Bordered Contact Invariant

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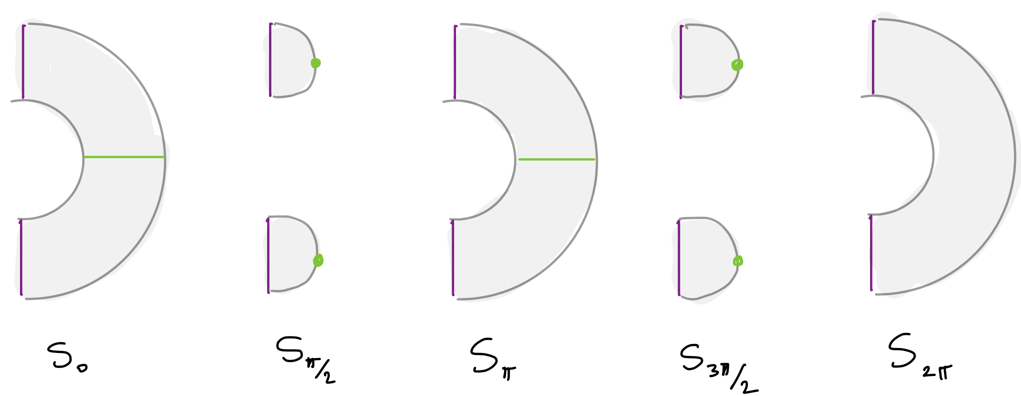
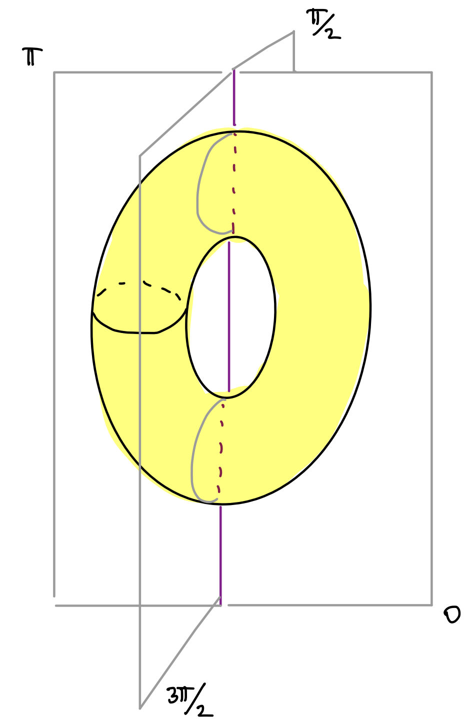




MOTIVATION

Contact invariant in Heegaard Floer homology

M^3 $\xleftarrow{1:\text{many}}$ Heegaard decomposition / $\rightsquigarrow$ HF(M^3) doubly graded vspace

/ diffeo $\xleftrightarrow{1:1}$; stab

$M^3 = U \cup V$

handlebodies

invariant under stab

contact Heegaard decomposition

$\uparrow$

ξ ctct str / contacto $\xleftarrow{1:\text{many}}$, $\xleftrightarrow{1:1}$ open book decomposition (B,π) / o.b. stab $\rightsquigarrow$ $c(\xi)$

$\rotatebox{90}{$\in$}$

invariant under o.b stab

MOTIVATION

Properties:

- Ozsváth - Szabó: ξ Stein fillable $\Rightarrow c(\xi) \neq 0$
- Ozsváth - Szabó: ξ overtwisted $\Longrightarrow c(\xi) = 0$
- Ghiggini - Honda - Van Horn-Morris:

  ξ contains Giroux torsion $\Longrightarrow c(\xi) = 0$

All local reasons ($\Leftarrow$ Honda - Kazez - Matic: partial open books)

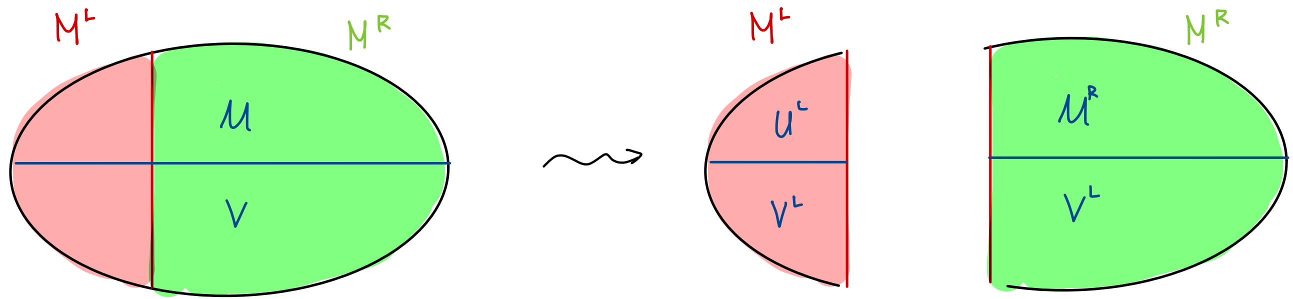
Conjecture (maybe: Ghiggini):

ξ contains ½ Giroux torsion along a separating torus

$\Rightarrow c(\xi) = 0$ (really global $c(T^3, \eta) \neq 0$)

MOTIVATION

bordered Heegaard Floer homology

$M =$ 

Σ ↖ separating surface

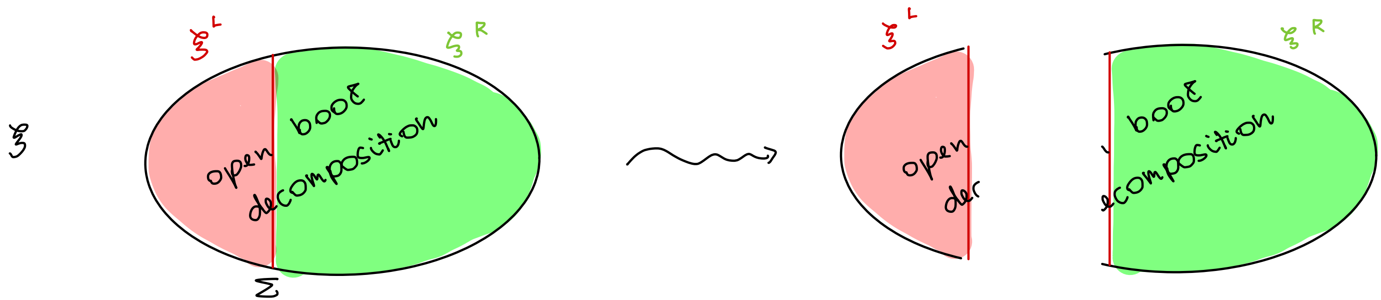
bordered 3-manifolds

(M, Σ) bordered 3-mfd $\xleftarrow{1:\text{many}}$ $\xleftrightarrow{1:1}$ bordered Heegaard decomp / stab $\rightsquigarrow$ indep. $CFA(M)$, $CFD(M)$

$\mathcal{A}^\infty$-modules over an $\mathcal{A}^\infty$-algebra associated to Σ

& $CFA(M^L, \Sigma) \tilde{\otimes} CFD(M^R, -\Sigma) = CF(M) \leftarrow H_*(CF(M)) = HF(M)$

MOTIVATION



ctct 3-mfds w/ foliated bdry

& foliated open books

$$(\xi, \mathcal{F}_\xi) \; \xleftarrow[\text{1: many}]{\text{Licata, V}} \; \text{foliated open book} \,/\, \text{stab} \; \overset{\text{AFHLPV}}{\rightsquigarrow} \; \begin{array}{l} c_A(\xi, \mathcal{F}_\xi) \\ c_D(\xi, \mathcal{F}_\xi) \end{array}$$

$$\longleftrightarrow$$

contact str w/ char. foliation on Σ

& $c_A(\xi^L, \mathcal{F}_\xi) \otimes c_D(\xi^R, \overline{\mathcal{F}_\xi}) = c(\xi)$

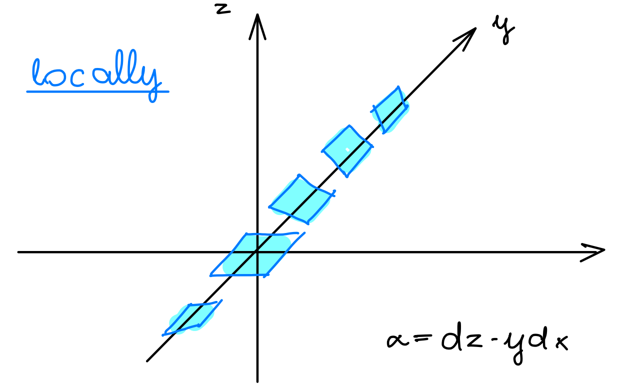
Contact Structures

$(M^3, \xi^2 = \ker\alpha)$ non-integrable 2-plane distribution

$$\Updownarrow$$

$$\alpha \wedge d\alpha > 0$$

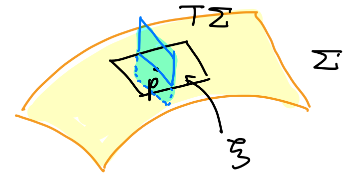
locally



$\alpha = dz - y\,dx$

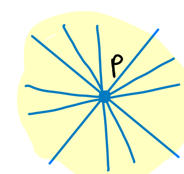
$\Sigma \hookrightarrow (M^3, \xi) \rightsquigarrow \alpha|_\Sigma$ induces a foliation:

characteristic foliation $\mathcal{F}_\xi$



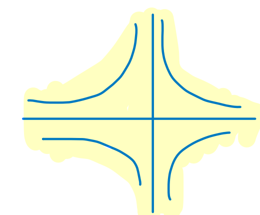
Fact (Giroux): $d(\alpha|_\Sigma) \neq 0$ at singular pts

$\Rightarrow$ the isolated singularities of $\mathcal{F}_\xi$ are either



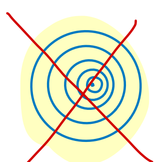
elliptic

or



hyperbolic

but no



center

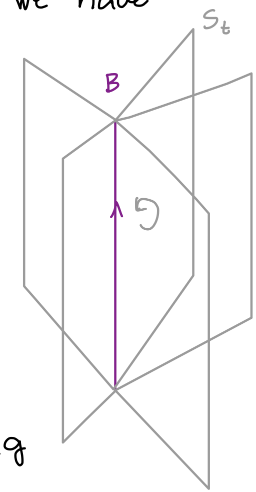
Thm (Giroux): $\mathcal{F}_\xi$ determines ξ in a nbhd of Σ.

Open Book Decompositions

M^3 $\qquad (B, \pi)$

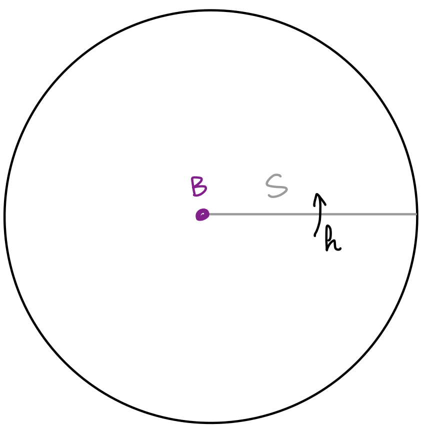
$B^1 \hookrightarrow M$ $\qquad \pi : M \setminus B \to S^1$ fibration, $S_t := \pi^{-1}(t)$ & near B we have

binding $\qquad$ page



$\Rightarrow$ We can think of $M \setminus B$ as a mapping cylinder of (S, h):

$$S \times I \big/ (x,1) \sim (h(x), 0)$$



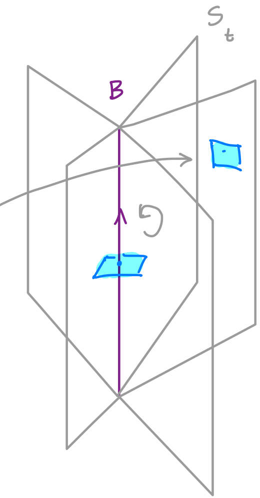
& we obtain M by further identifying

$$(x,t) \sim (x,t') \qquad \begin{array}{l} x \in \partial S \\ t, t' \in I \end{array}$$

COMPATIBILITY

(B,π) supports $\xi = \ker \alpha$ if
- $\alpha > 0$ on B
- $d\alpha > 0$ on S_t

ξ is 'nearby' $T S_t$

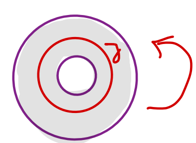


examples 1) & 2) both supports ξ_{st}

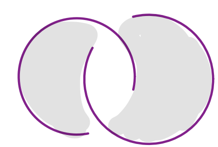
Giroux correspondence:

$$\{\xi \text{ ctct str}\}/\text{isotopy} \quad \xleftrightarrow{1:1} \quad \{\text{open book decomposition } (B,\pi)\}/\text{o.b. stab}$$

