING FROM PROJECTIONS OF A 1-PARAMETER FAMILY OF LEGENDRIAN KNOTS:



THEOREM: TWO FRONT PROJECTIONS CORRESPOND TO LEGENDRIAN ISOTOPIC LEGENDRIAN KNOTS $\Leftrightarrow$ THEY ARE RELATED BY A SEQUENCE OF MOVES RI–RIII.

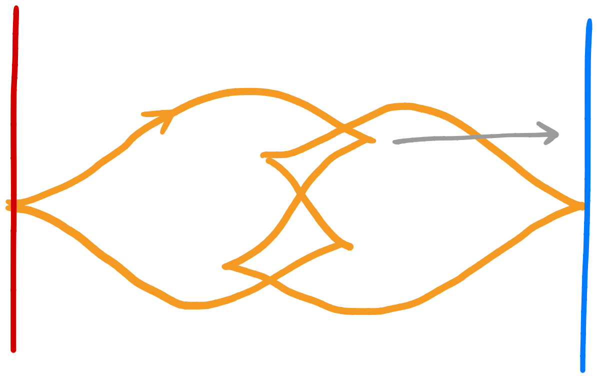
NORMALISING FRONT PROJECTIONS

GIVEN ANY FRONT PROJECTION

NORMALISING FRONT PROJECTIONS

GIVEN ANY FRONT PROJECTION

1) MOVE ALL RIGHT CUSPS TO THE RIGHT &
LEFT CUSPS TO THE LEFT



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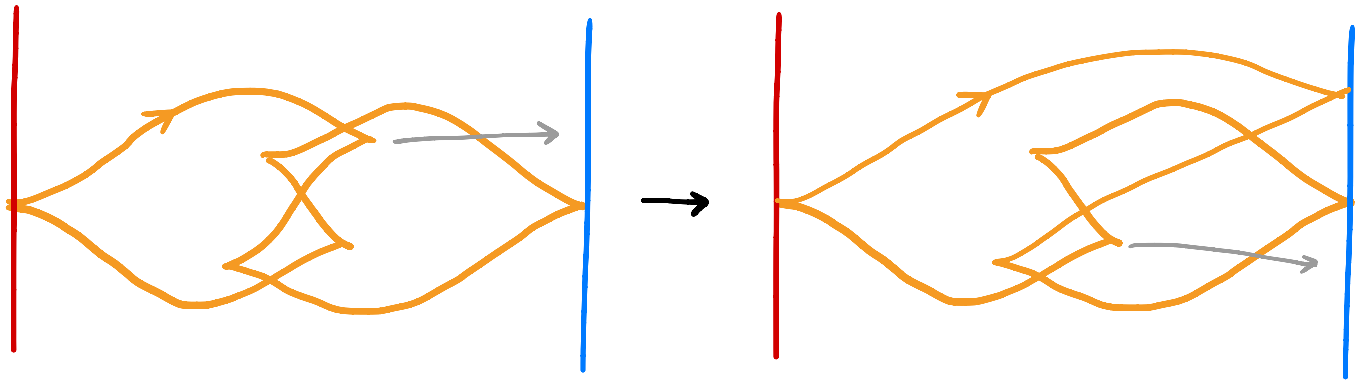

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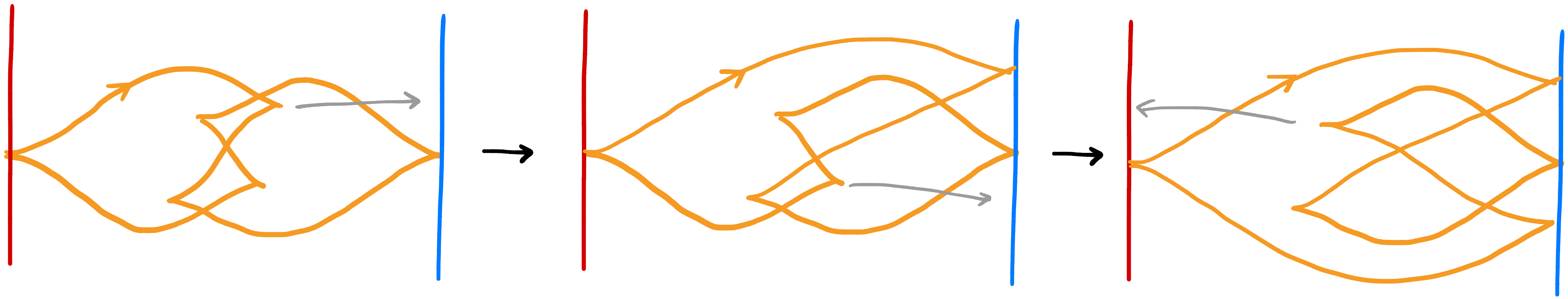

