7 FUNDAMENTAL SOLUTIONS

7.1. Intro In this final chapter of the lecture course we intrdouce and discuss in some detail the notion of fundamental solutions of linear PDO. Given such an operator $P(x, D)$ these are distributions E_y such that

$$P(x, D)\, E_y \;=\; \delta_y. \tag{7.1}$$

After properly introducing this notion is in section 7.1 we will see that fundamental solutions play an important role both in *solving* PDE and in deriving the *regularity* properties of the solutions. In 7.2 we prove the Malgrange-Ehrenpreis theorem which asserts that any linear PDO with constant coefficients possesses a fundamental solution. In section 7.3 we characterize hypoelliptic PDO with constant coefficients via the properties of their fundamental solutions. Finally in section 7.4 we explicitly calculate fundamental solutions for some well-known PDO.

§ 7.1. BASIC NOTIONS

7.2. Motivation (What are fundamental solutions and what are they good for?) The basic idea of fundamental solutions can be phrased in physical terms as follows. Let us consider electromagnetic fields as solutions of the Maxwell system of PDE. To begin with we derive the solution E_y for the point charge (cf. 0.4) at $y \in \mathbb{R}^3$, i.e.,

$$PE_y = \delta_y. \tag{7.2}$$

Next we consider a arbitrary charge density f as a "superposition" of point charges, i.e., $f \approx \int f(y)\delta_y\,dy$. Then we should expect the solution u to be a "superposition" of the E_y's, i.e.,

$$u \approx \int f(y)E_y\,dy. \tag{7.3}$$

A slightly more mathematical interpretation would be ($\varphi \in \mathcal{D}$)

$$\langle u, \varphi\rangle \approx \int\int f(y)E_y(x)\varphi(x)\,dy\,dx. \tag{7.4}$$

We will give an exact version of (7.4) in Proposition 7.6 below.

7.3. DEF (Fundamental solution) Let $P(x, D)$ be a linear PDO with coefficients in $\mathbb{C}^\infty(\Omega)$ and let $y \in \Omega$. A distribution $E_y \in \mathcal{D}'(\Omega)$ is called a fundamental solution of $P(X, D)$ at y if

$$P(x, D)E_y = \delta_y. \tag{7.5}$$

7.4. REM & DEF (The case of constant coefficients) If $P(x, D) = P(D)$ is a linear PDO with constant (complex) coefficients and E_0 is a fundamental solution for $P(D)$ at 0 then

$$E_y := \tau_y E_0 \tag{7.6}$$

is a fundamental solution of $P(D)$ at y. Indeed we have

$$P(D)E_y = P(D)\tau_y E_0 = \tau_y P(D)E_0 \overset{(7.5)}{=} \tau_y\delta \overset{3.15(ii)}{=} \delta_y. \tag{7.7}$$

Hence by definition/convention we call $E \in \mathcal{D}'(\Omega)$ a <u>fundamental solution</u> of $P(D)$ if

$$P(D)E = \delta. \tag{7.8}$$

<u>7.5. REM</u> (The fundamental role of fundamental solutions) Let $P = P(D)$ be a PDO with constant coefficients and consider the PDE

$$Pu = f \tag{7.9}$$

with $f \in \mathcal{E}'$, $u \in \mathcal{D}'$. Moreover let E be a fundamental solution of P then we have the following two important equations

$$E * Pu = u \quad \text{and} \quad P(E * f) = f. \tag{7.10}$$

Indeed to derive the first of these equations just write

$$E * Pu \overset{4.5(ii)}{\overset{\downarrow}{=}} PE * u \overset{(7.8)}{\overset{\downarrow}{=}} \delta * u \overset{4.5(iv)}{\overset{\downarrow}{=}} u. \tag{7.11}$$

Similarly to derive the second equation in (7.10) we write

$$P(E * f) \overset{4.5(ii)}{\overset{\downarrow}{=}} (PE) * f \overset{(7.8)}{\overset{\downarrow}{=}} \delta * f \overset{4.5(iv)}{\overset{\downarrow}{=}} f. \tag{7.12}$$

Note that (7.10) just means that E acts as a left as well as a right inverse of P. These equations play a pivotal role as we shall discuss now.

