Some more technical remarks prior to the course

Pseudodifferential Operators and Microlocal Analysis

What are pseudodifferential operators and what are they good for?
Pseudodifferential operators may be seen as natural generalizations of linear partial differential operators. The main idea is to use Fourier transform in the following way: The Fourier transform $\hat{\varphi}$ of a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ given by $\hat{\varphi}(\xi) = \int e^{-ix\xi}\varphi(x)\,dx$ has the property

$$\widehat{D^\alpha \varphi}(\xi) = \xi^\alpha \hat{\varphi}(\xi)$$

(here $D^\alpha := (-i)^\alpha \partial^\alpha$) which by the Fourier inversion formula may be exploited to give

$$D^\alpha \varphi(x) = (2\pi)^{-n} \int e^{ix\xi} \xi^\alpha \hat{\varphi}(\xi)\,d\xi.$$

For a linear partial differential operator $P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ this leads to the formula

$$P(x, D) = (2\pi)^{-n} \int e^{ix\xi} p(x, \xi) \hat{\varphi}(\xi)\,d\xi, \tag{1}$$

where the "symbol" $p(x, \xi)$ of the operator $P(x, D)$ is the polynomial $p(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha$.
The idea is now to replace the polynomial $p(x, \xi)$ by a more general "symbol" for example by smooth functions $p(x, \xi)$ satisfying estimates of the type

$$|\partial_x^\alpha \partial_\xi^\beta\, p(x, \xi)| \le C_{\alpha\beta}\, (1 + |\xi|^2)^{(m-|\beta|)/2} \tag{2}$$

for all $(x, \xi) \in \mathbb{R}^{2n}$ and some $m \in \mathbb{R}$. Note that every ξ-derivative improves the fall off at infinity by one order, a property which mimics the behavior of polynomials.
We now turn equation (1) into a definition associating with every symbol $p(x, \xi)$ satisfying (2) an operator $P(x, D)$ acting on $\mathcal{S}$ which is more general than a partial differential operator—hence called pseudodifferential operator. The action of $P(x, D)$ may subsequently be extended to $\mathcal{S}'$ and the L^2-based Sobolev spaces H^s.

It turns out that such operators are general enough to include approximative inverses to elliptic partial differential operators. Here approximative means up to a smooth function; the prime example being the construction of a parametrix for constant coefficient elliptic operators in the course of the proof of the elliptic regularity theorem (see e.g. G. Friedlander, *Introduction to the Theory of Distributions* (Cambridge University Press, 1998), Thm. 6.1.8). These approximative inverses are a key tool in the study of solvability and hypoellipticity of e.g. general elliptic operators.
The main technical benefit obtained from definition (1) is that computations on the operators (composition, adjoint, etc.) can be replaced by (easier) computations on the symbols leading to the introduction of the mighty pseudodifferential calculus. However, the technical complexity has to be stressed, e.g. the convergence of integrals as in equation (1) for symbols satisfying an estimate as in (2) has to be carefully analyzed; here the heavy machinery of oscillatory integrals—which lies at the technical core of pseudodifferential calculus—comes into the play.

Remarks on the aims and scope of the course.
Unfortunately many expositions of pseudodifferential calculus remain vague in giving the details of the technicalities and pretend to work with absolute convergent integrals whereas such well-known formulas as integration by parts or differentiation under the integral require quite involved proofs in case of oscillatory integrals. In this course we aim at giving full and detailed proofs of these kind of statements and introduce pseudodifferential calculus in a way that empowers the students to use it in their own work.
On the other hand we strive at presenting the main principles of the theory in a clear and transparent way keeping technicalities to the necessary minimum. These choices, however, have their drawback by necessitating that we restrict ourselves to the most basic class of symbols, i.e., S^m as defined by (2) (also known as $S^m_{0,1}$). We will only briefly comment on more general classes of symbols and, in particular, confine ourselves to global estimates (i.e., $(x, \xi) \in \mathbb{R}^{2n}$ in (2)). The hope of course is that the student following this course will acquire enough technical background and theoretical insight to find her/his own way through more advanced literature, in particular, treating the topics omitted here.

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