local operation, connect sum w/

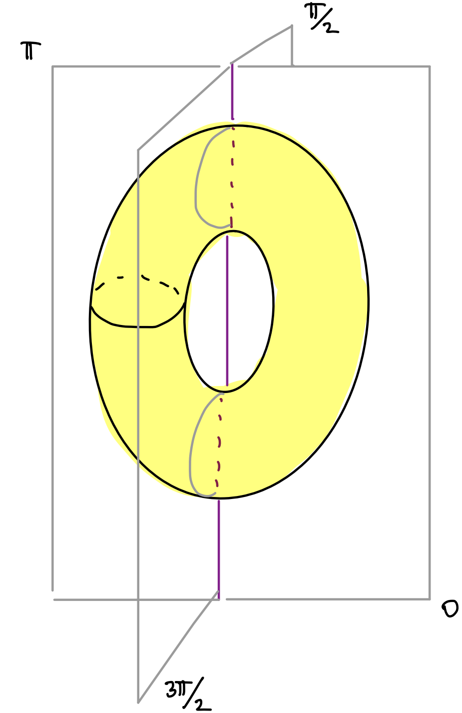


$h = D_\gamma$ on



$\subseteq \mathbb{R}^3$

FOLIATED OPEN BOOK — EXAMPLE

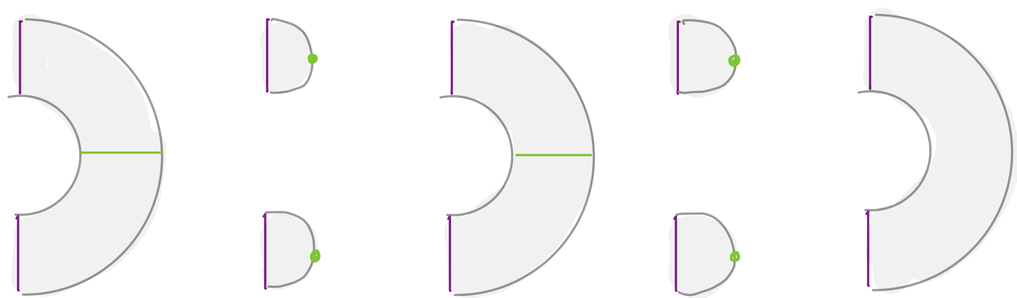


M = solid torus $\subseteq \mathbb{R}^3$

$B = \{z\text{-axis}\} \cap M$

π = angle : $M \setminus B \longrightarrow S^1$

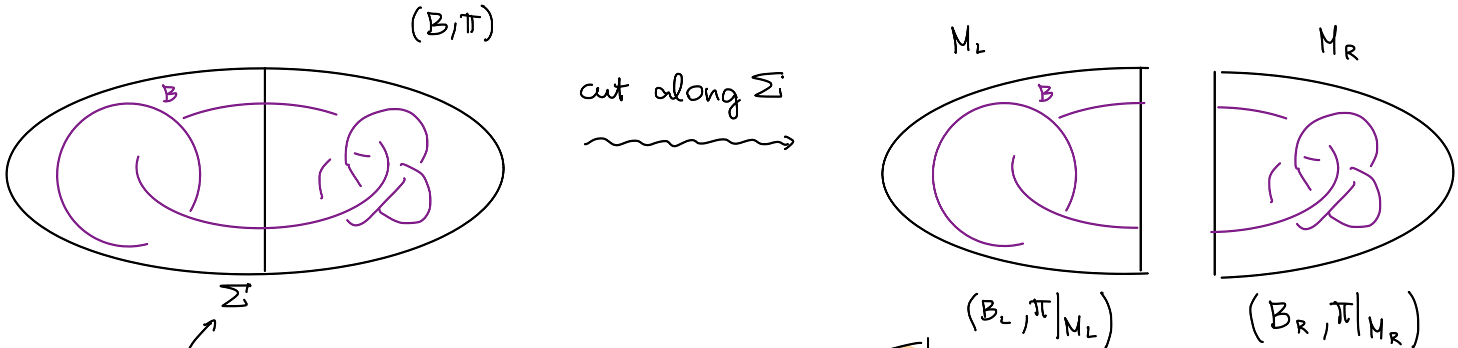
$\Rightarrow S_t = \pi^{-1}(t)$ changes in time



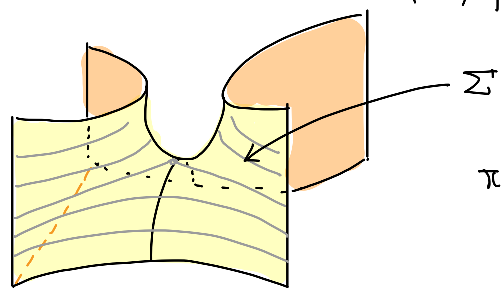
S_0 $\quad$ $S_{\pi/2}$ $\quad$ S_π $\quad$ $S_{3\pi/2}$ $\quad$ $S_{2\pi}$

$\pi|_{\partial M}$ gives a (singular) foliation $\mathcal{F}_\pi$ for ∂M

FOLIATED OPEN BOOKS - IDEA



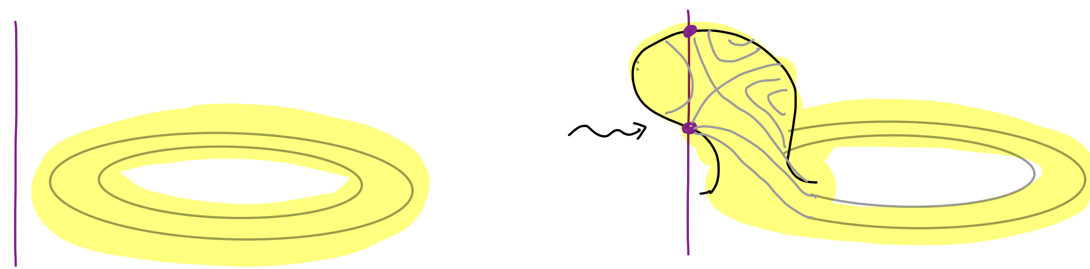
in "good position":



small isotopy
- $B \pitchfork \Sigma$
- $\pi|_\Sigma$ is Morse

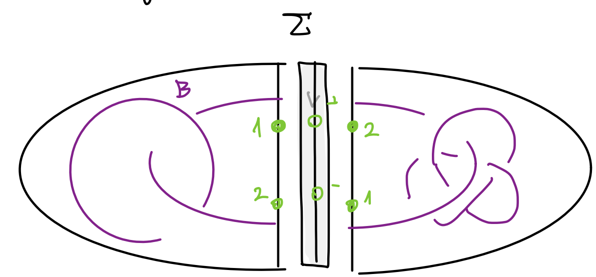
- level sets of $\pi|_\Sigma$ have no closed components

"big isotopy":



FOLIATED OPEN BOOKS

modify π near Σ by introducing a canceling pair of index 1 & 2 critical pts. for every critical pt of $\pi|_{\partial M}$



<u>Def</u>: (B,π) is a <u>foliated open book</u> for $(M,\partial M)$ if:

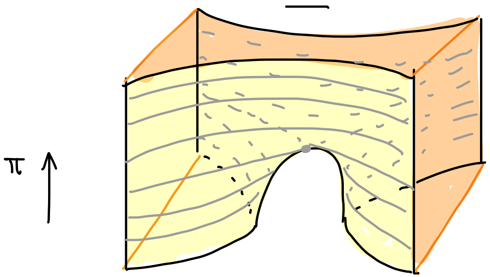
- $(B^1,\partial B) \hookrightarrow (M^3,\partial M)$
- $\pi : M \setminus B \longrightarrow S^1$ S^1-valued Morse function w/
  - $\rightarrow$ $\mathrm{Crit}(\pi) = \mathrm{Crit}(\pi|_{\partial M})$
  - $\rightarrow$ the level sets of $\pi|_{\partial M}$ have closed leaves

FOLIATED OPEN BOOKS

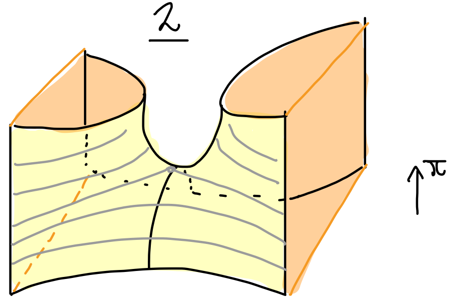
$\Rightarrow \pi|_{\partial M}$ has no min / max $\Rightarrow$ indices of critical pts of $\pi|_{\partial}$ are 1

$\Rightarrow$ ______ " ______ π are

1



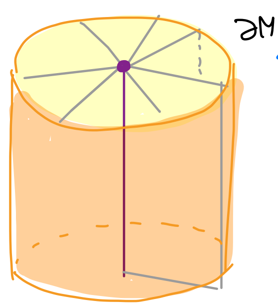
2



hyperbolic points

$\Rightarrow (\pi|_{\partial M})^{-1}(t)$ foliates ∂M : $\mathcal{F}_\pi$

near $B \cap \partial M$:

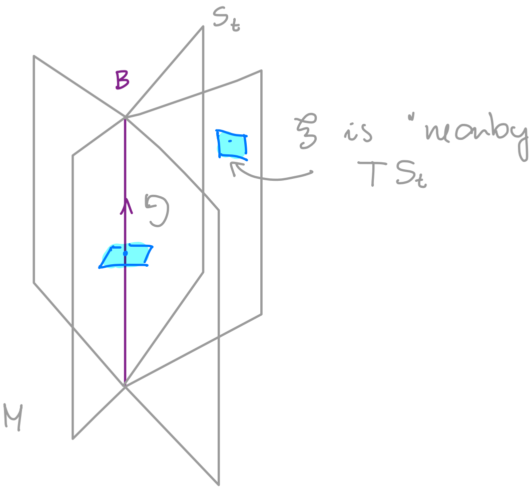


elliptic pt

COMPATIBILITY

(B,π) supports $\xi = \ker\alpha$ on $(M,\partial M)$ if

- $\alpha > 0$ on B
- $d\alpha > 0$ on $S_t = \pi^{-1}(t)$
- $\mathcal{F}_\xi \underset{\text{equiv.}}{\overset{\text{top}}{\sim}} \mathcal{F}_\pi$ ← this just says that ξ is "nearby" TS_t near ∂M



about $\mathcal{F}_\xi \underset{\text{equiv.}}{\overset{\text{top}}{\sim}} \mathcal{F}_\pi$:

We would like to say "the same" or $\exists$ small ~~diffeo~~ homeo isotopic to id that takes $\mathcal{F}_\xi$ to $\mathcal{F}_\pi$

but !

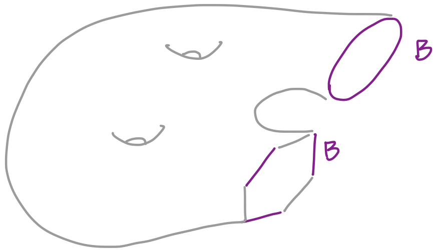
We can never have diffeo near the hyperbolic pts $(\operatorname{div}(\mathcal{F}_\xi) \neq 0 \leftrightarrow \operatorname{div}(\mathcal{F}_\pi) = 0)$

Thm (Licata-V)

$$\text{ctct str w/ fixed } \mathcal{F}_\xi \Big/ \text{isotopy rel } \partial M \quad \overset{1:1}{\longleftrightarrow} \quad \nexists\text{OB w/ fixed } \pi|_{\partial M} \Big/ \text{internal stab.}$$

Abstract Foliated Open Books - idea

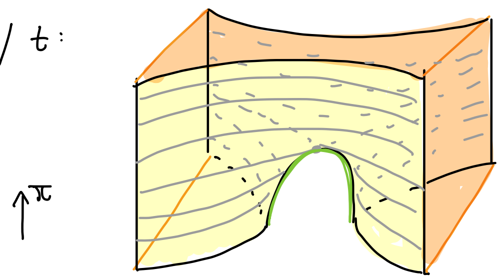
$\pi^{-1}(t) = S_t$

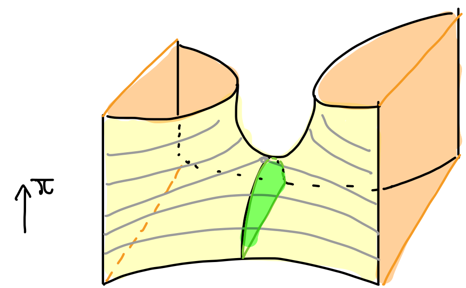


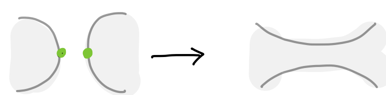
$\leftarrow (\pi|_{\partial M})^{-1}(t) =$ leaves of $\mathcal{F}_\pi$

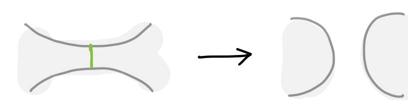
! no circles !

& S_t changes w/ t:



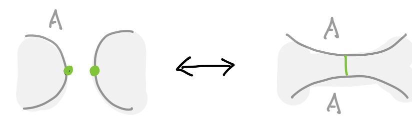


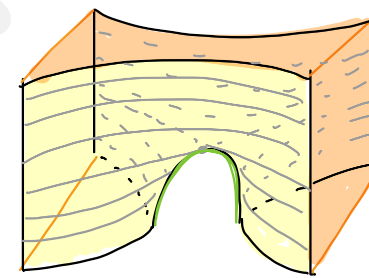




Abstract Foliated Open Books

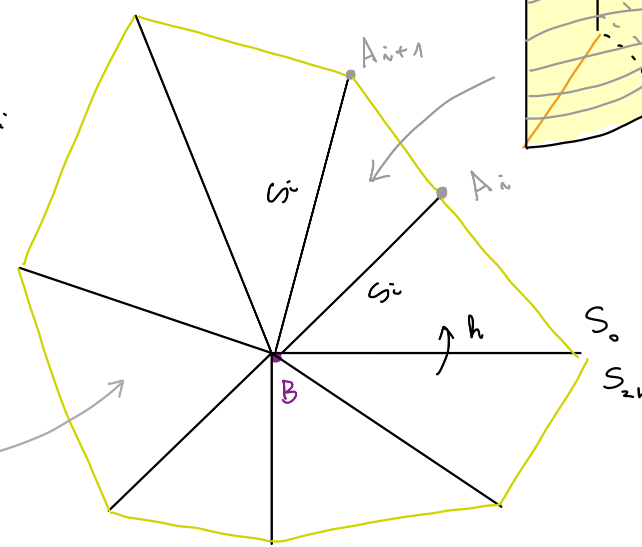
Def: an <u>abstract foliated open book</u> is $(S_0, S_1, \dots, S_{2k}, h)$ w/

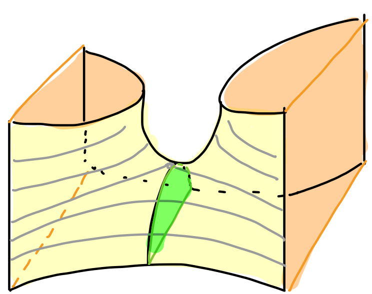
- S_i surfaces w/ polygonial bdry $= B \cup A_i$ w/ no circles in A_i
- S_{i-1} & S_i are related by 



- $h : S_{2k} \xrightarrow{\cong} S_0$ fixing B

$\rightsquigarrow$ Can reconstruct $(M, \partial M, \mathcal{F})$:

