Some prerequisites from integration theory

In this note we recall some basic facts from integration theory which will be used throughout the course. We consider the material a must in any advanced analysis course. For more details and further reading see the list of references at the end.

Integration in $\mathbb{R}^n$

Denote by $Q := \Pi_{j=1}^n [a_j, b_j]$ a compact cube in $\mathbb{R}^n$ and let $f : Q \to \mathbb{R}$ be a bounded function. For a partition $\mathfrak{Z}$ of Q into subcubes Q_i we define the upper and lower sums

$$U(\mathfrak{Z}) = \sum_i \sup_{Q_i} f\,|Q_i|, \quad L(\mathfrak{Z}) = \sum_i \inf_{Q_i} f\,|Q_i|,$$

where $|Q_i|$ denotes the volume of the cube Q_i. We say that f is **(Riemann-)integrable** over Q if

$$\inf_{\mathfrak{Z}} U(\mathfrak{Z}) = \sup_{\mathfrak{Z}} L(\mathfrak{Z}) =: \int_Q f(x)\,dx.$$

Any continuous f is integrable and Fubini's theorem tells us that

$$\int_Q f(x)\,dx = \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} f(x_1, \dots, x_n)\,dx_1 \dots dx_n.$$

For $\Omega \subset \mathbb{R}^n$ denote by χ_Ω the characteristic function of Ω, i.e., $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ for $x \notin \Omega$. A bounded set $\Omega \subseteq Q$ is called **(Jordan-)measurable** if χ_Ω is integrable over Q. In this case the **(Jordan-)volume** of Ω is given by

$$|\Omega| := \int_Q \chi_\Omega(x)\,dx = \int_\Omega dx.$$

Finally, for $f : \Omega \subseteq Q \to \mathbb{R}$ with $f\chi_\Omega$ integrable over Q we write

$$\int_\Omega f(x)\,dx := \int_Q f(x)\chi_\Omega(x)\,dx.$$

Next we discuss **changing of variables**. Let $G \subseteq \mathbb{R}^n$ be open and $T : G \to \mathbb{R}^n$ be an injective $\mathcal{C}^1$-function with $\det DT$ everywhere positive (or negative). Moreover, let $\Omega \subseteq G$ be compact and measurable and let $f : T(\Omega) \to \mathbb{R}$ be continuous. Then $T(G)$ is measurable and f is integrable over $T(\Omega)$ and we have that

$$\int_{T(\Omega)} f(y)\,dy = \int_\Omega f(T(x))\,|\det DT(x)|\,dx. \tag{1}$$

Polar and spherical coordinates

A simple and important special case of the above situation are **polar coordinates** in $\mathbb{R}^2$. We set $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$ or more formally

$$T_2 : (r, \varphi) \mapsto (r\cos\varphi, r\sin\varphi) \quad \text{for } 0 \le r \text{ and } 0 \le \varphi < 2\pi$$

and observe that T_2 is a diffeomorphism from $\Omega = \{(r,\varphi) :\ 0 < r,\, 0 < \varphi < 2\pi\}$ to $\mathbb{R}^2 \setminus \mathbb{R}_0^+$. Also we clearly have that

$$\det DT_2 = \det \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r.$$

Another important special case is **spherical coordinates** in $\mathbb{R}^3$, where we set

$$x_1 = r\cos\varphi\sin\theta_1,\ x_2 = r\sin\varphi\sin\theta_1,\ \text{and}\ x_3 = r\cos\theta_1.$$

More formally we may write $(0 \le r,\, 0 \le \varphi < 2\pi,\, 0 \le \theta_1 \le \pi)$

$$T_3 :\ (r,\varphi,\theta_1) \mapsto (r\cos\varphi\sin\theta_1, r\sin\varphi\sin\theta_1, r\cos\theta_1),$$

which can be seen to be a diffeomorphism on $\Omega = \{(r,\varphi,\theta_1) :\ 0 < r,\, 0 < \varphi < 2\pi,\, 0 < \theta_1 < \pi\}$ with image $\mathbb{R}^3 \setminus \{x :\ x_1 > 0,\, x_2 = 0\}$. For convenience we rewrite as $T_3 = T_z \circ \Psi$ with

$$\begin{array}{rcl} \Psi(r,\varphi,\theta_1) & := & (r\sin\theta_1, \varphi, r\cos\theta_1) \ =:\ (\rho,\varphi,x_3),\ \text{and} \\ T_z(\rho,\varphi,x_3) & := & (T_2(\rho,\varphi), x_3). \end{array}$$