(i) The second equation, in particular, implies *solvability* of the PDE (7.9) for any $f \in \mathcal{E}'$. We rephrase this important observation explicitly:

Let $P(D)$ be a PDO with constant coefficients and let E be a fundamental solution of P. Then the PDE $P(D)u = f$ has a solution $u \in \mathcal{D}'$ for all $f \in \mathcal{E}'$. It is given by $u = E * f$.

(ii) The first equation in (7.10) on the other hand allows to extract *regularity information* of the solution $u = E * f$ from the regularity of $f = Pu$.

A more general statement on solvability is given by the following statement.

<u>7.6. Prop</u> (Solvability using fundamental solutions) Let $P = P(x, D)$ be a linear PDO with $\mathcal{C}^\infty$-coefficients. Suppose that for all $y \in \Omega$ there is a fundamental solution $E_y \in \mathcal{D}'(\Omega)$ and

$$\forall \varphi \in \mathcal{D}(\Omega): \ y \mapsto \langle E_y, \varphi \rangle \quad \text{is } \mathcal{C}^\infty \text{ from } \Omega \text{ to } \mathbb{C} \tag{7.13}$$

$$\varphi \mapsto (y \mapsto \langle E_y, \varphi \rangle) \quad \text{is seq. continuous from } \mathcal{D}(\Omega) \text{ to } \mathcal{E}(\Omega). \tag{7.14}$$

Then for all $f \in \mathscr{E}'(\Omega)$ the PDE

$$P(x, D)u = f \tag{7.15}$$

has a solution $u \in \mathcal{D}'(\Omega)$. It is given by

$$\langle u, \varphi \rangle = \langle f(y), \langle E_y, \varphi \rangle \rangle. \tag{7.16}$$

Proof: [actually shorter than the statement] By (7.14) and the continuity of $f : \mathscr{E}(\Omega) \to \mathbb{C}$ we have that $u \in \mathcal{D}'(\Omega)$. Now for all $\varphi \in \mathcal{D}(\Omega)$ we have

$$\langle Pu, \varphi \rangle \overset{2.21}{\overset{\downarrow}{=}} \langle u, P^t \varphi \rangle \overset{(7.16)}{\overset{\downarrow}{=}} \langle f(y), \underbrace{\langle E_y, P^t \varphi \rangle}_{=\langle PE_y, \varphi\rangle \underset{(7.5)}{\underset{\uparrow}{=}} \langle \delta_y, \varphi\rangle = \varphi(y)} \rangle = \langle f, \varphi \rangle. \tag{7.17}$$

□

7.7. Remark (Difference of two fundamental solutions) Let $P = P(x, D)$ be a linear PDO with $\mathcal{C}^\infty$-coefficients and let E_y and F_y be two fundamental solutions of P at y. Then they differ only by a solution of the homogeneous equation, i.e., $E_y = F_y + h$ whith $h \in \mathcal{D}'$ and $Ph = 0$. Indeed we have

$$P(E_y - F_y) = PE_y - PF_y = \delta_y - \delta_y = 0. \tag{7.18}$$

7.8. Motivation (Main questions on fundamental solutions) Now that we have seen the fundamental importance of fundamental solutions (at least as the quest for existence of solutions of PDE is concerned), the most urging questions are the following:

(i) For which PDO do fundamental solutions exist?

(ii) How can we find a fundamental solution for a given PDO?

We will partly answer (i) in Section 7.2 and explicitly compute fundamental solutions for some prominent PDO in Section 7.4. We will, however, look at two instructing examples first.