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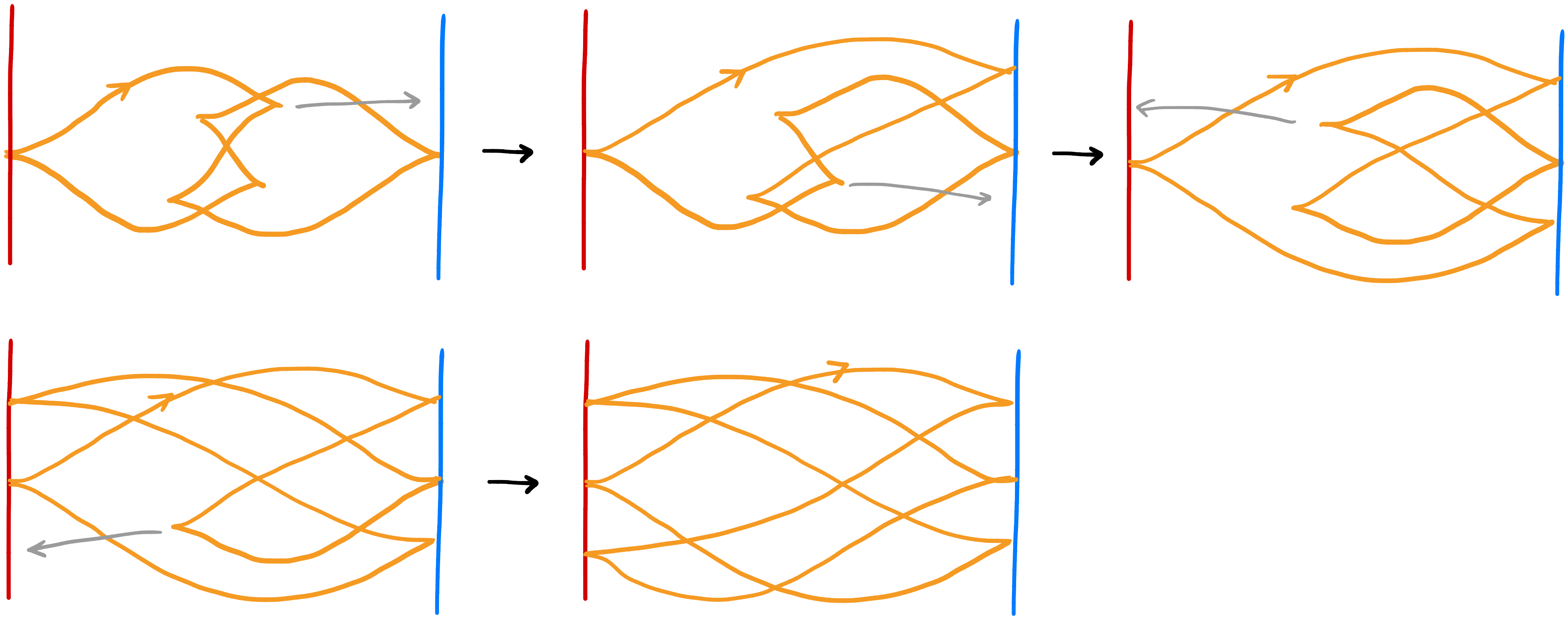
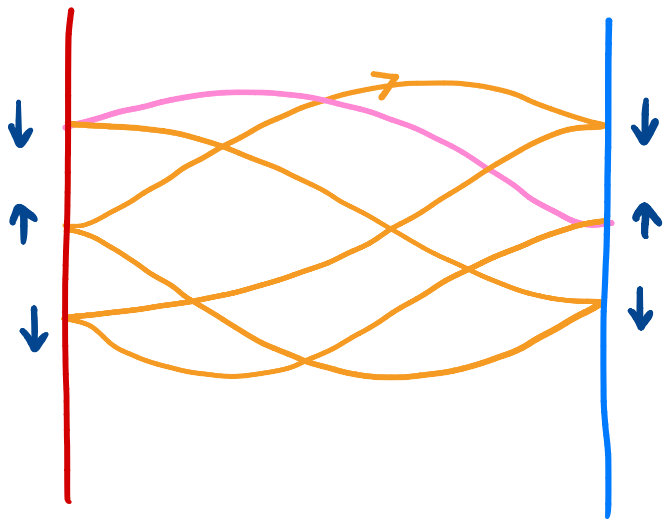


NORMALISING FRONT PROJECTIONS

LABEL UPWARDS/DOWNWARDS CUSPS W/ ↑ OR ↓

2) USE <u>RI</u> TO REMOVE CONSECUTIVE ↑↓ OR ↓↑



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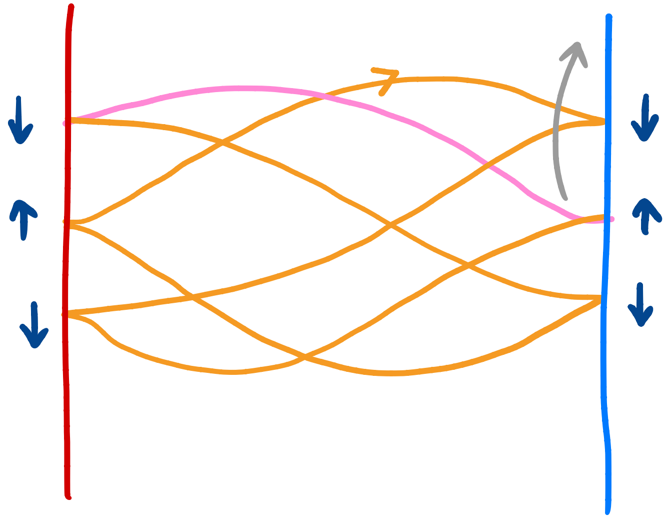