In this way we obtain by the chain rule $DT_3(r,\varphi,\theta_1) = DT_z(\rho,\varphi,x_3) \circ D\Psi(r,\varphi,\theta_1)$, hence

$$|\det DT_3(r,\varphi,\theta_1)| = |\det DT_z(\rho,\varphi,x_3)| \cdot |\det D\Psi(r,\varphi,\theta_1)| = \rho r = r^2\sin\theta_1.$$

This now generalizes easily to the **case of arbitrary** n. Indeed we set $(0 \le r,\, 0 \le \varphi < 2\pi,$ $0 \le \theta_i \le \pi,\, i = 1,\ldots,n-2)$

$$\begin{array}{rcr} x_1 & = & r\cos\varphi\sin\theta_1\sin\theta_2\sin\theta_3 \ldots\ldots\ldots\ldots \sin\theta_{n-2} \\ x_2 & = & r\sin\varphi\sin\theta_1\sin\theta_2\sin\theta_3 \ldots\ldots\ldots\ldots \sin\theta_{n-2} \\ x_3 & = & r\cos\theta_1\sin\theta_2\sin\theta_3 \ldots\ldots\ldots\ldots \sin\theta_{n-2} \\ x_4 & = & r\cos\theta_2\sin\theta_3 \ldots\ldots\ldots\ldots \sin\theta_{n-2} \\ & \vdots & \\ x_{n-1} & = & r\cos\theta_{n-3}\sin\theta_{n-2} \\ x_n & = & r\cos\theta_{n-2}, \end{array} \tag{2}$$

or for short

$$x = T_n(r,\varphi,\theta_1,\ldots,\theta_{n-2}). \tag{3}$$

Similar to the above we now have that T_n is a diffeomorphism on $(0,\infty) \times (0,2\pi) \times (0,\pi)^{n-2}$ with image $\mathbb{R}^n \setminus \{x :\ x_1 \ge 0,\, x_2 = 0\}$ and that

$$|\det DT_n| = r^{n-1}\sin\theta_1(\sin\theta_2)^2 \ldots (\sin\theta_{n-2})^{n-2}. \tag{4}$$

Indeed the latter statement can be proved by induction. To this end we write similar to the above $T_n = T_z \circ \Psi$ with

$$\begin{array}{rcl} \Psi(r,\varphi,\theta_1,\ldots,\theta_{n-2}) & := & (r\sin\theta_{n-2}, \varphi, \theta_1, \ldots, \theta_{n-3}, r\cos\theta_{n-2}) \ =:\ (\rho,\varphi,\theta_1,\ldots,\theta_{n-3},x_n), \\ T_z(\rho,\varphi,\theta_1,\ldots,\theta_{n-3},x_n) & := & (T_{n-1}(\rho,\varphi,\theta_1,\ldots,\theta_{n-3}), x_n). \end{array}$$

Now by the induction hypothesis $|\det DT_z| = \rho^{n-2}\sin\theta_1 \ldots (\sin\theta_{n-3})^{n-3}$ and observing that $|\det D\Psi| = r$ as well as the definition of ρ we obtain

$$|\det T_n| = |\det DT_z|\ |\det D_\psi| = r^{n-1}\sin\theta_1(\sin\theta_2)^2 \ldots (\sin\theta_{n-2})^{n-2}.$$

Surface integrals

The next topic we discuss is integration over m-dimensional surfaces in $\mathbb{R}^n$. We start with some linear algebra and recall that given m vectors $v_1, \ldots, v_m$ in $\mathbb{R}^n$ the **Gram determinant** $G(v_1, \ldots, v_m)$ of $v_1, \ldots, v_m$ is defined by

$$G(v_1, \ldots, v_m) := \det \begin{pmatrix} \langle v_1, v_1 \rangle & \ldots & \langle v_1, v_m \rangle \\ \vdots & & \vdots \\ \langle v_m, v_1 \rangle & \ldots & \langle v_m, v_m \rangle \end{pmatrix},$$

where $\langle \ , \ \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. Also recall that the volume of the parallelepiped spanned by $v_1, \ldots, v_m$ equals the square root of the Gram determinant, i.e.,

$$\mathrm{Vol}(v_1, \ldots, v_m) = \big(G(v_1, \ldots, v_m)\big)^{1/2}.$$

In particular, if $n = 3$ and $m = 2$ then we have that

$$\mathrm{Vol}(v_1, v_2) = \sqrt{||v_1||^2 ||v_2||^2 - \langle v_1, v_2 \rangle^2},$$

the surface of the parallelogram spanned by v_1, v_2.

Let now $\Omega \subseteq \mathbb{R}^m$ be open and let $S \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$ be an **immersion**, that is

$$\mathrm{rank}(DS(x)) = m \quad \text{for all } x \in \Omega.$$

Recall that the latter condition is equivalent to linear independence of the vectors $D_1 S(x), \ldots, D_m S(x)$, which tells us that they span an m-dimensional subspace in $\mathbb{R}^n$. So we call $S(\Omega)$ or more sloppy S an m-dimensional (parametrized) **surface** in $\mathbb{R}^n$.