7.9. Example On $\mathbb{R}^2$ we consider the operator

$$P = P(D_x, D_y) = \partial_x + a\partial_y = i(D_x + aD_y) \tag{7.19}$$

with $a \in \mathbb{C}$. We are looking for $E \in \mathcal{D}'(\mathbb{R}^2)$ such that

$$\partial_x E + a\partial_y E = \delta(x,y) = \delta(x) \otimes \delta(y). \tag{7.20}$$

We will use the partial Fourier transform

$$(\mathcal{F}_2\varphi)(x,\eta) := \mathcal{F}_{y\to\eta}(\varphi(x,.)) = \int_R e^{-iy\eta}\varphi(x,y)dy \tag{7.21}$$

which obviously is a well defined concept on $\mathscr{S}(\mathbb{R}^2)$ with the same resp. analogous properties as the Fourier transform itself. So we also have $\mathcal{F}_2$ on $\mathscr{S}'(\mathbb{R}^2)$ and for $E \in \mathscr{S}'(\mathbb{R}^2)$ we will write $\tilde{E} := \mathcal{F}_2(E)$.
Now acting with $\mathcal{F}_2$ on (7.20) gives

$$\partial_x \tilde{E} + ia\eta\tilde{E} \overset{5.26(i)}{\overset{\downarrow}{=}} \delta(x) \otimes \hat{\delta}(y) \overset{5.27(i)}{\overset{\downarrow}{=}} \delta(x), \tag{7.22}$$

which is an ODE for $\tilde{E}$ in the variable x with parameter η. We solve it via variation of constants, that is using the ansatz

$$\tilde{E}(x,\eta) = c(x,\eta)e^{-ia\eta x}, \tag{7.23}$$

where $c \in \mathscr{S}'(R^2)$. Observe that $e^{-ia\eta x}$ is a solution of the homogeneous version of equation (7.22). We obtain

$$\partial_x \tilde{E} = \partial_x c e^{-ia\eta x} - ia\eta\tilde{E} \overset{(7.22)}{\overset{\downarrow}{=}} \delta(x) - ia\eta\tilde{E} \tag{7.24}$$

and hence

$$\partial_x c = e^{ia\eta x}\delta(x) \overset{(2.6)}{\overset{\downarrow}{=}} \delta(x). \tag{7.25}$$

So by (2.2) we obtain

$$c = H(x) + d(\eta), \qquad d \in D'(\mathbb{R}) \tag{7.26}$$

and we will use the ansatz

$$\tilde{E}(x,\eta) = \Big(H(x) + d(\eta)\Big)\, e^{-ia\eta x}, \tag{7.27}$$

where d should be chosen such that for all $x \neq 0$ we have that $\tilde{E}(x,.) \in \mathscr{S}'(\mathbb{R})$.
We now discuss the cases $a \in \mathbb{R}$ and $a \in \mathbb{C} \setminus \mathbb{R}$ separately.

(i) $\boxed{a \in \mathbb{R}}$ We set $d = 0$ so that $\tilde{E}(x, \eta) = H(x)e^{-ia\eta x}$. Then $\tilde{E}$ is bounded and hence in $\mathscr{S}'(\mathbb{R}^2)$. So we find

$$E(x,y) = \mathcal{F}_2^{-1}(e^{-iax\cdot})(y)H(x) \overset{5.27(ii)}{=} \tau_{-ax}\delta(y) \otimes H(x). \tag{7.28}$$

[Indeed we have for $\varphi \in \mathcal{D}(\mathbb{R}^2)$

$$\begin{aligned} \langle E, \varphi\rangle &= \langle \mathcal{F}_2^{-1}\tilde{E}, \varphi\rangle = \langle \tilde{E}, \mathcal{F}_2^{-1}\varphi\rangle = \int_{\mathbb{R}^2} H(x)e^{-ia\eta x}\Big(\mathcal{F}_2^{-1}\varphi\Big)(x,\eta)\,dx\,d\eta \\ &= \int_{\mathbb{R}} H(x) \underbrace{\int_{\mathbb{R}} e^{-ia\eta x}\Big(\mathcal{F}_2^{-1}\varphi\Big)(x,\eta)\,d\eta}_{=\varphi(x,ax) \text{ by } 5.15}\,dx = \int_0^\infty \varphi(x,ax)\,dx. \quad] \end{aligned}$$