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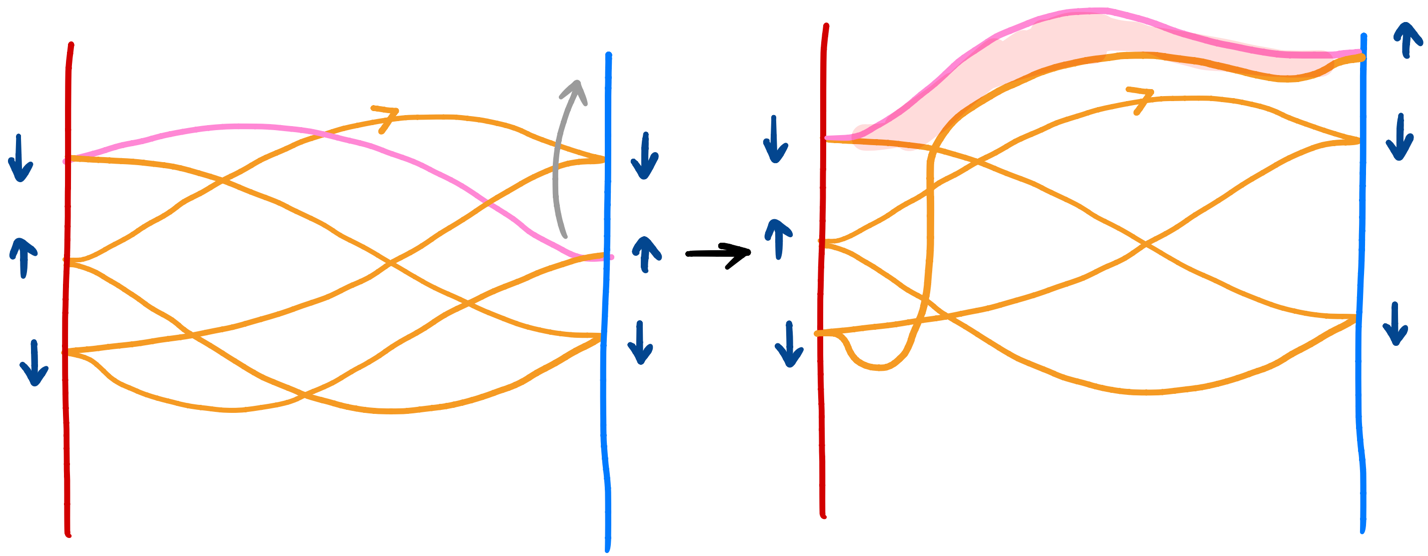

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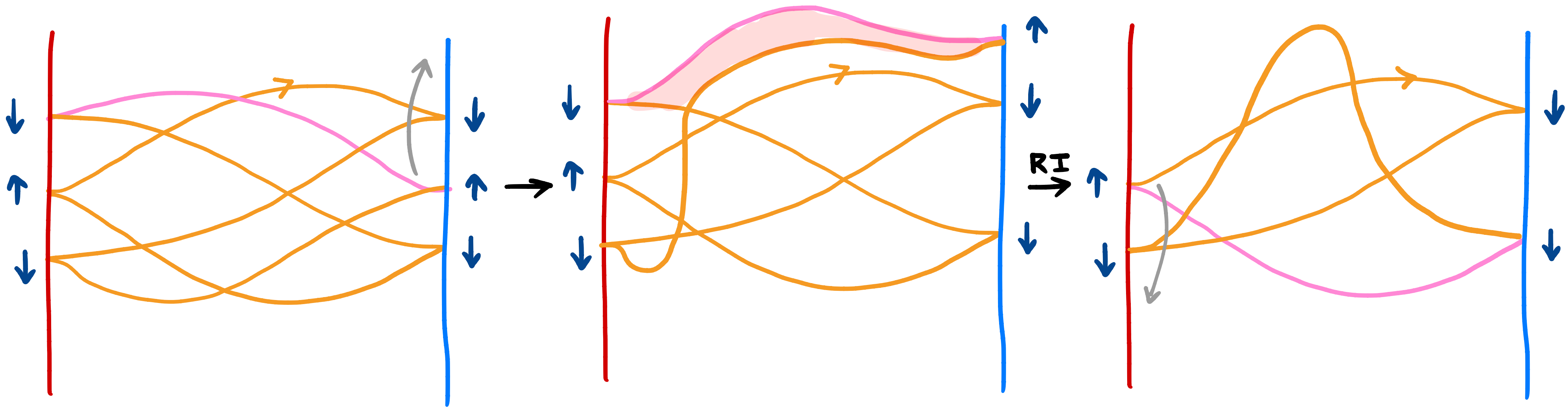