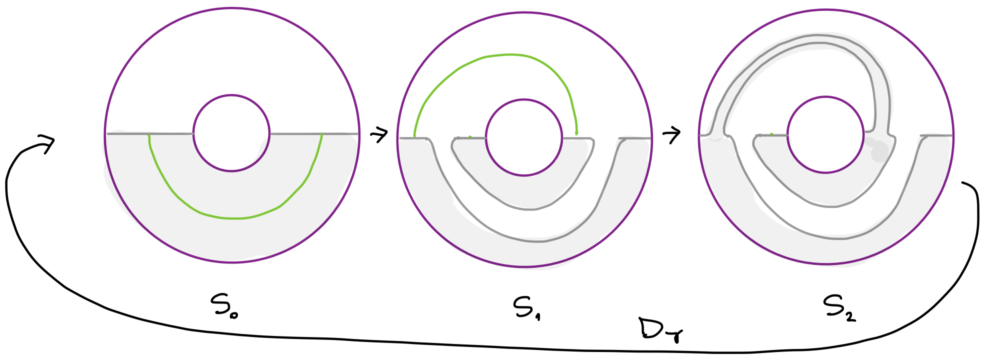




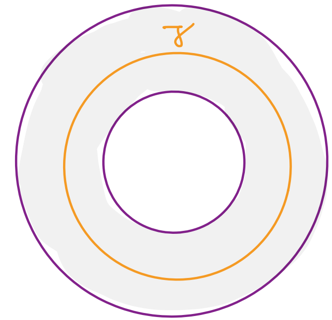
Abstract Foliated Open Books - Examples

1) take any abstract open book (S,h)

& $2k$ subsets $S_i \subseteq S$ w/ $h(S_{2k}) = S_0$



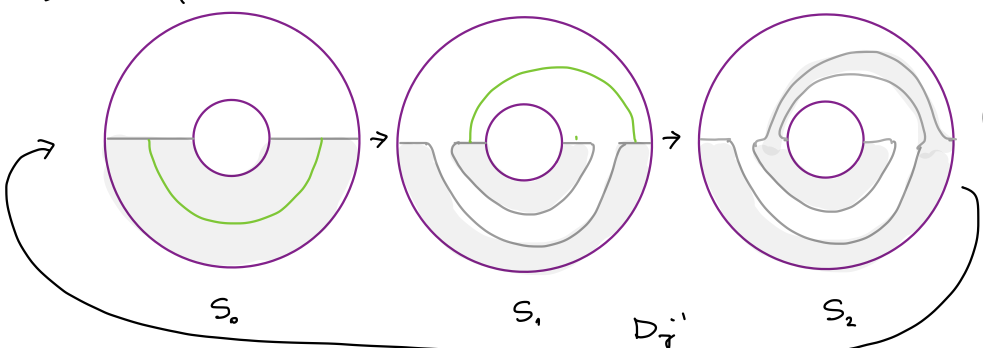
$S =$ annulus

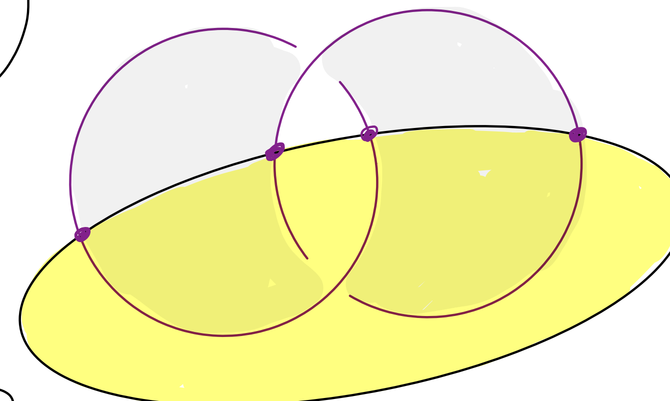


$h = D_\gamma$

|| Dehn twist along γ

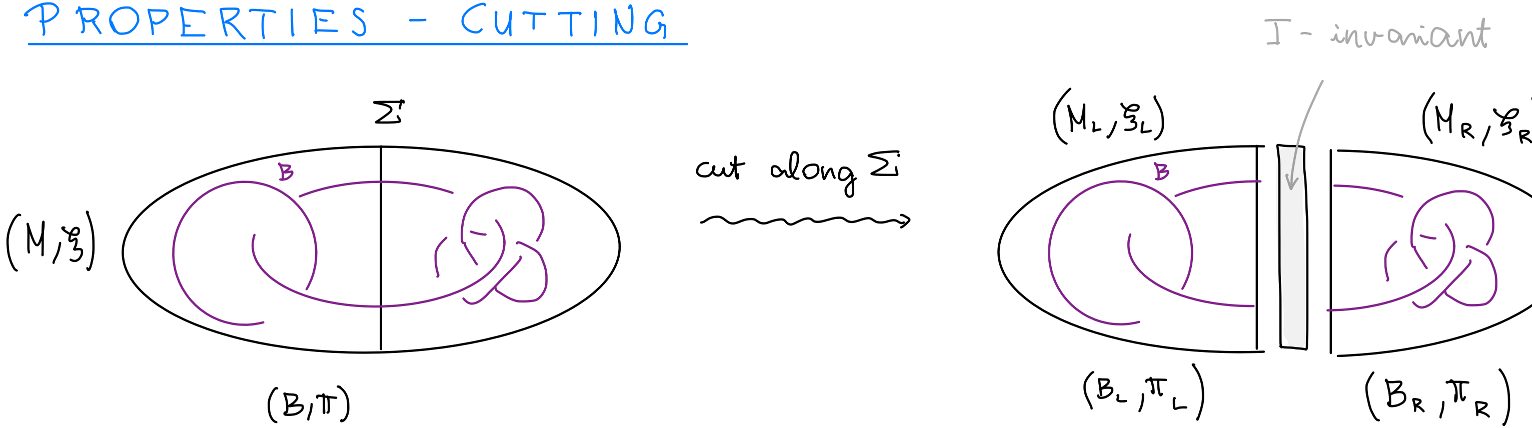
2) for (anulus, D_γ^{-1})





← neighbourhood of OT disc

PROPERTIES - CUTTING



If
- $B \pitchfork \Sigma$
- $\pi|_\Sigma$ is Morse
- the level sets of $\pi|_\Sigma$ has no circles

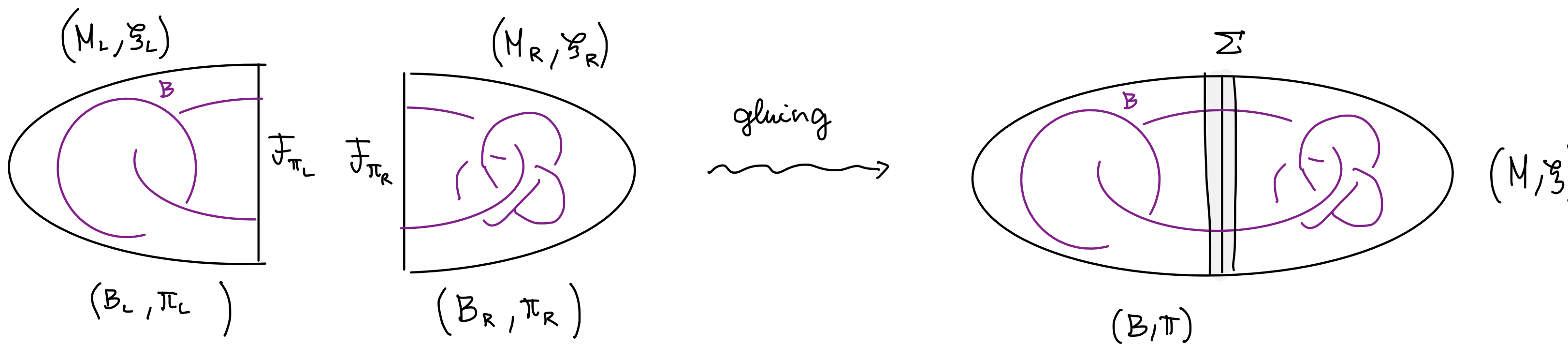
$\Rightarrow$ after modifying π near Σ & cutting out a nbhd of Σ then

$(M_L, \xi_L = \xi|_{M_L})$ is supported by $(B_L = B \cap M_L, \pi_L = \pi'|_{M_L})$

&

$(M_R, \xi_R = \xi|_{M_R})$ is supported by $(B_R = B \cap M_R, \pi_R = \pi'|_{M_R})$

PROPERTIES - GLUING



If $\partial M_L \xrightarrow[\text{rev}]{\text{or.}} \partial M_R$

$\mathcal{F}_{\pi_L} \longrightarrow -\mathcal{F}_{\pi_R}$

$\Longrightarrow$ there is a standard piece on $\Sigma \times I$.
(depends on $\mathcal{F}$)

s.t. $M = M_L \cup \Sigma \times I \cup M_R$

has an open book (B, π) w/

$(B,\pi)|_{M_L} = (B_L, \pi_L)$ supports $\xi|_{M_L} = \xi_L$

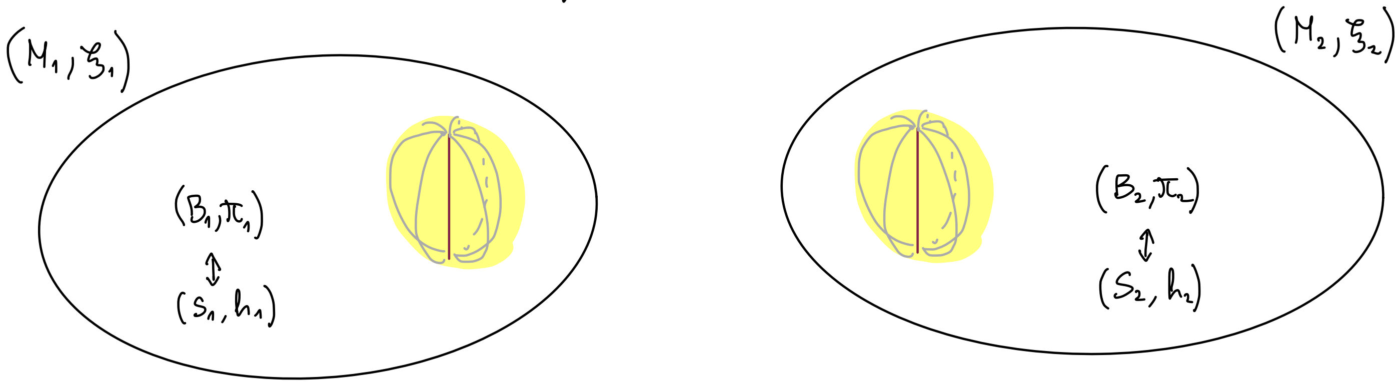
&

$(B,\pi)|_{M_R} = (B_R, \pi_R)$ supports $\xi|_{M_R} = \xi_R$

APPLICATIONS - ADDITIVITY OF THE SUPPORT NORM

(M^3, ξ) closed contact : $sn(\xi) = \min\{-\chi(S) - 1 : (S, h) \text{ supports } \xi\}$

Question: How does $sn(\xi)$ behave under connected sum?



$\rightsquigarrow$ the pages for the o.b for $(M_1 \# M_2, \xi_1 \# \xi_2)$ after gluing are $S_1 \natural S_2$

$\Longrightarrow \quad sn(\xi) \leq sn(\xi_1) + sn(\xi_2)$

Question: Is $sn(\xi) = sn(\xi_1) + sn(\xi_2)$?

APPLICATIONS - ADDITIVITY OF THE SUPPORT NORM

Example (Özbagci) : M $\mathbb{Z}$-homology sphere, ξ overtwisted

$$(M,\xi) \# (S^3, \xi_{-1/2}) \cong (M,\xi)$$

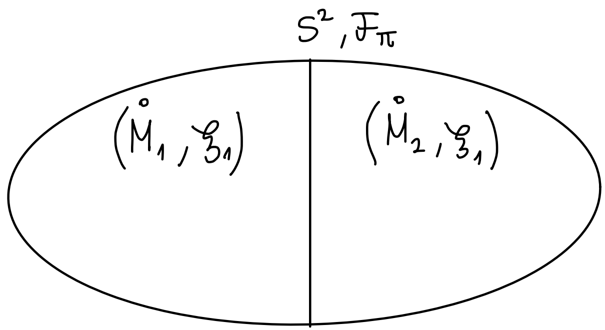
$d_3 = -\frac{1}{2}$ compute

but $sn(S^3, \xi_{-1/2}) \neq 0$

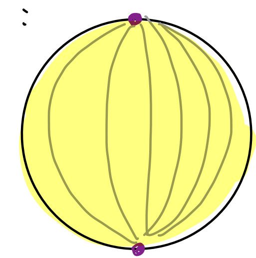
$$\boxed{\text{Thm (V): } (M_1,\xi_1) \ \& \ (M_2,\xi_2) \text{ tight } \Longrightarrow sn(\xi) = sn(\xi_1) + sn(\xi_2)}$$

$(M,\xi) =$

(B,π)



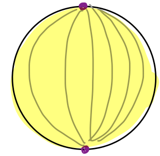
if $\mathcal{F}_\pi$ on S^2 is :

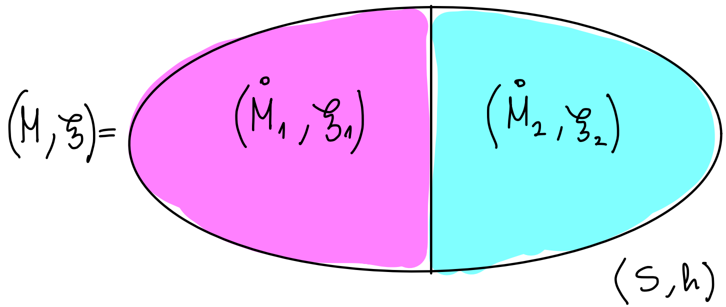


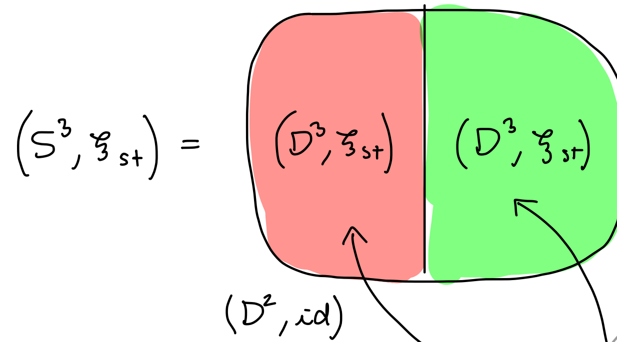
$\Rightarrow \checkmark$

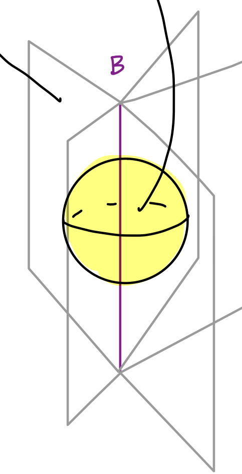
Applications - Additivity of the Support Norm

Why are we done ?