One often writes

$$E(x,y) = \delta(y - ax)H(x) \tag{7.29}$$

since it actually is the Lebesgue measure on $\{x > 0, y = ax\} \subseteq \mathbb{R}^2$. ♣ insert figure ♣

(ii) $\boxed{a \in \mathbb{C} \setminus \mathbb{R}}$ We write $a = \alpha + i\beta$ with $\beta \neq 0$. Then we distinguish the cases

$$\begin{aligned} x < 0: &\quad \tilde{E}(x,\eta) = d(\eta)e^{-i\alpha\eta x}\,e^{\beta x\eta} \\ x > 0: &\quad \tilde{E}(x,\eta) = \Big(1 + d(\eta)\Big)e^{-i\alpha\eta x}\,e^{\beta x\eta}. \end{aligned}$$

Now set $d(\eta) = -H(\beta\eta)$ to achieve $|\tilde{E}(x,\eta)| \leqslant Ce^{-|\beta x\eta|}$ and hence $\tilde{E} \in L^1(\mathbb{R}_\eta)$ for all fixed $x \neq 0$. Now we calculate in case $\boxed{x < 0}$

$$\begin{aligned} E(x,y) &= -\frac{1}{2\pi}\int_{\mathbb{R}} e^{iy\eta}H(\beta\eta)e^{-iax\eta}\,d\eta \\ &= -\frac{\operatorname{sign}(\beta)}{2\pi}\int_0^{\operatorname{sign}(\beta)\infty} e^{i\eta(y-ax)}\,d\eta \\ &\underset{\beta=\operatorname{Im}(a)\neq 0}{=} -\frac{\operatorname{sign}(\beta)}{2\pi}\,\frac{1}{i(y-ax)}\operatorname{sign}(\beta)(0-1) = \frac{1}{2\pi i}\,\frac{1}{y-ax}. \end{aligned}$$

We now make the following claim: $\boxed{\dfrac{1}{2\pi i}\,\dfrac{1}{y - ax} \in \mathscr{S}'(\mathbb{R}^2)}$

To prove the claim we start by observing that

$$y - ax = 0 \Leftrightarrow \ y - \alpha x = 0 \wedge \beta x = 0 \Leftrightarrow x = 0 \wedge y = \alpha x = 0 \Leftrightarrow (x, y) = (0, 0).$$

Hence we have for all $(x, y) \neq (0, 0)$

$$|y - ax| = r|y/r - ax/r| \geqslant \mu|(x, y)|, \tag{7.30}$$

where $r = \sqrt{x^2 + y^2} = |(x, y)|$ and μ is defined the minimum of $|y/r - ax/r|$ on S^1. So we write for $\varphi \in \mathscr{S}(\mathbb{R}^2)$

$$\begin{aligned} |\int_{\mathbb{R}^2} \frac{\varphi(x,y)}{y - ax}\, dx\, dy| &\leqslant \frac{1}{\mu} \int \frac{|\varphi(x,y)|}{|(x,y)|}\, dx\, dy \\ &\leqslant C \int \frac{(1 + |(x,y)|)^{-l}}{|(x,y)|}\, dx\, dy \\ &\leqslant C \int_0^\infty \int_{S^1} \frac{r\, d\theta\, dr}{r(1+r)^l} \leqslant \tilde{C} \qquad \text{for } l > n, \end{aligned} \tag{7.31}$$

which establishes the claim.

Similarly we calculate for $\boxed{x > 0}$

$$\begin{aligned} E(x,y) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{iy\eta} \big(1 - H(\beta\eta)\big) e^{-iax\eta}\, d\eta \\ &= \frac{\operatorname{sign}(\beta)}{2\pi} \int_{\operatorname{sign}(\beta)\infty}^{0} e^{i\eta(y - ax)}\, d\eta \\ &= \frac{\operatorname{sign}(\beta)}{2\pi} \frac{1}{i(y - ax)} (1 - 0) = \frac{1}{2\pi i} \frac{1}{y - ax}. \end{aligned} \tag{7.32}$$