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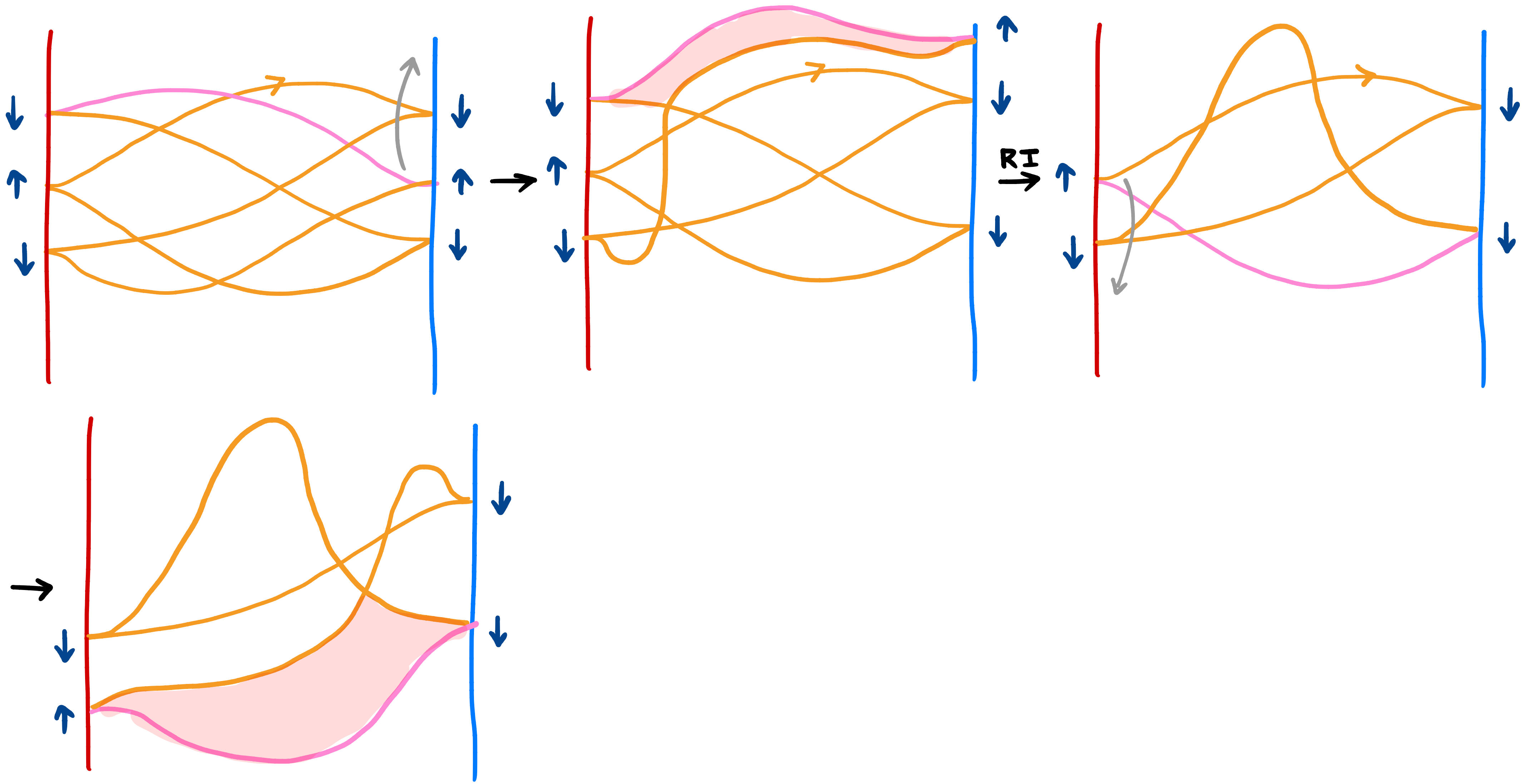