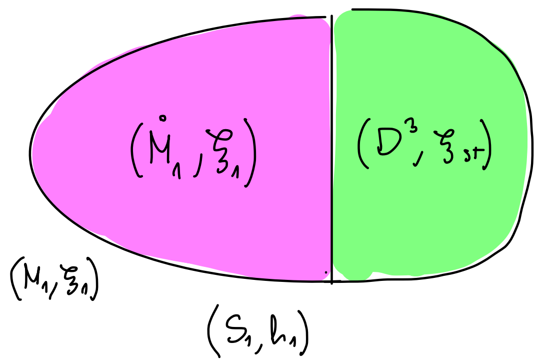


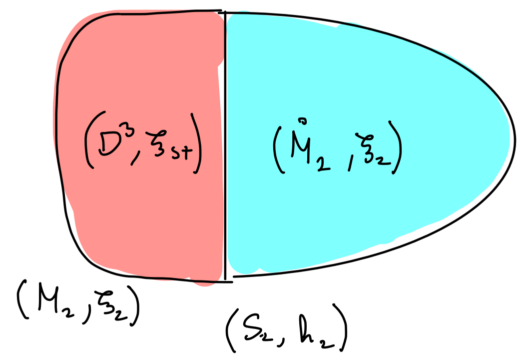






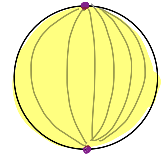
cut both $\rightsquigarrow$ & glue



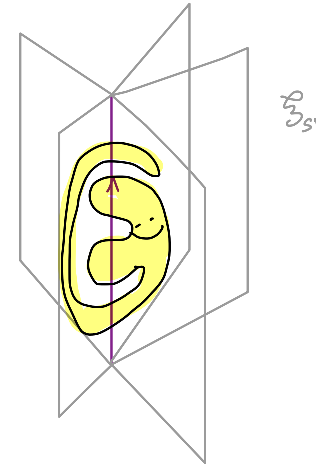


$$\Rightarrow \quad S = S_1 \natural S_2 \quad \Longrightarrow \quad \chi(S) = \chi(S_1) + \chi(S_2) - 1 \quad \Longrightarrow \quad sn(\xi) \geq sn(\xi_1) + sn(\xi_2)$$

APPLICATIONS - ADDITIVITY OF THE SUPPORT NORM

But! We cannot alway simplify to  .

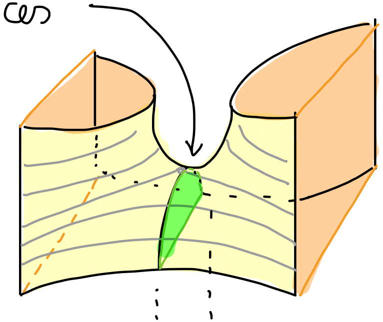
instead:



Lemma: Any foliation $\mathcal{F}$ (coming from a tight ξ) can be realised in (D^2, id)

idea of Proof: tight $\Rightarrow$ the graph on elliptic pts formed by these separatrices is a tree.

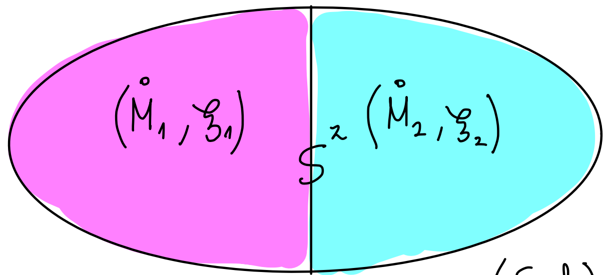
$\rightsquigarrow$ can do induction ...



elliptic point

APPLICATIONS - ADDITIVITY OF THE SUPPORT NORM

Proof: Given $(M,\xi) =$

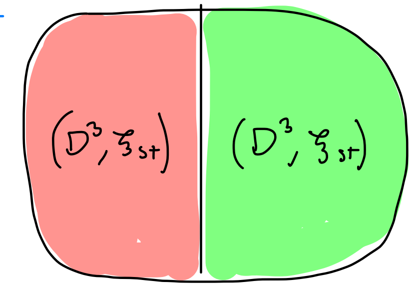


$\mathcal{F}$

(S,h)

Lemma $\rightsquigarrow$

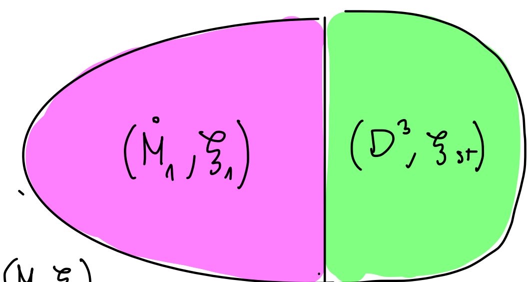
$\mathcal{F}$



(S^3, ξ) (D^2, id)

cut both $\rightsquigarrow$ & glue

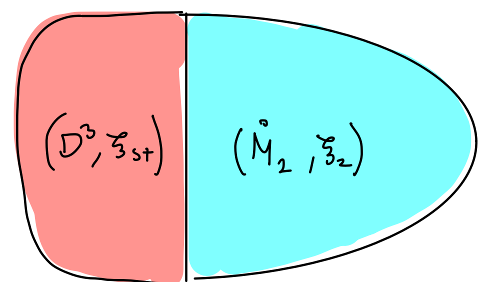
$\mathcal{F}$



(M_1, ξ_1)

(S_1, h_1)

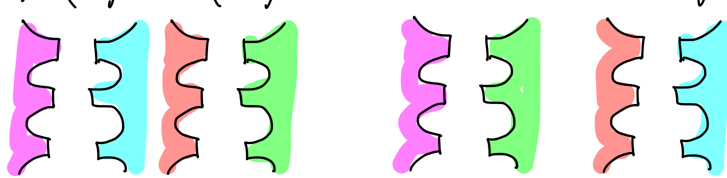
$\mathcal{F}$



(M_2, ξ_2) (S_2, h_2)

$$\Rightarrow \quad \chi(S) + \chi(D^2) = \chi(S_1) + \chi(S_2) \qquad \Longrightarrow sn(\xi) \geq sn(\xi_1) + sn(\xi_2)$$

pages



∎

CONTACT INVARIANTS (work in progress)

(M,ξ) closed ctct 3-manifold $\underset{\text{Szabó}}{\overset{\text{Ozsváth}}{\rightsquigarrow}}$ $c(\xi)\in HF(-M)$

(M,ξ,Γ) ctct 3-mfd w/ convex ∂ $\underset{\text{-Matic}}{\overset{\text{Honda-Kazez}}{\rightsquigarrow}}$ $EH(\xi,\Gamma)\in SFH(-(M,\Gamma))$

$\boxed{\begin{array}{ll} (M,\xi,\mathcal{F}_\xi) \text{ ctct 3-mfd w/ char. foliation on } \partial & \\ \underset{\text{Petkova, Licata, V}}{\overset{\text{Alishahi, Földvári, Hendricks}}{\rightsquigarrow}} & c_A(\xi)\in CFA(-(M,\partial M)) \\ & c_D(\xi)\in CFD(-(M,\partial M)) \end{array}}$

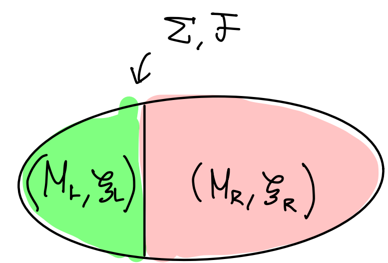
(this is also cx)

$\mathcal{F}_\xi$ parametrizes ∂M

Thm (AFHLPV) $(M,\xi)=(M_L,\xi_L,\mathcal{F})\underset{\Sigma}{\cup}(M_R,\xi_R,\overline{\mathcal{F}})$

$\Rightarrow$ $CFA(-M_L,\Sigma)\,\tilde{\otimes}\,CFD(-M_R,\Sigma) = CF(-M)$

& $c_A(\xi_L)\,\tilde{\otimes}\,c_D(\xi_R) = c(\xi)$



CONNECTION TO SUTURED INVARIANT

Thm: (AFHLPV) $CFA(-M) \xrightarrow{\text{R. Zarev}} SFH(-M)$ & $CFD(-M) \xrightarrow{\text{R. Zarev}} SFH(-M)$

$c_A(\xi) \longmapsto EH(\xi)$ $\qquad c_D(\xi) \longmapsto EH(\xi)$

&

$$\begin{tikzcd}
CFA(-M_L) \arrow[d, "\text{Zarev}"] & \tilde{\otimes} & CFD(-M_R) \arrow[d, "\text{Zarev}"] & = & CF(-M) \arrow[d, "H_*(\ )"] \\
SFH(-M) & \otimes & SFH(-M) \arrow[rr, "HKM"] & & HF(-M)
\end{tikzcd}$$

$c_A(\xi_L) \ \tilde{\otimes} \ c_D(\xi_R)$ $\qquad c(\xi)$

$EH(\xi_L) \otimes EH(\xi_R)$ $\qquad c(\xi)$

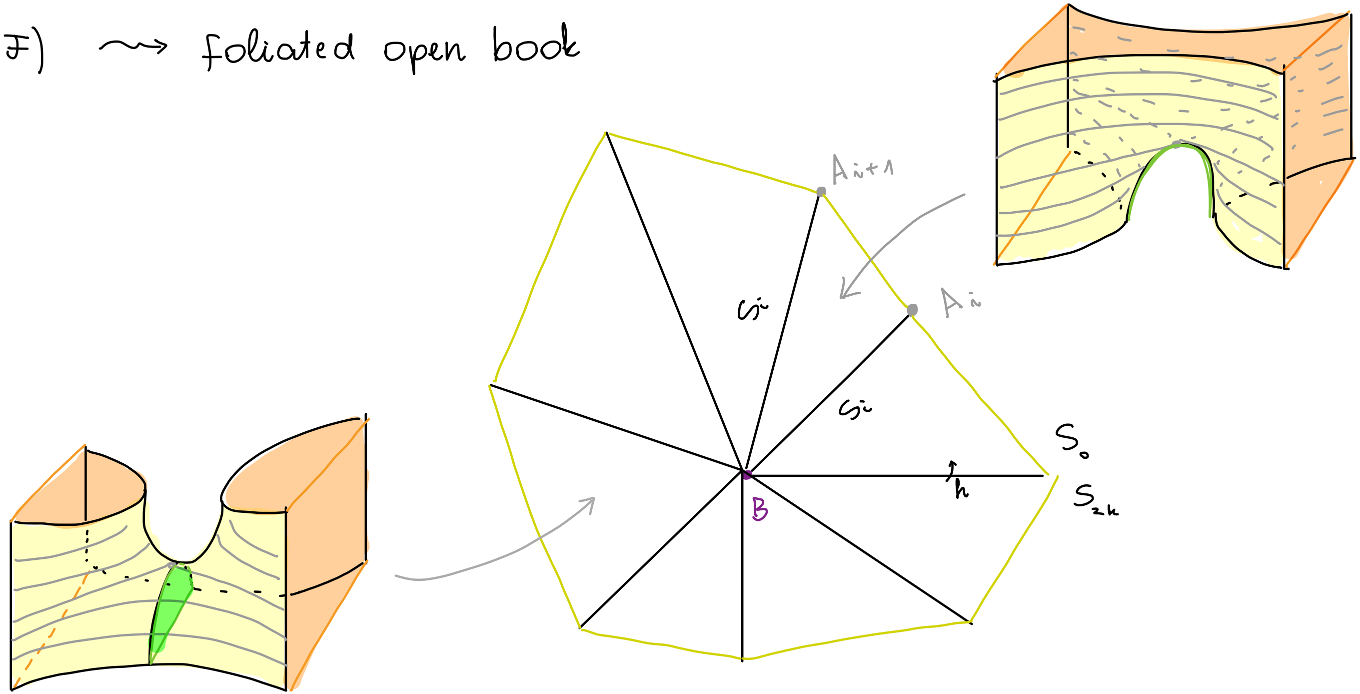
commutes

the definition of this map includes a "sufficiently complicated" so it is impossible to compute it even in concrete examples

this diagram makes this computation possible

Contact Invariant in Bordered Floer Homology - IDEA

$(M, \xi, \mathcal{F}) \rightsquigarrow$ foliated open book

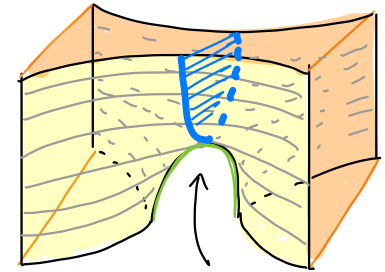


CONTACT INVARIANT IN BORDERED FLOER HOMOLOGY – IDEA

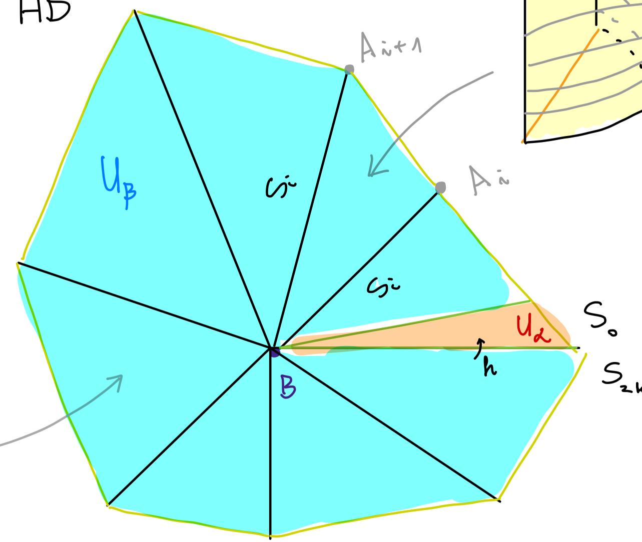
$(M, \xi, \mathcal{F}) \rightsquigarrow$ foliated open book

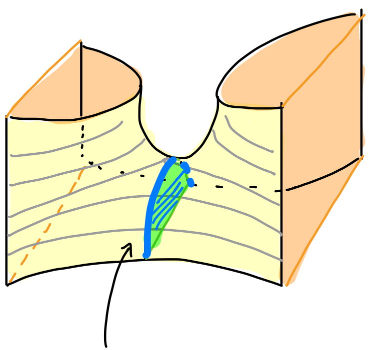
$\rightsquigarrow$ "multipointed" β-bordered HD

(= sutured bordered HD)



parametrisation





parametrisation

basepoints $= N(\partial S_0 \cap \partial M) = N(\tilde{\pi}^{-1}(0)) \subset \partial M$

! depends on the time-parametrisation of $\pi : M \setminus B \to S^1$!