Summing up we have derived that a fundamental solution of $\partial_x + a\partial_y$ is given by

$$E(x,y) = \begin{cases} H(x)\delta(y - \alpha x) & a \in \mathbb{R} \\ \frac{1}{2\pi i} \frac{1}{y - ax} & a \in \mathbb{C} \setminus \mathbb{R} \end{cases} \tag{7.33}$$

As an important special case we have for $a = i$, that is $\frac{1}{2}P = \frac{1}{2}(\partial_x + i\partial_y) = \frac{\partial}{\partial \bar{z}}$ the Cauchy-Riemann operator (cf. 6.44(ii)). It has a fundamental solution given by

$$F = 2E = \frac{1}{i\pi} \frac{1}{y - ix} = \frac{1}{\pi} \frac{1}{x + iy} = \frac{1}{\pi z} \qquad (z \in \mathbb{C}). \tag{7.34}$$

Exercise: Show that F really is a fundamental solution of the Cauchy-Riemann operator by directly calculating PF. Or more generally show by direct calculation that E is a fundamental solution of $P = \partial_x + a\partial_y$.
♣ insert later ♣

7.10. Example (An ordinary DO without a fundamental solution) On $\Omega = \mathbb{R}$ we consider

$$P\Big(x, \frac{d}{dx}\Big) = (1-x^2)^3 \frac{d}{dx} - 4x. \tag{7.35}$$

We claim that P has no fundamental solution at 0. Suppose to the contrary that there is $E \in \mathcal{D}'(\mathbb{R})$ with $(1-x^2)^3 E' - 4xE = \delta$. Then we have in $\mathbb{R} \setminus \{\pm 1, 0\}$

$$E' = \frac{4x}{(1-x^2)^3}\, E \;\Rightarrow\; E(x) = Ce^{\frac{1}{(1-x^2)^2}}. \tag{7.36}$$

Hence in any of the intervals $J_1 = (-\infty, -1)$, $J_2 = (-1, 0)$, $J_3 = (0, 1)$, and $J_4 = (1, \infty)$ we have $E_k = C_k e^{1/(1-x^2)^2}$ $(k = 1, 2, 3, 4)$. ♣ insert figure ♣
Next as in 2.30(i) it follows that

$$\begin{aligned} &\nexists v \in \mathcal{D}'(-\infty, 0):\ v|_{J_k} = E_k\ (k = 1, 2) \quad \text{with } C_1 \neq 0 \neq C_2 \\ &\nexists w \in \mathcal{D}'(-\infty, 0):\ w|_{J_k} = E_k\ (k = 3, 4) \quad \text{with } C_3 \neq 0 \neq C_4 \end{aligned}$$

So $E|_{\mathbb{R}\setminus\{0\}} = 0$ and hence $\mathrm{supp}(E) = \{0\}$ (since otherwise $E = 0 \Rightarrow PE = 0$). Therefore by 1.70 there is m such that

$$E = \sum_{j=0}^{m} \lambda_j\, \delta^{(j)} \qquad \text{with } \lambda_j \in \mathbb{C},\ \lambda_m \neq 0. \tag{7.37}$$

Now we may calculate

$$\begin{aligned} \delta &= (1-x^2)^3 \Big(\sum_{j=0}^{m} \lambda_j\, \delta^{(j)}\Big)' - 4x \sum_{j=0}^{m} \lambda_j\, \delta^{(j)} \\ &= \sum_{j=0}^{m} \lambda_j\, (1-x^2)^3\, \delta^{(j+1)} - \sum_{j=0}^{m} 4x\lambda_j\, \delta^{(j)} \\ &= \sum_{j=0}^{m} \lambda_j\, \delta^{(j+1)} + \underbrace{\sum_{j=0}^{m} \lambda_j\, (-3x^2 + 3x^4 - x^6)\delta^{(j+1)} - 4\sum_{j=0}^{m} \lambda_j\, x\, \delta^{(j)}}_{(*)}. \end{aligned} \tag{7.38}$$

The terms in $(*)$ are of the form $x^k\delta^{(l)}$ and as in formula $(***)$ in 2.30(i) one finds that they are distributions of order $l - k$. So these terms are at most proportional to $\delta^{(m-1)}$. So λ_m has to vanish and inductively also $\lambda_j = 0$ for all $j \in \{0, 1, \dots, m\}$. So $E = 0$, a contradiction.