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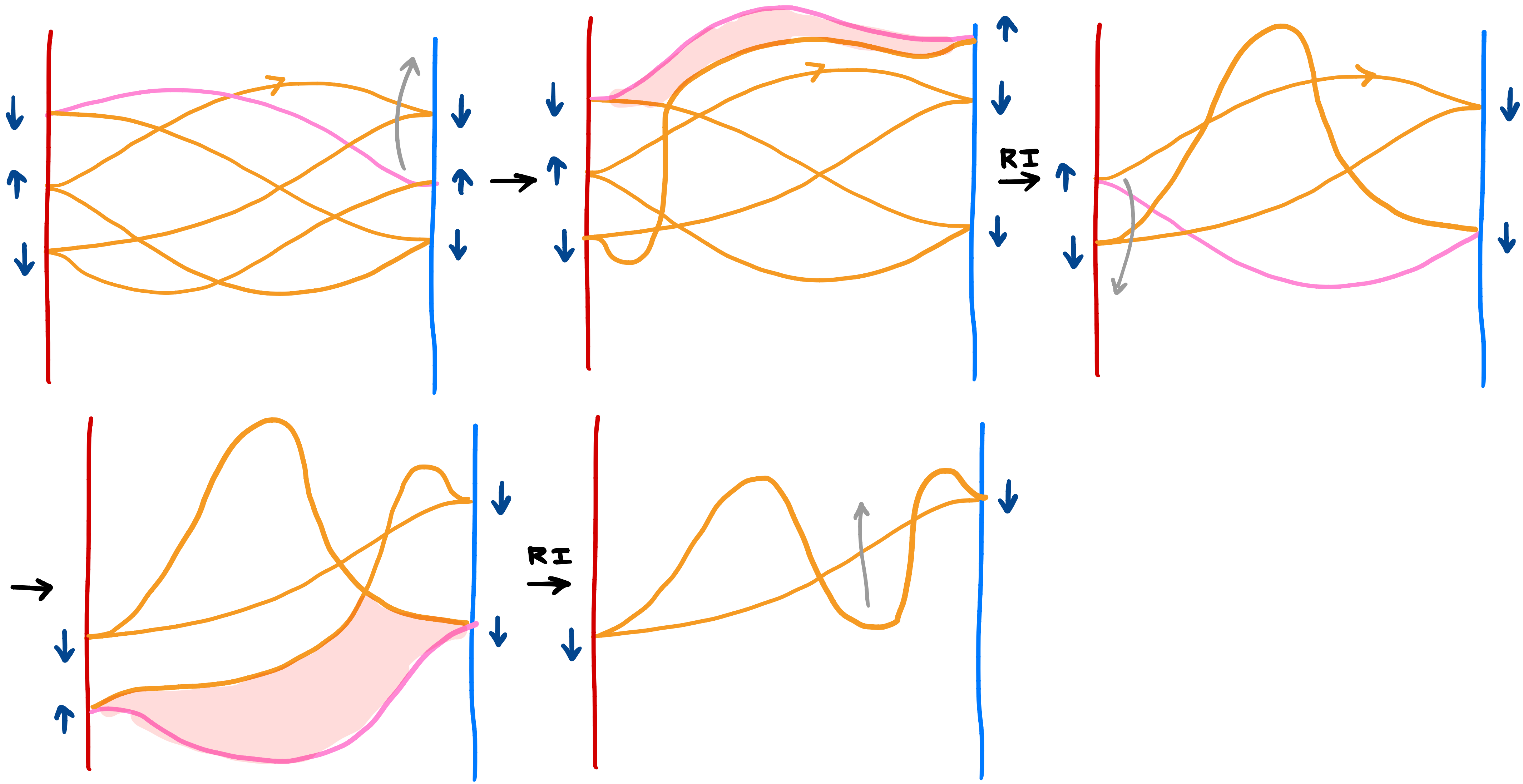

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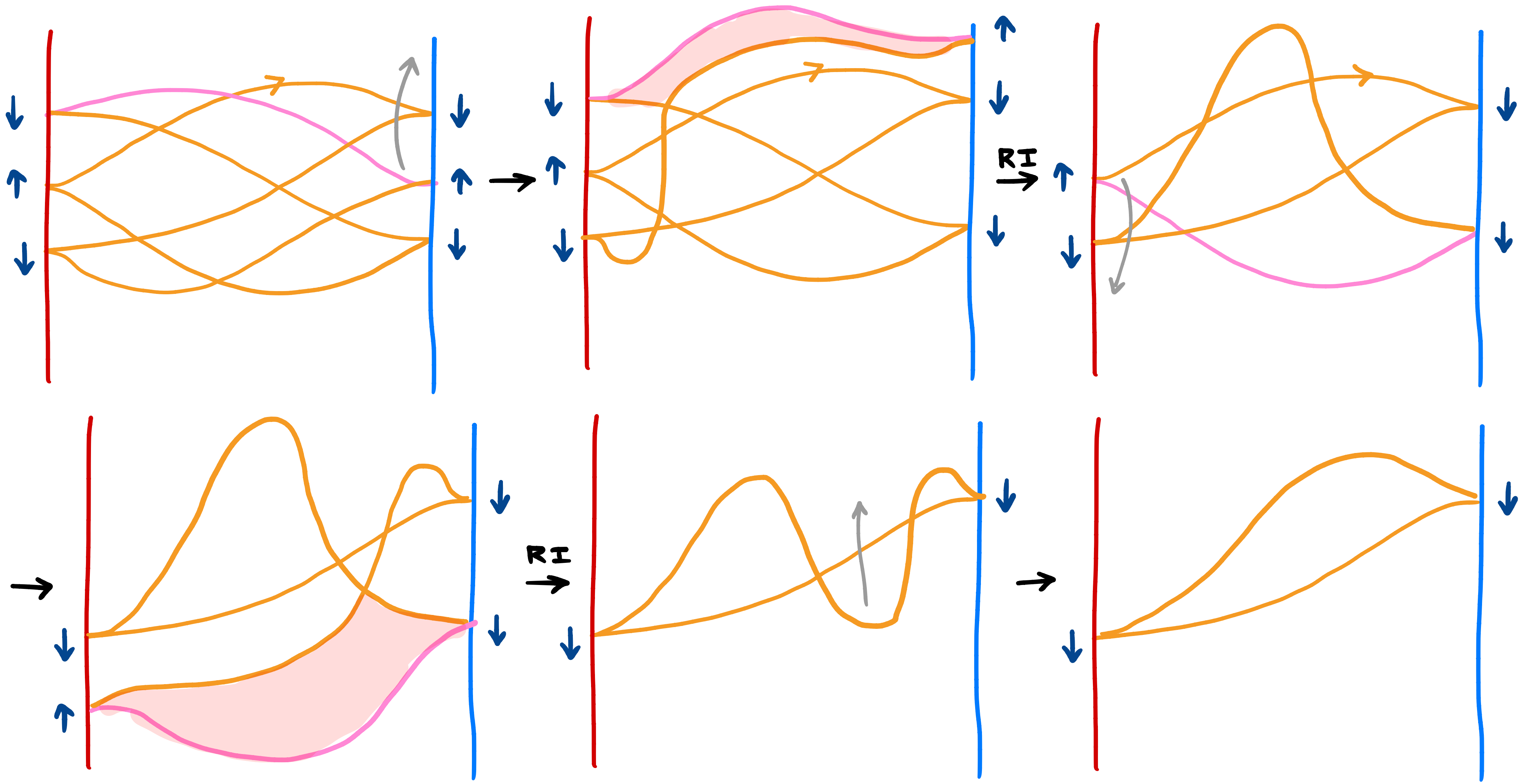