FURTHER DIRECTIONS

- contact invariant in $CFDA(-M)$

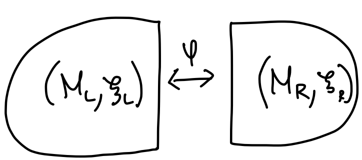
  $\rightsquigarrow$ description of $CFA(M)$ & $CFD(M)$ in terms of SFH:

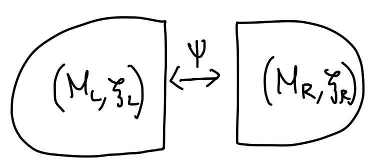
  Thm (Zarev): $CFA(M) = \bigoplus_{I} SFH(M, \Gamma_I)$

  elementary div curves

  Conjecture (Zarev): the $\mathcal{A}_\infty$-maps can be described by gluing standard $(\partial M \times I, \Gamma_I \cup -\Gamma_J)$-pieces.

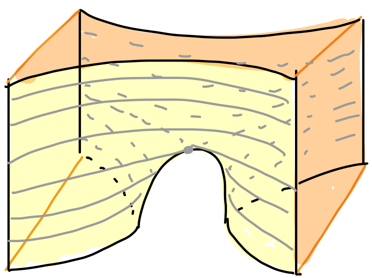
- gives a good guess of what bordered ECH should be.
- can be used to give global results as well:

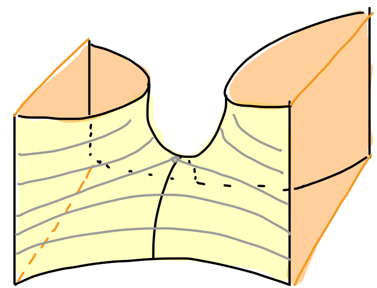


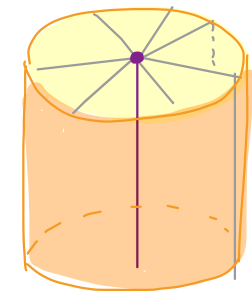


$\leftarrow$ can distinguish these

THANKS FOR YOUR ATTENTION!

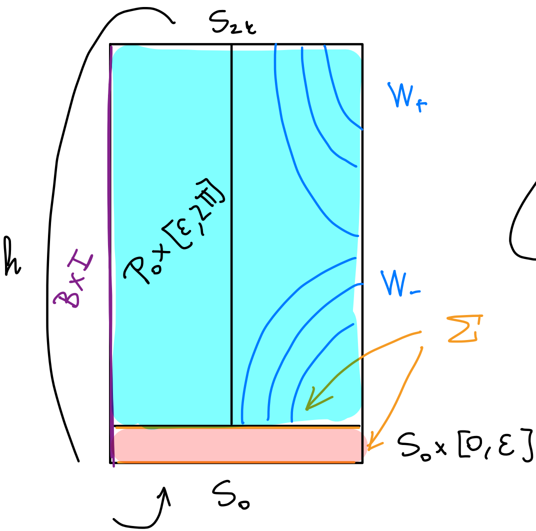


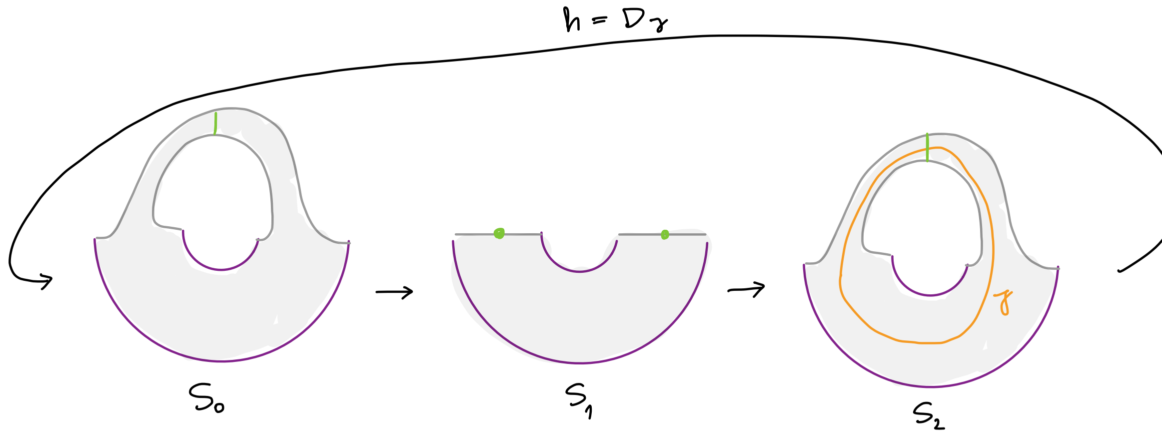




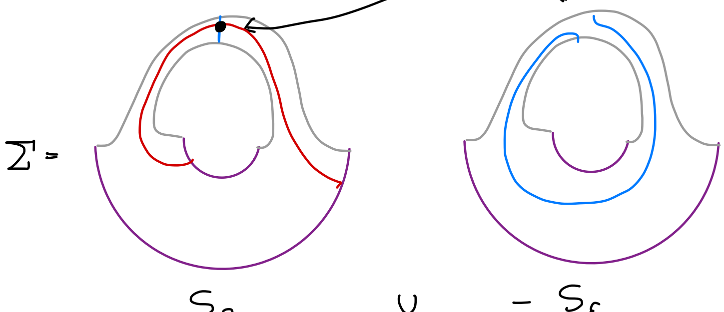
QUESTIONS ?

Contact Invariant in Bordered Floer Homology – IDEA





give the bdry parametrisation



$\rightsquigarrow$

$$\boxed{\begin{aligned}[x] &\in CFD(-M)\\ &\in CFA(-M)\end{aligned}}$$

FURTHER DIRECTIONS

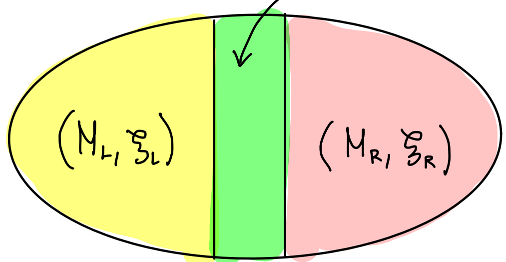
Conjecture (maybe: Ghiggini):

ξ contains $1/2$ Giroux torsion along a separating torus

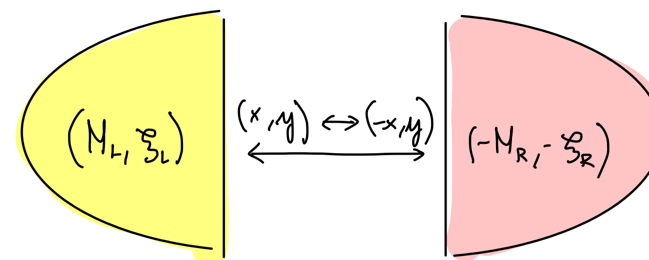
$\Rightarrow c(\xi) = 0$

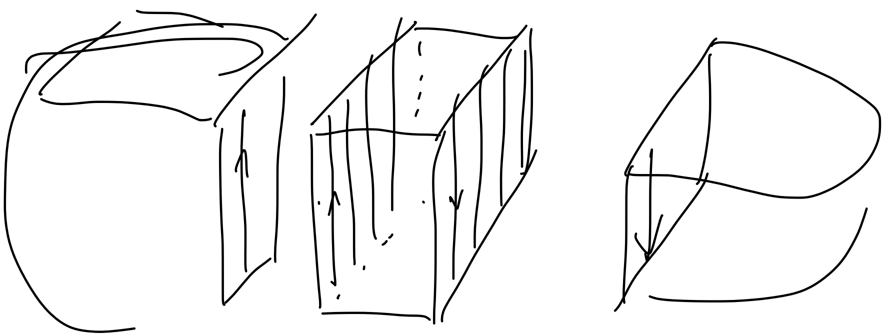
$(T^2 \times I,\ \eta_{1/2} = \ker(\cos(\pi t)dx - \sin(\pi t)dy))$

$(M_1, \xi_1) =$



$(M_2, \xi_2) =$





$f\alpha$

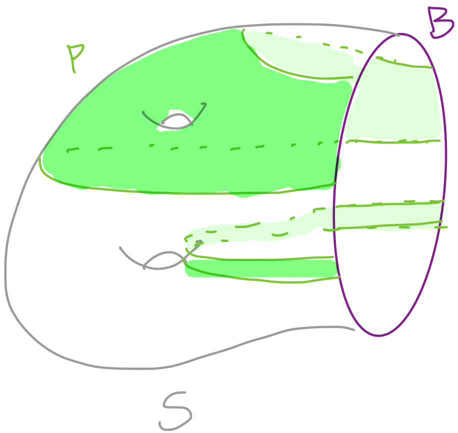
$d\alpha = \alpha\, df \wedge \alpha + f\, d\alpha$

$f^2\, \alpha \wedge d\alpha$

Partial Open Books (Honda-Kazez-Matić)

abstract version (S, P, h) $\quad h : P \hookrightarrow S \quad h|_B = id$

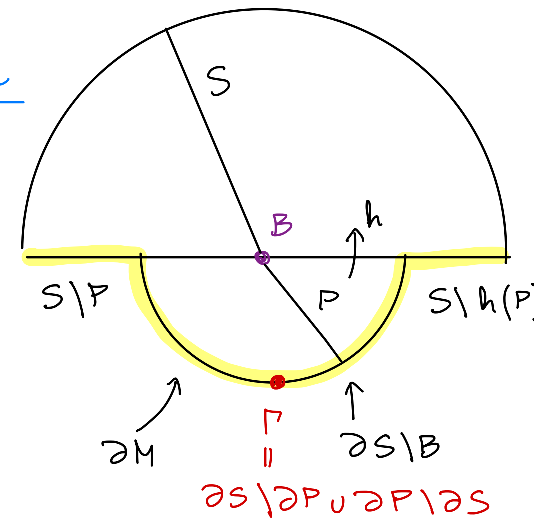
surface w/ $\partial = B$ $\quad$ subsurface



$\rightsquigarrow$

dividing curve

(M, Γ)

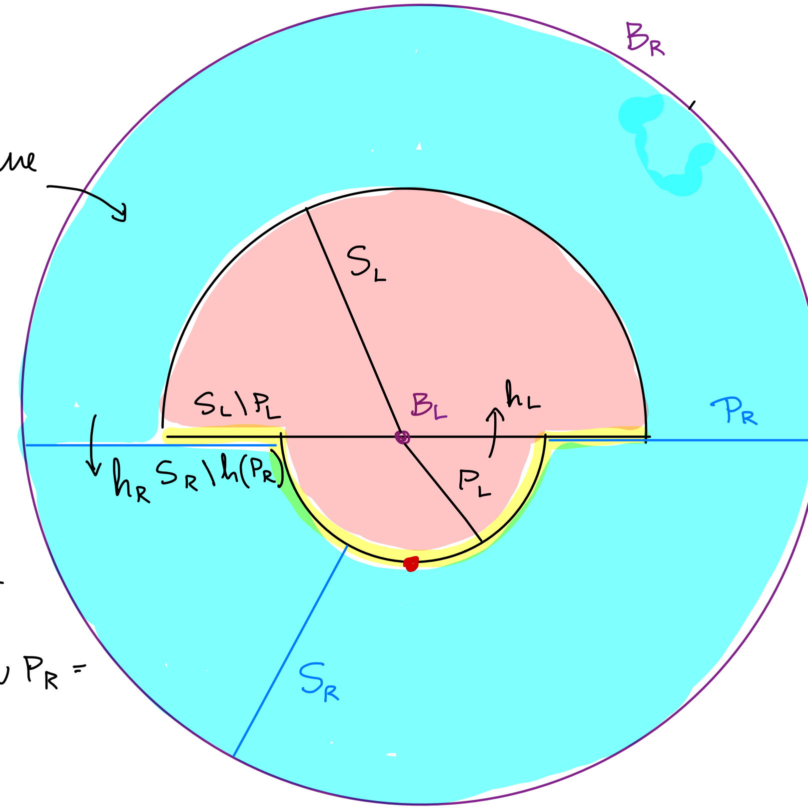


One can define supported ctct str & then

Thm (HKM) $\quad$ contact str / contactomorphism $\longleftrightarrow$ POB / stabilisation

GLUING PARTIAL OPEN BOOKS

same schematic picture inside-out



new page: $S_L \cup P_R$

monodromy:

$$S_L\cup P_R \xrightarrow{h_R} S_L \cup \overset{\overset{S_R}{\cup}}{h_R(P_R)} = P_L \cup (S_L\setminus P_L) \cup h_R(P_R) \xrightarrow{\text{glue}}$$

$$\to P_L \cup (S_R\setminus h_R(P_R)) \cup h_R(P_R) \to h(P_L) \cup S_R =$$

$$= h(P_L) \cup (S_R\setminus P_R) \cup P_R \xrightarrow{\text{glue}} h(P_L) \cup S_L\setminus h(P_L) \cup P_R =$$

$$= S_L \cup P_R$$

Fix: glue a "standard" POB of $\partial M \times I$ in between

$\rightsquigarrow$ FOBs exactly do this

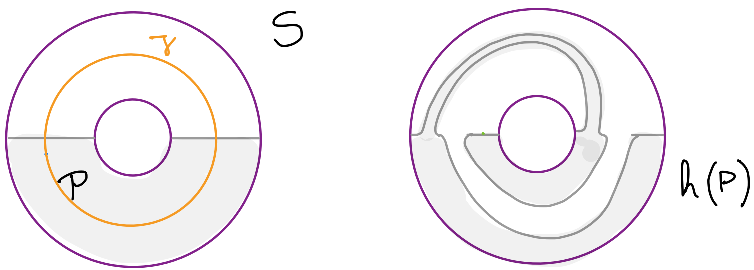
Relation to Partial Open Books (POB $\rightsquigarrow$ FOB)