§ 7.2. THE MALGRANGE-EHRENPREIS THEOREM

7.11. Intro In this paragraph we are going to answer the question raised in 7.8(i) for a large class of PDO. More precisely we state and prove the theorem of Malgrange and Ehrenpreis which asserts that any linear PDO with constant coefficients has a fundamental solution in $\mathcal{D}'$.
First proofs of this statement were independently given by Bernard Malgrange and Leon Ehrenpreis in 1953/54. We give a short historical overview of different proofs and results in this realm in the following table.

1911, Nils Zeilon: first definition of the notion of fundamental solution in L^1

1950/51 Laurent Schwartz: general definition of fundamental solutions

1953/54 Berndard Malgrange & Leon Ehrenpreis: existence of a fundamental solution for any PDO with constant coefficients in $\mathcal{D}'$ using the Hahn-Banach theorem

1957/58 Stanisław Łojasiewicz, Lars Hörmander: existence even in $\mathscr{S}'$

late 1950-ies Lars Hörmander, Jean-François Treves: constructive proofs, continuos dependence on the coefficients of the operator

mid 1990-ies Heinz König, Norbert Ortner and Peter Wagner: short explicit formulae

2009 Peter Wagner: very simple formula

The Malgrange-Ehrenpreis theorem may be regarded as one of the big successes of distribution theory and is a central result. However, soon after its proof it turned out that a generalization to the case of PDO with non-constant coefficients—which was actually regarded as a not-too-hard problem (Louis Nierenberg suggested it as a Ph.D topic to Jean-François Treves)—is *not* possible. Hans Lewy in 1957 gave the first counterexample which became very famous and now bears his name; explicitly for the operator

$$Lu(x, y, z) = -u_x - iu_y + 2i(x + iy)u_z \tag{7.39}$$

there exist $f \in \mathcal{C}^\infty(\mathbb{R}^3)$ such that $Lu = f$ has no distributional solution in any open set $\Omega \subseteq \mathbb{R}^3$. For proofs see [Agm10] (non-solvability in L^2), [Joh78] (non-solvability in Hölder-continuous functions) and, for the ultimative treatment [Hör63].
Before heading to the theorem we briefly discuss fundamental solutions in $\mathscr{S}'$.

7.12. Remark (On $\mathscr{S}'$-fundamental solutions) We consider a linear PDO $P(D)$ with constant coefficients and with $P(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. Then $1/P(\xi) \in \mathcal{O}_M(\mathbb{R}^n) \subseteq \mathscr{S}'(\mathbb{R}^n)$. Then a fundamental solution is given by $E = \mathcal{F}^{-1}(1/P) \in \mathscr{S}'$. Indeed we have

$$P(D)E = P(D)\mathcal{F}^{-1}\Big(\frac{1}{P}\Big) = \mathcal{F}^{-1}\Big(\underbrace{P(\xi)\frac{1}{P(\xi)}}_{=1}\Big) = \delta. \tag{7.40}$$

Moreover, in this case E is the only fundamental solution in $\mathscr{S}'$ since we have

$$P(D)E = \delta \quad \Rightarrow \quad P(\xi)\hat{E} = 1 \quad \overset{1/P \in \mathcal{O}_M}{\Rightarrow} \quad \hat{E} = \frac{1}{P}. \tag{7.41}$$

7.13. Theorem (Malgrange-Ehrenpreis) Let $P(D)$ be a linear PDO with constant coefficients which is non-trivial (i.e., $P \not\equiv 0$). Then $P(D)$ has a fundamental solution in $\mathcal{D}'$, i.e.,

$$\exists E \in \mathcal{D}'(\mathbb{R}^n): \quad P(D)E = \delta. \tag{7.42}$$