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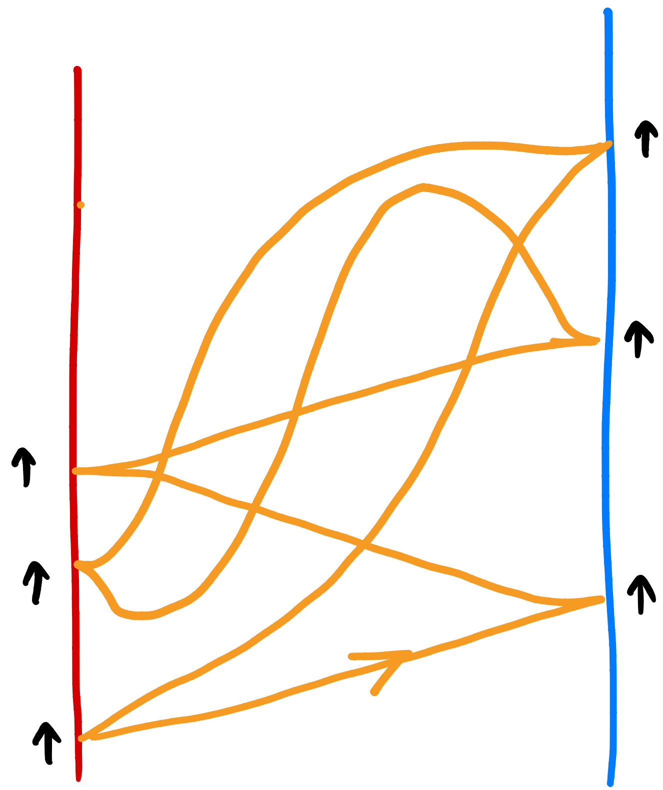

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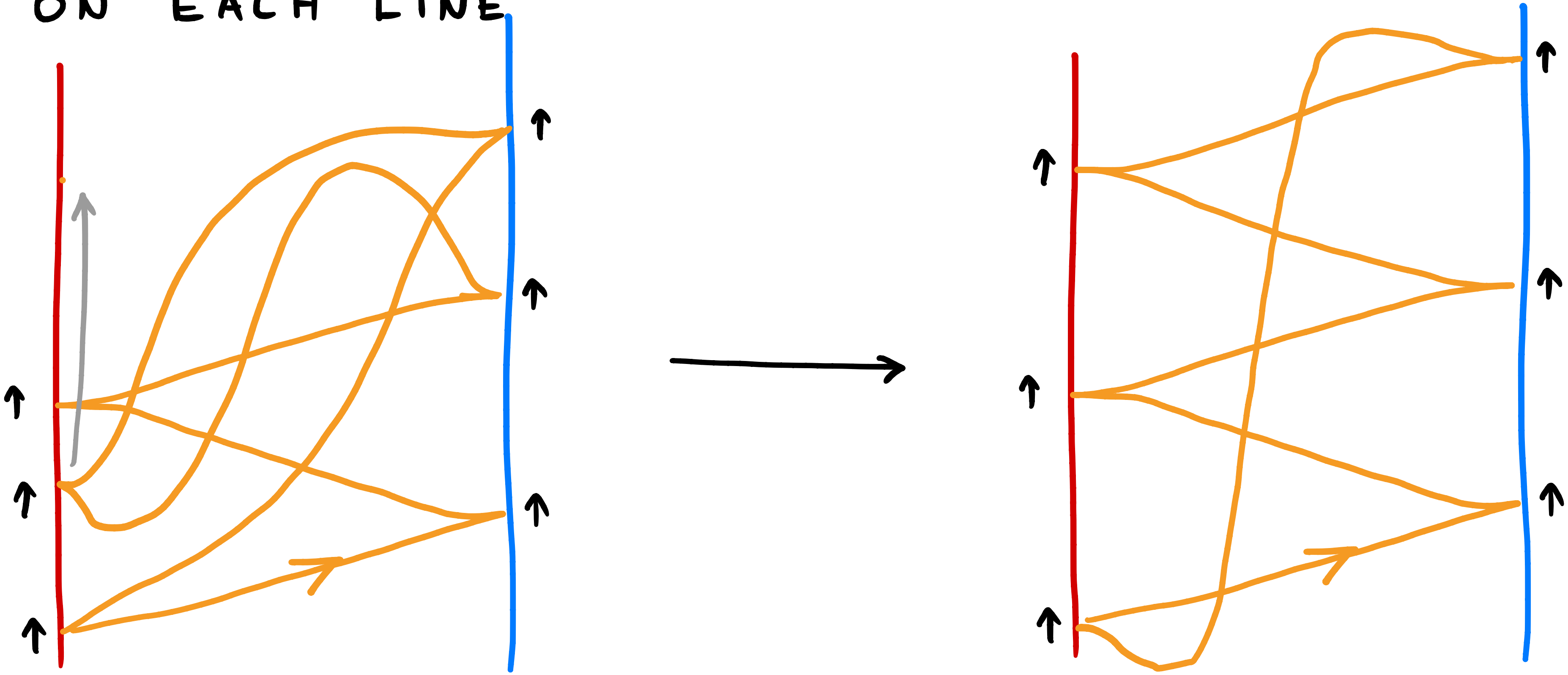
3) REARRANGE THE CUSPS SO THAT THEY INCREASE ON EACH LINE