Thm (Licata-V): (S, P, h) compatible w/ (M, ξ, Γ) $\mathcal{F}$ divided by Γ

$\Longrightarrow \exists$ $S_i \subseteq S$ w/ $S_0 = h(P)$ $S_k = S$ $S_{2k} = P$

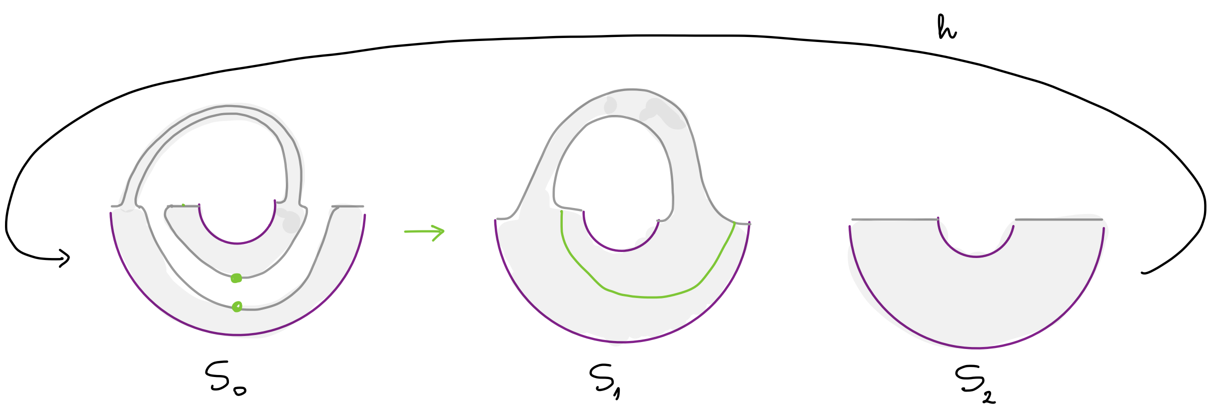
& $(S_0, \dots, S_{2k}, h)$ FOB compatible w/ $(M, \xi, \mathcal{F})$

e.g. POB



choice of handle decomposition for $S \setminus P$ & $S \setminus h(P)$ determines $\mathcal{F}$

FOB



Relation to Partial Open Books (FOB $\rightsquigarrow$ POB)

Thm (Licata-V): $(S_0, \dots, S_{2k}, h)$ FOB compatible w/ $(M, \xi, \mathcal{F})$

$\Longrightarrow$ can stabilise it sufficiently to $(S_0^{st}, \dots, S_{2k}^{st}, h^{st})$

s.t. (S', P', h') supports (M, ξ, Γ) & $S' = S_0$, $P' \overset{i}{\hookrightarrow} S_i$ & $h' = h|_{i_{2k}(P')}$

dividing curve for $\mathcal{F}$

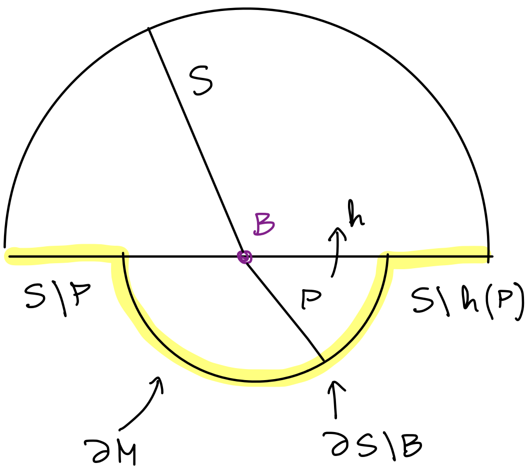
In summary:

$$\{\text{FOB for } (M, \xi, \mathcal{F})\}/_{\text{stab}} \longleftrightarrow \{\text{POB for } (M, \xi, \Gamma)\}/_{\text{stab}}$$

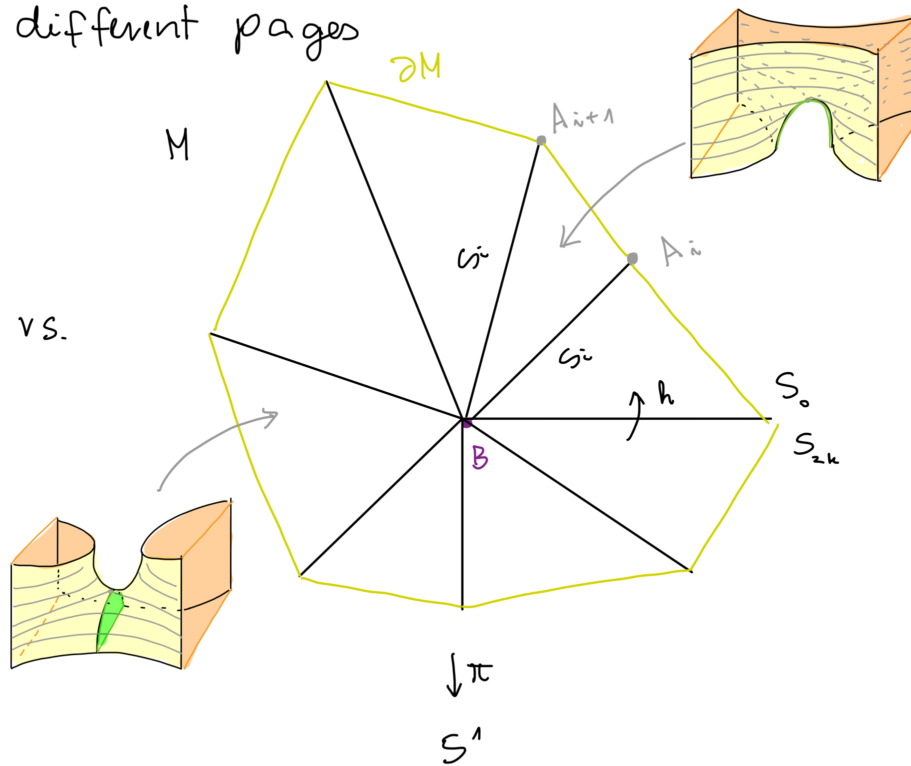
Relation to Partial Open Books

POBs have "big" (S) & "small" (P) pages.

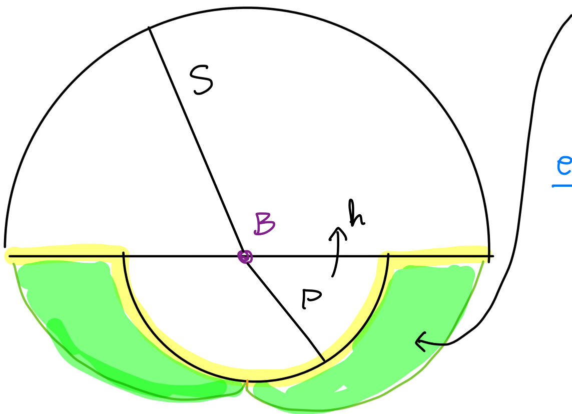
FOBs have lots of different pages



vs.

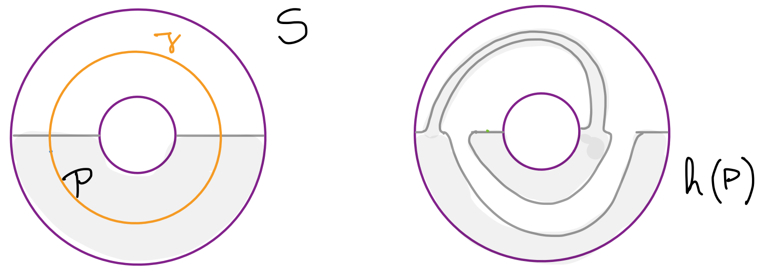


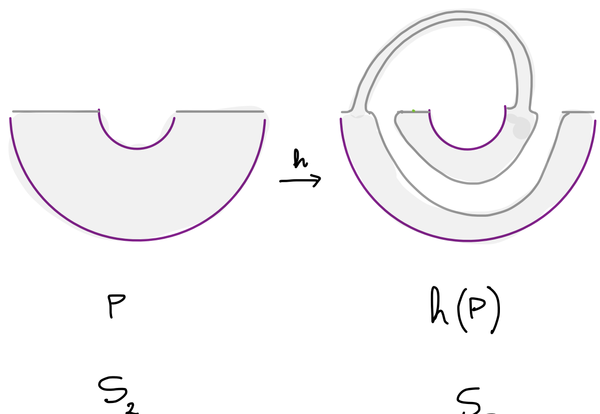
POB ⇝ FOB



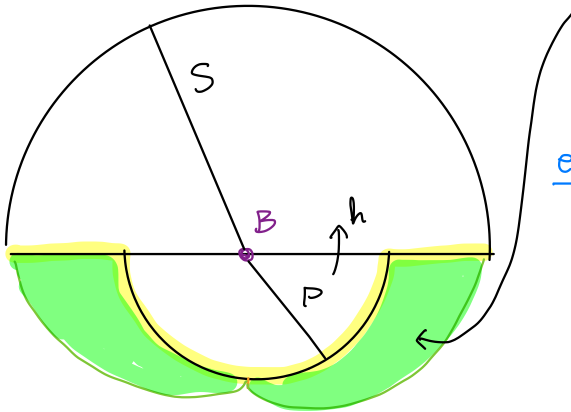
add an extra piece to interpolate between the pages :

e.g. :



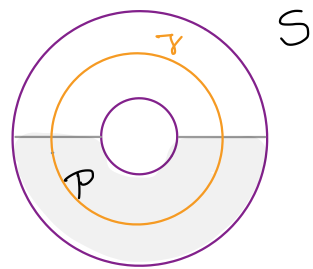


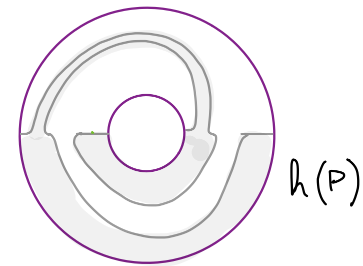
POB $\rightsquigarrow$ FOB

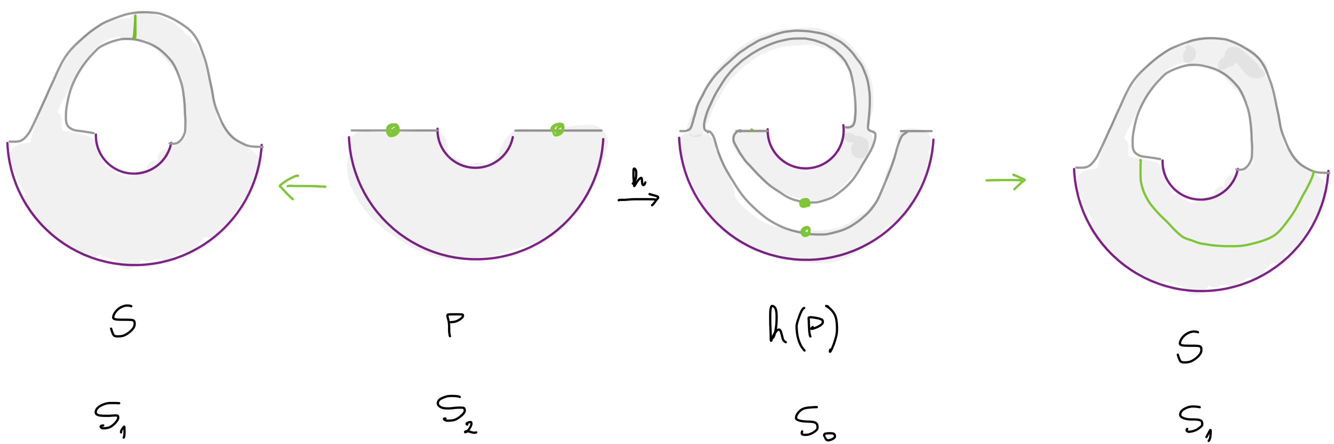


add an extra piece to interpolate between the pages:

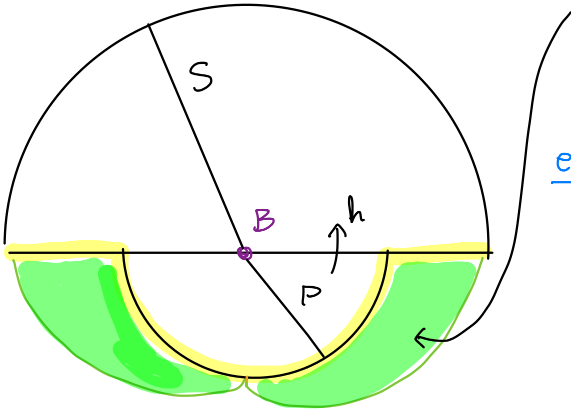
e.g.:





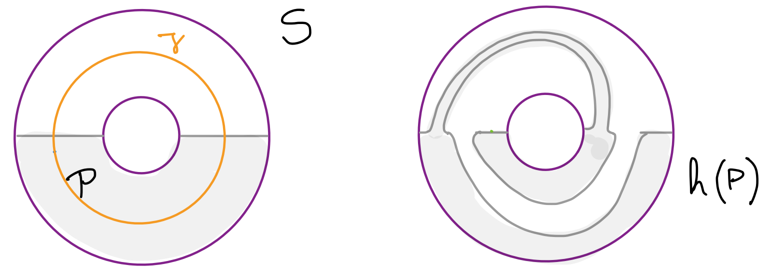


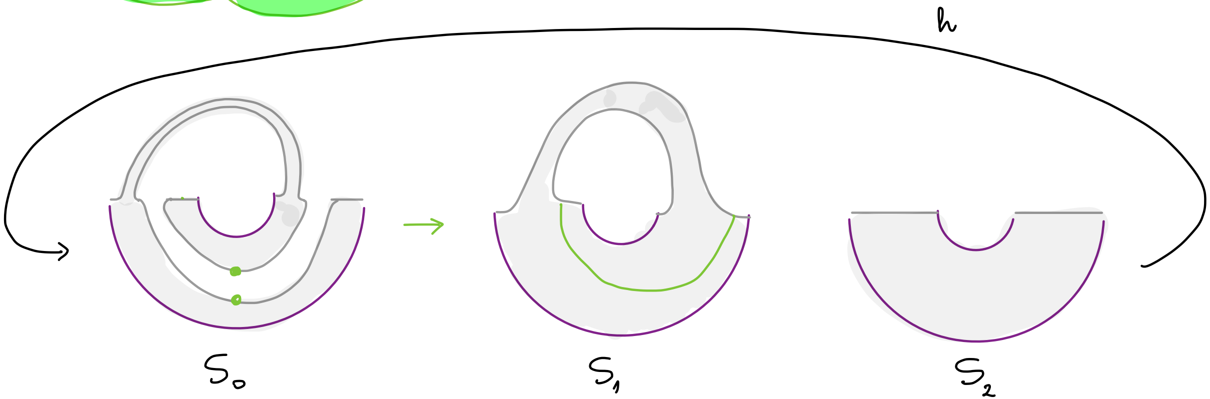
POB ⇝ FOB



add an extra piece to interpolate between the pages :

e.g. :





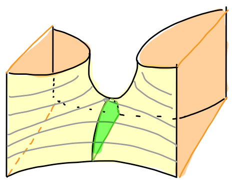
the resulting FOB & $\mathcal{F}$ on ∂M depends on the choices of handles

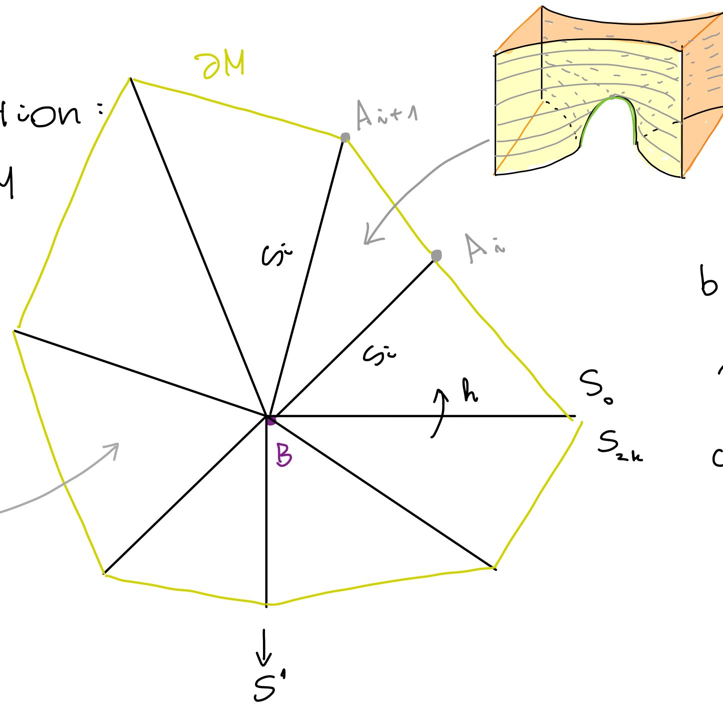
Prop (Licata - V) We can acchieve any $\mathcal{F}$ divided by Γ

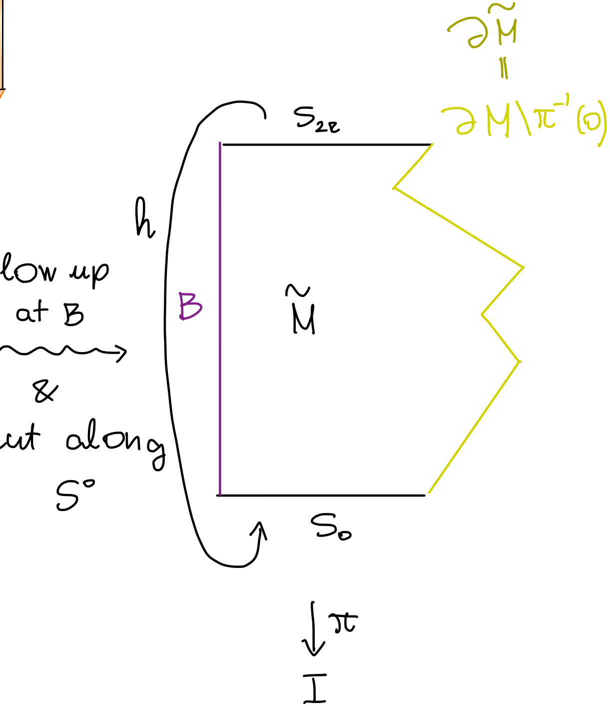
FOB → POB

easier notation:

M



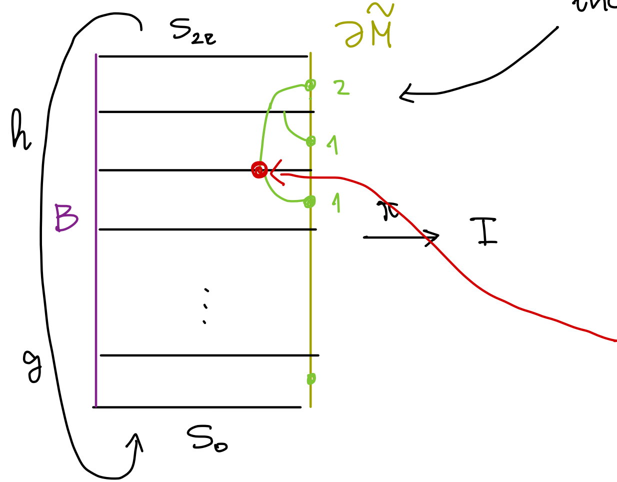


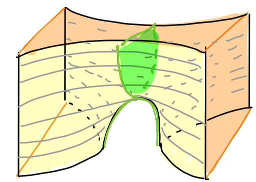
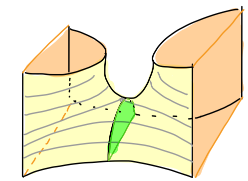


Want to identify a "biggest" & "smallest" page

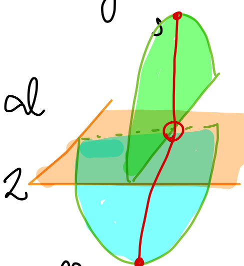
idea: use Morse theory: choose gradient like vectorfield $\nabla\pi$ & follow the flowlines.

FOB → POB



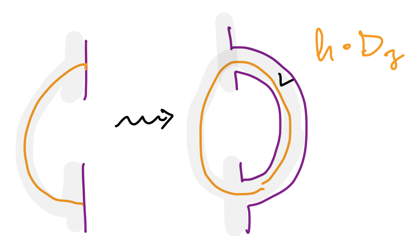
index 1  & 2  critical pts

- draw the 2-dim critical submanifold & try to continue in the other layers
- we cannot continue if the critical submanifolds of an index 1 & 2



Prop (Licata-V): We can always stabilise the FOB to avoid all the intersections

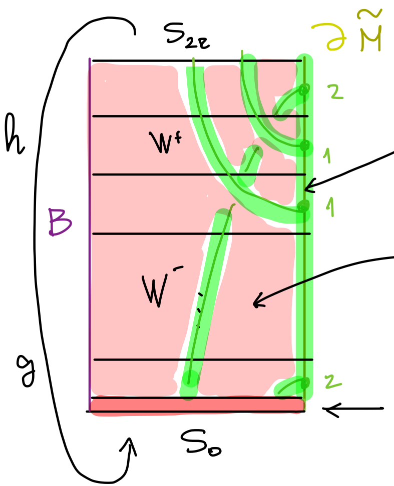
stabilisation: add an extra handle to each page



! Depends on which page we stabilise !

FOB → POB

Assume that the ascending & descending submanifolds do not intersect



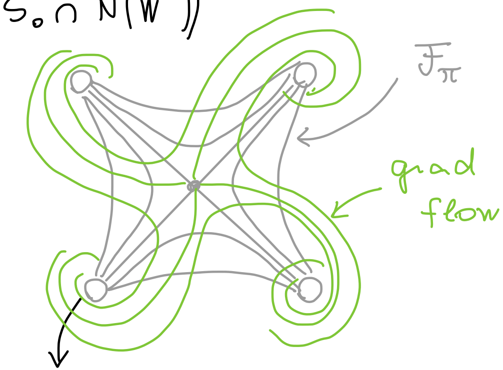
$\cong \partial\tilde{M} \times I$

$\cup$ of descending manifolds

$$P_0 \overset{\text{def}}{=} S_0 \setminus \big(S_0 \cap N(\partial M)\big) \cup \big(S_0 \cap N(W^-)\big)$$

$\cong P_0 \times I$

$S_0 \times I$

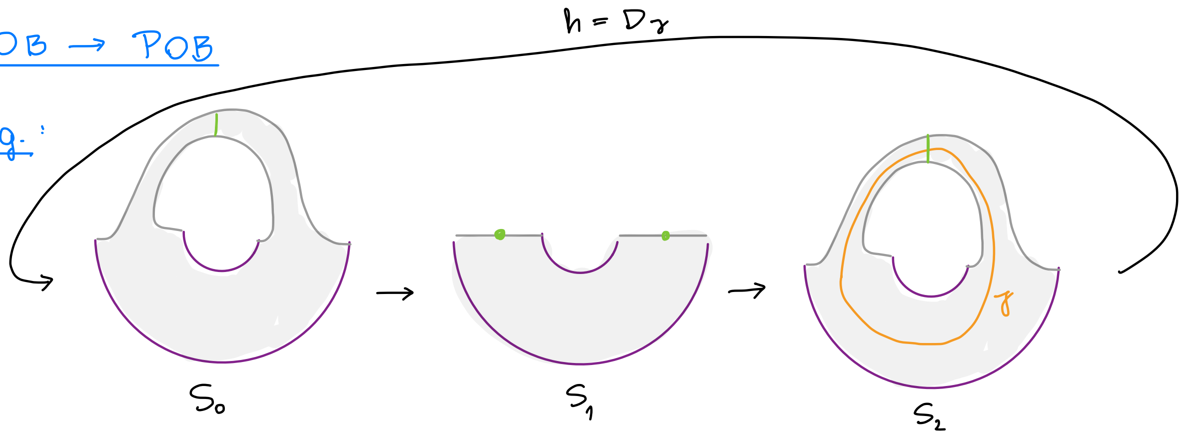


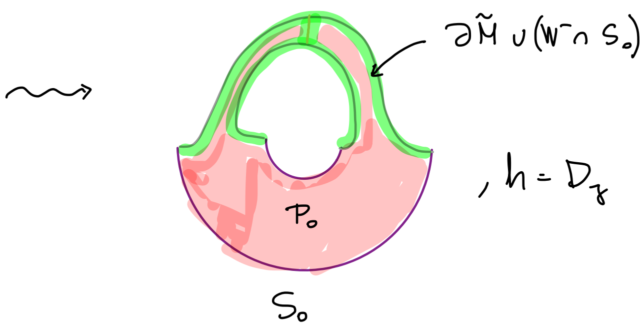
Thm (Licata - V) If the gradient flow on $\partial\tilde{M}$ is "adapted" to $\mathcal{F}_\pi$ then (after enough stabilisations) $(S_0, P_0, h|_{P_0})$ is a POB for (M, ξ).

"sorted FOB"

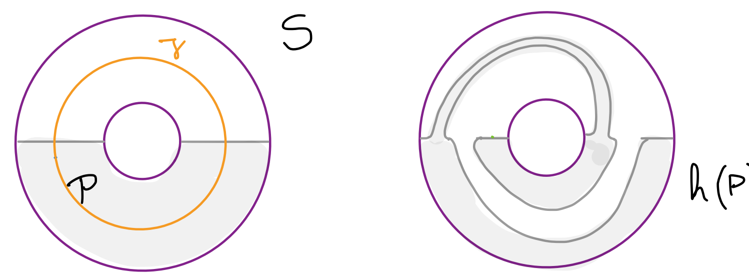
FOB → POB

e.g.:



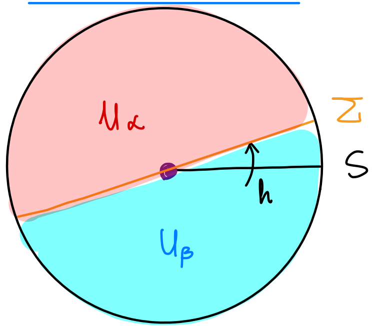


$, h = D_\gamma$:

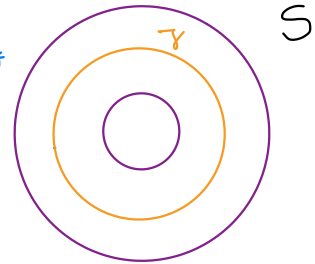


Contact Invariant in Heegaard Floer Homology

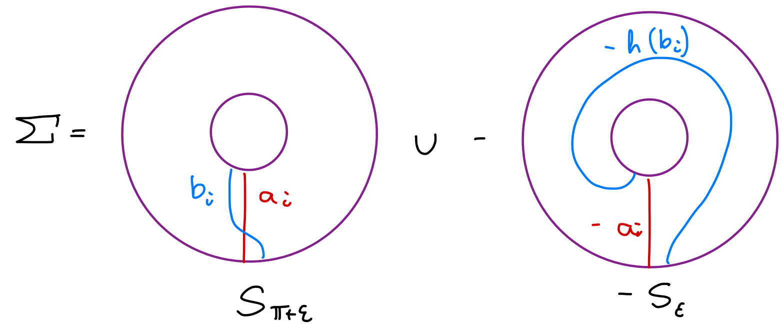
- closed case



e.g



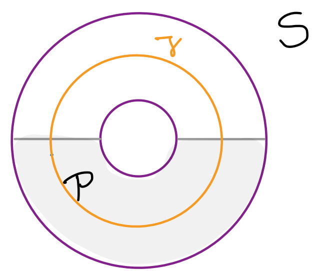
$\Sigma' =$

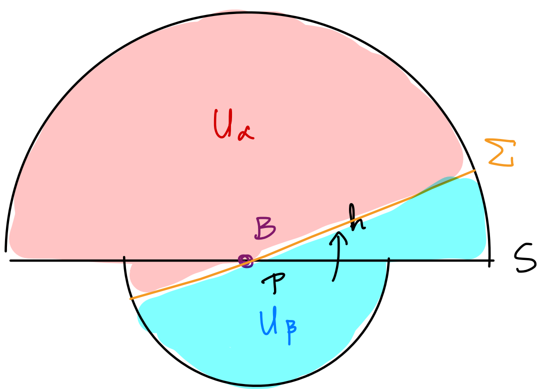


$\rightsquigarrow \boxed{[x] \in HF(-M)}$

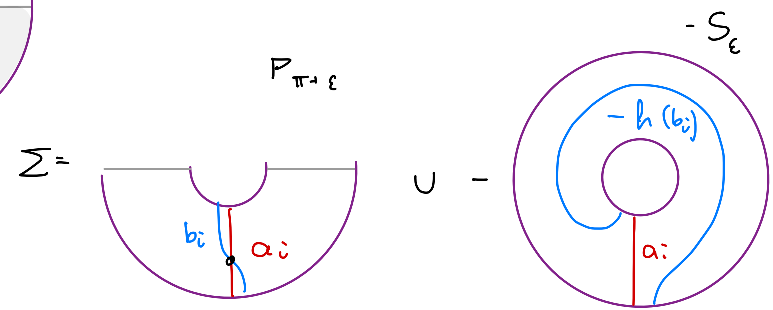
- partial case

e.g.:





$\Sigma =$



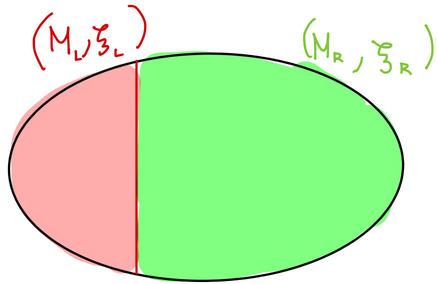
$\rightsquigarrow \boxed{[x] \in SFH(-M, \Gamma)}$

Contact Invariant in Bordered Floer Homology – Properties

Thm: (AFHLPV) $c_A(\xi) \in CFA(-M)$ & $c_D(\xi) \in CFD(-M)$

are well-defined elements (up to A_∞-homotopy equivalence)

Thm: (AFHLPV)

$(M, \xi) =$ 

$\Longrightarrow c_A(\xi_L) \tilde{\otimes} c_D(\xi_R) = c(\xi)$

$\mathcal{F}$ ⇝ boundary parametrisation

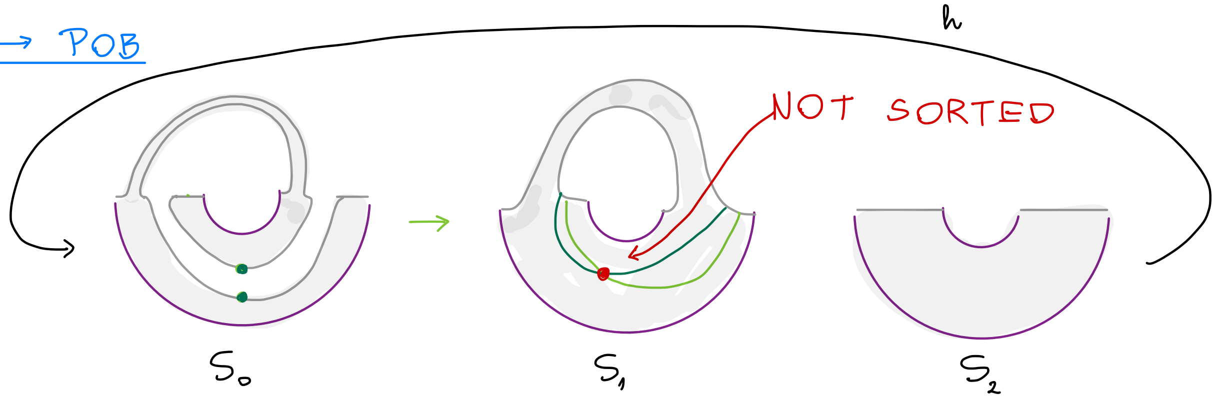
Thm: (AFHLPV)
$$\begin{array}{ccc} CFA(-M) & \xrightarrow{\text{R. Zarev}} & SFH(-M) \\ c_A(\xi) & \longmapsto & EH(\xi) \end{array} \quad \& \quad \begin{array}{ccc} CFD(-M) & \xrightarrow{\text{R. Zarev}} & SFH(-M) \\ c_D(\xi) & \longmapsto & EH(\xi) \end{array}$$

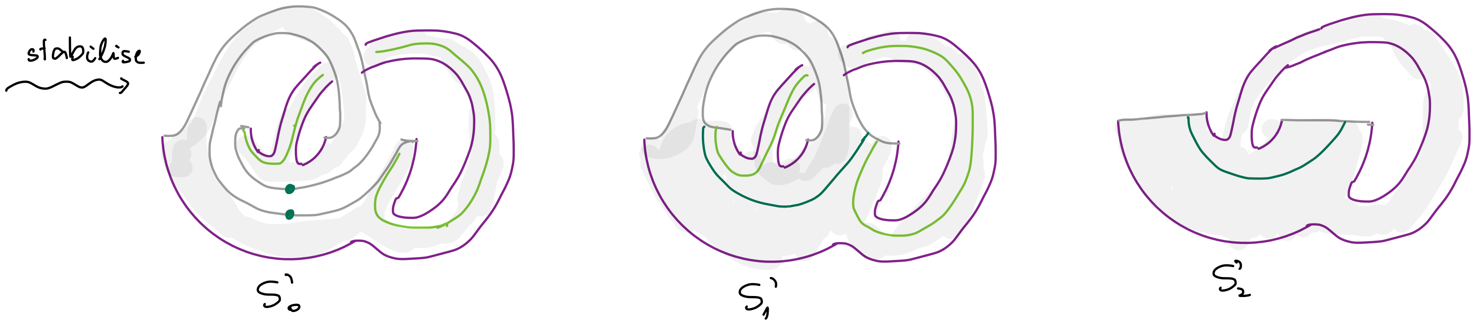
Moreover:
$$\begin{tikzcd} CFA(-M_L) \arrow[d, "\text{Zarev}"] & \tilde{\otimes} & CFD(-M_R) \arrow[d, "\text{Zarev}"] & = & CF(M) \arrow[d, dashed, "\text{homology}"] \\ SFH(-M_L) & \otimes & SFH(-M_R) \arrow[rr, "\text{HKM}"] & & HF(M) \end{tikzcd}$$
$c_A(\xi_L)$, $c_D(\xi_R)$, $c(\xi)$ commutes

$EH(\xi_L)$, $EH(\xi_R)$, $c(\xi)$

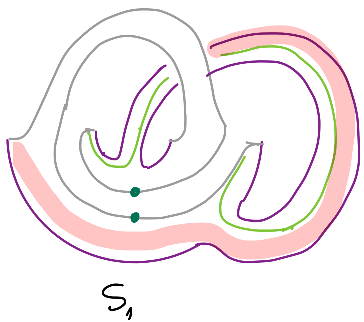
FOB → POB

e.g.:





POB



INTRODUCTION

New tools to study contact structures w/ boundary.

- w/o bdry: open books (Giroux)
- w/ bdry there was already one: partial open books (Honda-Kazez-Matić)

Why do we care?

- gluable
- more natural
- & has Applications: → Thm (V): the support norm (= minimal contact genus) is additive for tight contact structures

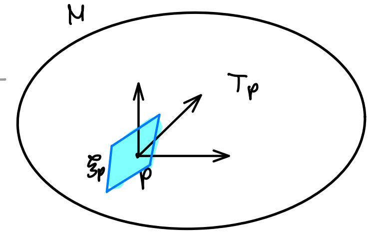
  → bordered contact invariant (A. Alishahi, V. Földvári, K. Hendricks, I. Licata, V)

  → ...

Contact Structures

(M^3, ξ) non-integrable 2-plane distribution

$\nexists\ S^2 \subset M^3$ s.t. $\xi = TS^2$



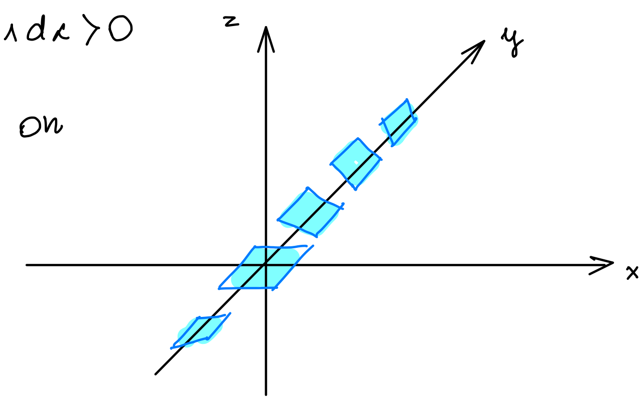
locally: $\xi = \ker \alpha$ ← 1 form on M

non-integrability of $\xi \iff \alpha \wedge d\alpha \neq 0$

We will work with cooriented contact structures:
- $\xi = \ker \alpha$ globally
- $\alpha \wedge d\alpha > 0$

Darboux Thm: locally contact structures are modelled on

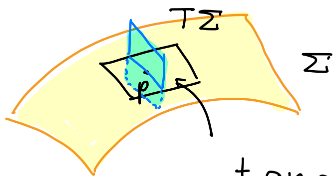
$(\mathbb{R}^3, \xi_{st} = \ker(dz - y\,dx))$



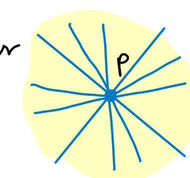
Characteristic Foliations

$\Sigma \hookrightarrow (M,\xi)$ embedded surface $\rightsquigarrow$ $\xi_p \cap T_p\Sigma$ induces the *characteristic foliation* $\mathcal{F}_\xi$

(more precisely $\mathcal{F}_\xi$ is given by $\alpha|_\Sigma$)

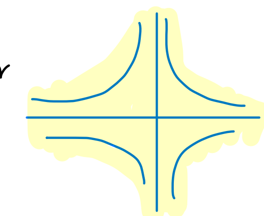


Fact (Giroux): if $\xi_p = T_p\Sigma \Rightarrow$ near p $\mathcal{F}_\xi$ is either



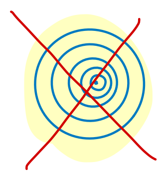
elliptic point

or



hyperbolic point

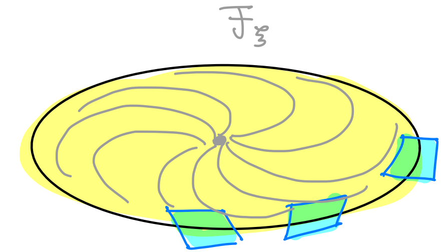
So it cannot be e.g.



$(\Leftarrow d(\alpha|_\Sigma) \neq 0)$

Thm (Giroux): $\mathcal{F}_\xi$ determines ξ in a nbhd of Σ.

TIGHT - OVERTWISTED DICHOTOMIE



overtwisted disc : $D^2 \hookrightarrow (M,\xi)$ w/ $T(\partial D^2) \subset \xi$

(M,ξ) → Overtwisted
(M,ξ) → tight = not overtwisted

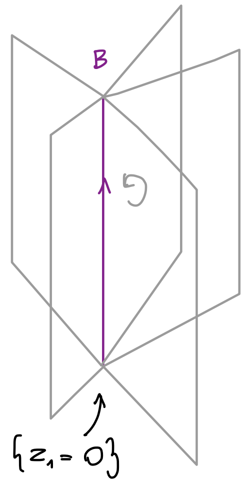
Thm (Eliashberg) (overtwisted ctct structures are flexible)

ξ_1 homotopic to ξ_2 & OT $\Longrightarrow$ $\xi_1 \overset{isot}{\sim} \xi_2$

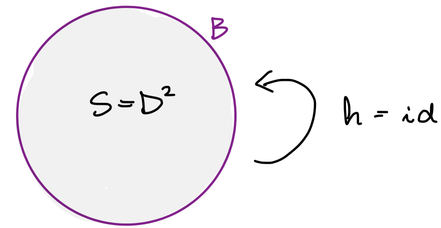
EXAMPLES

1) $S^3 \subseteq \mathbb{C}^2$
(z_1, z_2)

$$\pi = \frac{z_1}{|z_1|}$$

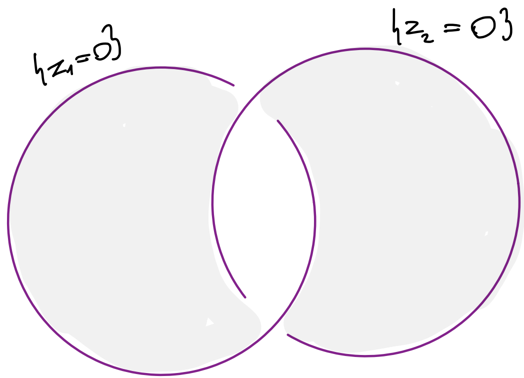


$\rightsquigarrow$

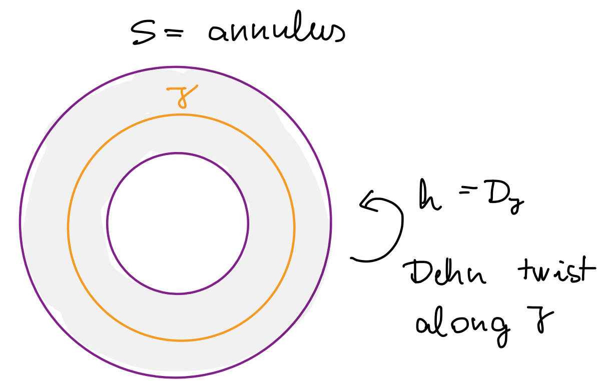


2) $S^3 \subseteq \mathbb{C}^2$

$$\pi = \frac{z_1 z_2}{|z_1 z_2|}$$



$\rightsquigarrow$



3) $\pi = \frac{z_1 \overline{z_2}}{|z_1 \overline{z_2}|}$ $\quad$ (annulus, D_γ^{-1})