there exist $f \in \mathcal{C}^\infty(\mathbb{R}^3)$ such that $Lu = f$ has no distributional solution in any open set $\Omega \subseteq \mathbb{R}^3$. For proofs see [Agm10] (non-solvability in L^2), [Joh78] (non-solvability in Hölder-continuous functions) and, for the ultimative treatment [Hör63].
Before heading to the theorem we briefly discuss fundamental solutions in $\mathscr{S}'$.

7.12. Remark (On $\mathscr{S}'$-fundamental solutions) We consider a linear PDO $P(D)$ with constant coefficients and with $P(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. Then $1/P(\xi) \in \mathcal{O}_M(\mathbb{R}^n) \subseteq \mathscr{S}'(\mathbb{R}^n)$. Then a fundamental solution is given by $E = \mathcal{F}^{-1}(1/P) \in \mathscr{S}'$. Indeed we have

$$P(D)E = P(D)\mathcal{F}^{-1}\Big(\frac{1}{P}\Big) = \mathcal{F}^{-1}\Big(\underbrace{P(\xi)\frac{1}{P(\xi)}}_{=1}\Big) = \delta. \tag{7.40}$$

Moreover, in this case E is the only fundamental solution in $\mathscr{S}'$ since we have

$$P(D)E = \delta \quad \Rightarrow \quad P(\xi)\hat{E} = 1 \quad \overset{1/P\in\mathcal{O}_M}{\Rightarrow} \quad \hat{E} = \frac{1}{P}. \tag{7.41}$$

7.13. Theorem (Malgrange-Ehrenpreis) Let $P(D) \not\equiv 0$ be a linear PDO with constant coefficients. Then $P(D)$ has a fundamental solution in $\mathcal{D}'$, i.e.,

$$\exists E \in \mathcal{D}'(\mathbb{R}^n): \quad P(D)E = \delta. \tag{7.42}$$

We will actually give a simple and explicit form of E following the approach of [Wag09]. We first need a preparatory statement from linear algebra, which we (funnily) prove by complex analysis methods.

7.14. Lem Let $\lambda_0, \ldots, \lambda_m \in \mathbb{C}$ be pairwise different. Then the unique solution of the linear system of equations

$$\sum_{j=0}^{m} a_j \lambda_j^k = \begin{cases} 0 & (1 \leqslant k \leqslant m-1) \\ 1 & (k = m) \end{cases} \tag{7.43}$$

is given by $a_j = \prod_{l=0, l\neq j}^{m} (\lambda_j - \lambda_l)^{-1}$.

Proof. The coefficient matric of the system is just Vandermonde's matrix and since its determinant $\prod_{0\leqslant j<k\leqslant 0}(\lambda_j - \lambda_k)$ does not vanish for pairwise different λ's the solution vector $(a_0, \ldots, a_m) \in \mathbb{C}^{m+1}$ is uniquely determined.
Next we set $p(z) = \prod_{l=0}^{m}(z - \lambda_l)$ and calculate the following complex contour integral over a circle of radius $N > |\lambda_j|$ for all j using the residue theorem (observe that by our

assumption all λ_j are simple poles)

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{|z|=N}\frac{z^k}{p(z)} &= \sum_{l=0}^{m}\operatorname{Res}\Big(\frac{z^k}{p(z)},\lambda_l\Big) = \sum_{l=0}^{m}\lim_{z\to\lambda_l}(z-\lambda_l)\,\frac{z^k}{\prod_{l=0}^{m}(z-\lambda_l)}\\ &= \sum_{l=0}^{m}\frac{\lambda_j^k}{\prod_{l=0,l\neq j}^{m}(\lambda_j-\lambda_l)} = \sum_{j=0}^{m}a_j\lambda_j^k.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\int\limits_{|z|=N}\frac{z^k}{p(z)}\,dz &= \int\limits_0^{2\pi}\frac{N^k e^{ikt}}{\prod_{l=0}^{m}Ne^{it}(1-\frac{\lambda_l}{N}e^{-it})}\,iNe^{it}\,dt\\ &= iN^{k-m}\int\limits_0^{2\pi}\frac{e^{i(k-m)t}}{\prod_{l=0}^{m}(1--\frac{\lambda_l}{N}\,e^{-it})}\,dt \overset{(N\to\infty)}{\to} \begin{cases}0 & (1\leqslant k\leqslant m-1)\\ i\int_0^{2\pi}dt = 2\pi i & (k=m)\end{cases}\end{aligned}$$

and we are done. □

Proof of Theorem 7.13. We write down explicitly a fundamental solution E of the PDO P which we assume to be of order m: Let $\eta\in\mathbb{R}^n$ be such that $\sigma_p(\eta)\neq 0$, choose pairwise different real numbers $\lambda_0,\dots,\lambda_m$ and set $a_j = \prod_{l=0,l\neq j}^{m}(\lambda_j-\lambda_l)^{-1}$ (cf. Lemma 7.14). Finally define

$$E = \frac{1}{\overline{\sigma_P(2\eta)}}\sum_{j=0}^{m}a_j e^{\lambda_j\eta x}\,\mathcal{F}_\xi^{-1}\left(\frac{\overline{P(i\xi+\lambda_j\eta)}}{P(i\xi+\lambda_j\eta)}\right). \tag{7.44}$$

First note that for any fixed $\lambda\in\mathbb{R}$ the set

$$N = \{\xi\in\mathbb{R}^n:\ P(i\xi+\lambda\eta = 0\} \tag{7.45}$$

has Lebesgue measure zero. In fact, applying a linear transformation we may assume that $\sigma_p(1,0,\dots,0)\neq 0$ and then

$$\int\limits_N d\xi = \int\limits_{\mathbb{R}^{n-1}}(\int\limits_{N_{\xi'}}d\xi_1)d\xi' = 0 \tag{7.46}$$

by Fubini's theorem and the fact that the sets $N_{\xi'} = \{\xi_1\in\mathbb{R}:\ P(i(\xi_1,\xi')+\eta\lambda) = 0\}$ are finite for $\xi' = (\xi_2,\dots,\xi_n)\in\mathbb{R}^{n-1}$. So

$$S(\xi) := \frac{\overline{P(i\xi+\lambda_j\eta)}}{P(i\xi+\lambda_j\eta)}\ \in L^\infty(\mathbb{R}^n)\subseteq\mathscr{S}'(\mathbb{R}^n) \tag{7.47}$$

and (7.44) is meaningful.