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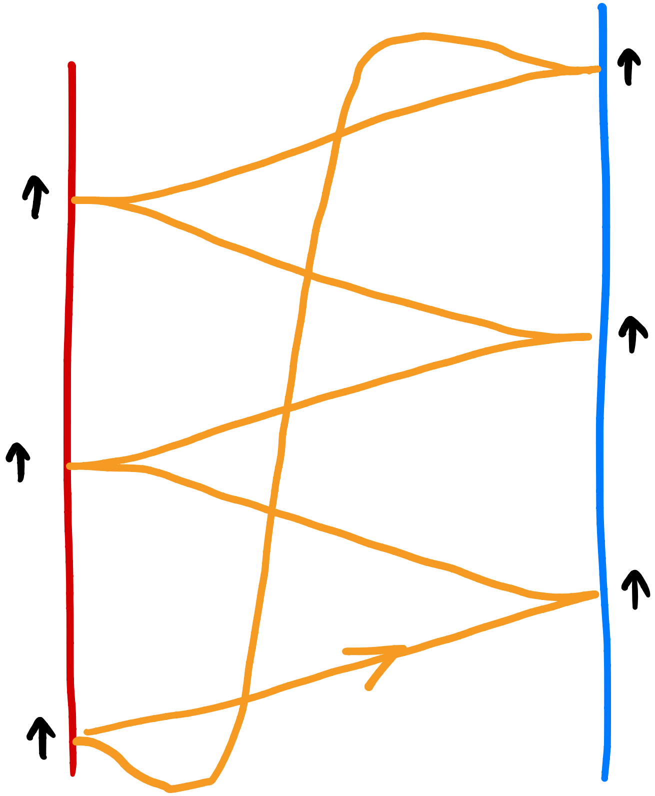
⇒ WE GET A <u>STANDARD FORM</u> THAT ONLY DEPENDS ON $\boxed{\text{ROT}(L)}$

(ROT DOES NOT CHANGE DURING 1) 2) 3))



BACK TO WHITNEY-GRAUSTEIN THEOREM

WHITNEY-GRAUSTEIN THEOREM

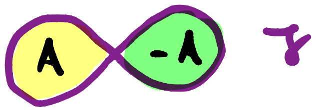
$$\{\text{REGULAR PLANE CURVES}\} \Big/ \text{REGULAR HOMOTOPY} \xrightarrow{W} \mathbb{Z}$$ IS A BIJECTION.

PROOF: • W SURJECTIVE ✓

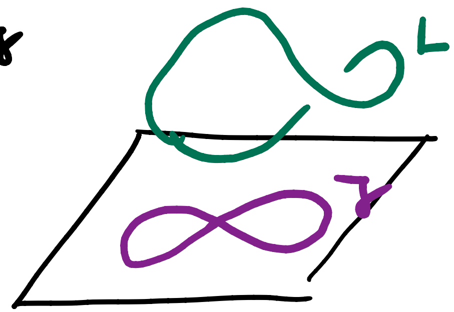
• **W INJECTIVE**

$\gamma : S^1 \to \mathbb{R}^2$ REGULAR PLANECURVE

1) DO REGULAR HOMOTOPY ON γ SO THAT $\oint y\dot{x}\,ds = 0$



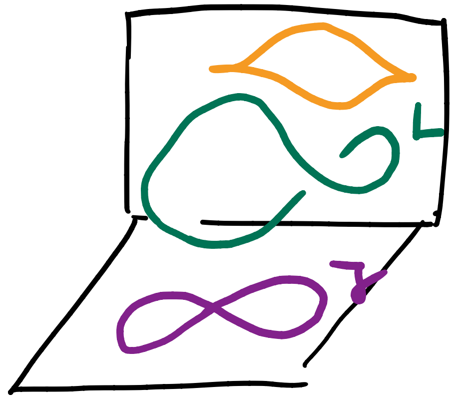
2) LIFT UP γ TO A LEGENDRIAN CURVE L SO THAT THE LAGRANGIAN PROJECTION $\pi_{(x,y)}(L) = \gamma$



$$\boxed{ROT(L) = W(\gamma)}$$

Back to Whitney-Graustein Theorem

(up to regular homotopy) we have $\gamma = \pi_{(x,y)}(L)$



3) Take the front projection of L
& normalise it

4) The steps give a 1-parameter
family of (immersed) Legendrians $\{L_t\}$
where

$L_1 =$ 

5) The Lagrangian projection $\gamma_t = \pi_{(x,y)}(L_t)$
gives a regular homotopy of

$\gamma = \gamma_0$ to $\gamma_1 = \pi_{(x,y)}(L_1) =$ 

BACK TO WHITNEY-GRAUSTEIN THEOREM

SO: ANY REGULAR PLANE CURVE $\gamma : S^1 \to \mathbb{R}^2$

IS REGULARLY HOMOTOPIC TO  γ_n

FOR $n = W(\gamma)$

$\Rightarrow$ W IS INDEED INJECTIVE $\square$

WHITNEY-GRAUSTEIN IN OTHER DIMENSIONS

CLASSIFY IMMERSED SPHERES $\gamma : S^k \looparrowright \mathbb{R}^n$

UP TO REGULAR HOMOTOPY : $\mathrm{IMM}(S^k, \mathbb{R}^n)$

- **THEOREM** (SMALE, 1959): $\mathrm{IMM}(S^k, \mathbb{R}^n) = \pi_k(V_k(\mathbb{R}^n))$

STIEFEL MANIFOLD

(ORTHONORMAL FRAMES IN $\mathbb{R}^n$)

- FOR $k = n-1$: $S^{n-1} \looparrowright \mathbb{R}^n$

$$\mathrm{IMM}(S^{n-1}, \mathbb{R}^n) = \pi_{n-1}(V_{n-1}(\mathbb{R}^n)) = \pi_{n-1}(SO(n)) \cong \begin{cases} 0 & n = 1, 3, 7 \\ \mathbb{Z} & n = 2 \\ \mathbb{Z}^2 & n = 4 \\ \vdots & \end{cases}$$

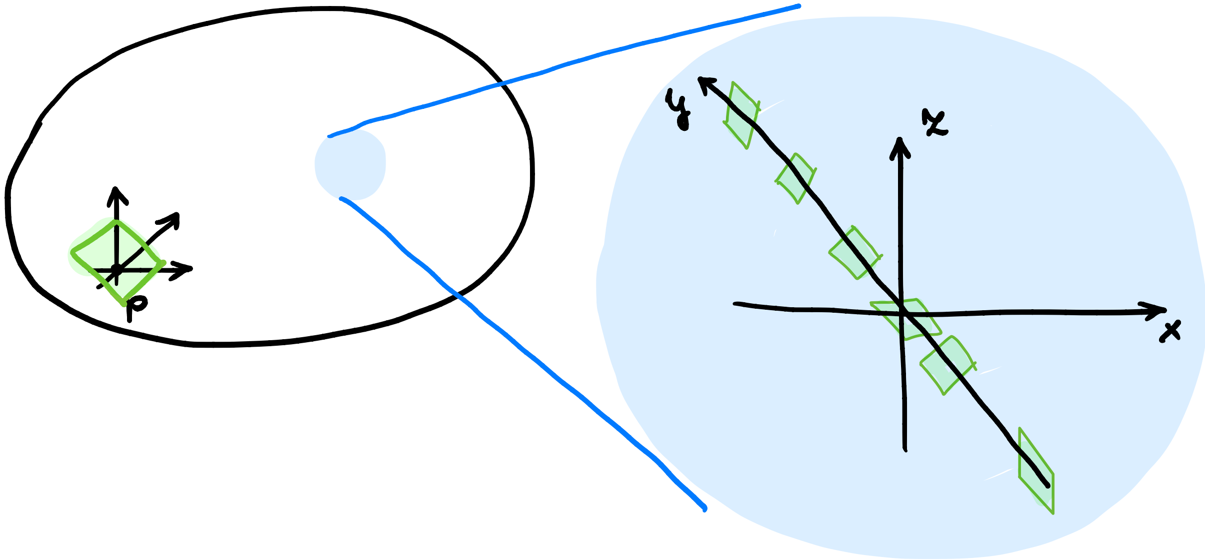
- $\underline{n=2}$ $\quad \{S^1 \looparrowright \mathbb{R}^2\}/_{\simeq_{REG}} \overset{W}{\longleftrightarrow} \pi_1(SO(2)) \cong \mathbb{Z}$

- $\underline{n=3}$ $\quad \{S^2 \looparrowright \mathbb{R}^3\}/_{\simeq_{REG}} \longleftrightarrow \pi_2(SO(3)) = 0$

! WE CAN TURN THE SPHERE INSIDE-OUT !

EPILOGUE

- CONTACT STRUCTURES CAN BE DEFINED ON ANY ODD-DIMENSIONAL MANIFOLD



- LEGENDRIAN SUBMANIFOLDS ARE USED AS TOOLS IN PHYSICS (OPTICS, THERMODYNAMICS, MECHANICS, ...)
- THEY ARE USED IN TOPOLOGY & GEOMETRY
- THE TOOLS TO STUDY CONTACT MANIFOLDS INCLUDE:
  - COMPLEX GEOMETRY
  - FLOER HOMOLOGIES
  - COMBINATORICS, ...

THANKS FOR YOUR ATTENTION!

QUESTIONS?

FUN VIDEOS ON YOUTUBE:

- 1972 REGULAR HOMOTOPIES ON THE PLANE
- 1994 OUTSIDE IN