Now we calculate $P(\partial)E$ using the exchange formulae. We have

$$\begin{aligned} P(\partial)(e^{\lambda_j\eta x}\mathcal{F}_\xi^{-1}S) &= e^{\lambda_j\eta x}P(\partial+\lambda_j\eta)\mathcal{F}_\xi^{-1}S \\ &= e^{\lambda_j\eta x}\mathcal{F}_\xi^{-1}\Big(P(i\xi+\lambda_j\eta)S\Big) = e^{\lambda_j\eta x}\mathcal{F}_\xi^{-1}\left(\overline{P(i\xi+\lambda_j\eta)}\right) \end{aligned} \tag{7.48}$$

and

$$\mathcal{F}_\xi^{-1}(\overline{P(i\xi+\lambda_j\eta)}) = \mathcal{F}_\xi^{-1}\left(\overline{P}(-i\xi+\lambda_j\eta)\,1\right) = \overline{P}(-\partial+\lambda_j\eta)\;\delta. \tag{7.49}$$

So we finally obtain

$$\begin{aligned} P(\partial)\left(e^{\lambda_j\eta x}\mathcal{F}_\xi^{-1}\left(\frac{\overline{P(i\xi+\lambda_j\eta)}}{P(i\xi+\lambda_j\eta)}\right)\right) &= e^{\lambda_j\eta x}\overline{P}(-\partial+\lambda_j\eta)\,\delta \;=\; \overline{P}(-\partial+2\lambda_j\eta)(e^{\lambda_j\eta x}\delta) \\ &= \overline{P}(-\partial+2\lambda_j\eta)\delta \;=\; \lambda_j^m\overline{\sigma_P(2\eta)}\delta+\sum_{k=0}^{m-1}\lambda_j^kT_k \end{aligned} \tag{7.50}$$

for certain distributions $T_k \in \mathscr{E}'(\mathbb{R}^n)$. In the last equality we have used the homogeneity of the principal symbol (cf. (6.28)). Now by our choice of the a_j we are done thanks to Lem 7.14. $\square$

§ 7.3. HYPOELLIPTICITY OF PDO WITH CONSTANT COEFFICIENTS

7.15. Motivation In section 6.4 we have introduced the fundamental notion of hypoellipticity and highlighted its importance. In this short section we show that hypoellipticity of constant coefficient PDO is characterized by properties of its fundamental solution. The main idea is similar to the one in the third step in the proof of Thm. 6.45.

7.16. THM (Characterizing hypoellipticity) Let $P(D) \not\equiv 0$ be a PDO with constant coefficients. Then P is hypoelliptic iff it has a fundamental solution E with $\operatorname{singsupp}(E) \subseteq \{0\}$.

7.17. Rem If one (and hence both) conditions in the theorem hold true then every fundamental solution E of P has the property that its singular support is contained in the origin. Indeed by Remark 7.7 two fundamental solutions only differ by a solution h of $P(D)h = 0$, which by hypoellipticity is smooth.

Proof of Theorem 7.16. Suppose that P is hypoelliptic. Set $\Omega := \mathbb{R}^n \setminus \{0\}$ and let E be some fundamental solution of P (which exists by theorem 7.13). We have $P(D)E = \delta$ and hence $P(D)E = 0$ in Ω. But now hypoellipticity guarantees E to be smooth in Ω hence $\operatorname{singsupp}(E) \subseteq \{0\}$.

To prove the converse direction we use a slimmed down version of the proof of Thm. 6.45. Indeed we have (6.29) with $\rho = 0$ and the regularity property established there in step 2 now holds by assumption. Therefore we can directly jump to step 3 there and repeat the argument to establish hypoellipticity. □

7.18. Ex (Hypoellipticity of $\partial_x + a\partial_y$) We consider the operator $P(D) = \partial_x + a\partial_y$ on $\mathbb{R}^2$ already encountered in Ex. 7.9.

(i) If $a \in \mathbb{R}$ then we have constructed the fundamental solution $E = H(x)\delta(y - ax)$ which clearly has $\operatorname{singsupp}(E) = \{(x, ax) : x \geqslant 0\}$. Since $P(D)E = \delta$ we have $P(D)E$ smooth on $\Omega := \mathbb{R}^2 \setminus \{0\}$ but clearly $E \notin \mathcal{C}^\infty(\Omega)$, so $P(D)$ is *not* hypoelliptic.

(ii) If $a \in \mathbb{C} \setminus \mathbb{R}$ we have found the fundamental solution $E = \frac{1}{2\pi i} \frac{1}{y - ax}$ which clearly is smooth outside $(0, 0)$ and so by 7.16 $P(D)$ is hypoelliptic.

Summing up we have seen that

$$P(D) = \partial_x + (\alpha + i\beta)\partial_y \quad \text{is hypoelliptic iff } \beta \neq 0. \tag{7.51}$$