Now we want to determine the area of S. To this end we imagine that the surface consist of many infinitesimal parallelepipeds spanned by $D_1 S(x), \ldots, D_m S(x)$ and we have to sum all of them. Mathematically this sum has to be turned into an integral and so the following definition is justified: We define the **area** $|S|$ of S to be

$$|S| := \int\limits_{\Omega} \sqrt{G(DS(y))}\, dy = \int\limits_{\Omega} \sqrt{\det \Big(\langle \frac{\partial S}{\partial y_i}, \frac{\partial S}{\partial y_j} \rangle \Big)_{i,j=1}^{m}}\, dy.$$

One often writes $g_{ij}(S) = \langle \frac{\partial S}{\partial y_i}, \frac{\partial S}{\partial y_j} \rangle$ and $dS(y) = \sqrt{G(DS(y))} dy = \sqrt{\det g_{ij}(S)} dy$ and moreover $\int_S dS(y) = \int_\Omega \sqrt{G(DS(y))} dy$. With this notation we define for any continuous $f : \mathbb{R}^n \to \mathbb{R}$ the **integral of f over** S by

$$\int\limits_S f(y)\, dS(y) := \int\limits_{\Omega} f(S(y)) \sqrt{G(DS(y))}\, dy.$$

Spheres and balls

As an application of the above we calculate the **surface of the unit sphere** S^{n-1} in $\mathbb{R}^n$ as well as the volume of the unit ball $B(0, 1)$. We again start with some linear algebra and observe that in the case $m = n$, hence for vectors $v_1, \ldots v_n$ in $\mathbb{R}^n$ we have with $A := (v_1, \ldots, v_n)$ that

$$G(v_1, \ldots, v_n) = \det A^2. \tag{5}$$

Next we parametrize the unit sphere S^{n-1} in $\mathbb{R}^n$ using (3), i.e,

$$\begin{aligned} S := T_n \,|_{r=1} \colon \ (0, 2\pi) \times (0, \pi)^{n-2} &\ \to \ \mathbb{R}^n \\ y = (\varphi, \theta_1, \ldots, \theta_{n-2}) &\ \mapsto \ T_n(1, \varphi, \theta_1, \ldots, \theta_{n-2}). \end{aligned}$$

To determine $dS(y)$ we have to calculate $G(DS(y))$. To do so we first show that $G(DS) = G(DT_n)|_{r=1}$ and then use (5) to obtain $G(DT_n)|_{r=1}$ via $\det DT_n^2$ which we already know from (4). Indeed from (2) we see that

$$\langle \partial_r T_n, \partial_r T_n \rangle = 1,\ \langle \partial_r T_n, \partial_\varphi T_n \rangle = 0 = \langle \partial_r T_n, \partial_{\theta_i} T_n \rangle \qquad (1 \le i \le n-2),$$

hence

$$G(DT_n)\mid_{r=1} = \det \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & g_{ij}(S) & \\ 0 & & & \end{pmatrix} = \det(g_{ij}(S)) = G(DS)$$

and so we obtain

$$\begin{aligned} dS(y) &= \sqrt{G(DS(y))}dy = \sqrt{G(DT_n)}\mid_{r=1} dy = \sqrt{(\det DT_n)^2}\mid_{r=1} dy = |\det DT_n|_{r=1} dy \\ &= \sin\theta_1 (\sin\theta_2)^2 \dots \sin(\theta_{n-2})^{n-2} d\varphi d\theta_1 \dots d\theta_{n-2}. \end{aligned}$$

Now to compute the area of the unit sphere we just have to evaluate

$$\begin{aligned} |S^{n-1}| &\equiv |\partial B(0,1)| = \int\limits_{\partial B(0,1)} dS(y) \\ &= \int\limits_0^{2\pi} \int\limits_0^{\pi} \dots \int\limits_0^{\pi} \sin\theta_1 \dots \sin(\theta_{n-2})^{n-2} d\varphi d\theta_1 \dots d\theta_{n-2} =: n\alpha(n), \end{aligned}$$

and an explicit calculation (cf. e.g. [1, Bsp. (5.7)]) shows that

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)},$$

where Γ denotes Euler's gamma function.

Now it is easy to calculate the **volume of the unit ball** $B(0,1)$ in $\mathbb{R}^n$: Observe that its interior $B^\circ(0,1)$ is given by $B^\circ(0,1) = T_n((0,1) \times (0,2\pi) \times (0,\pi)^{n-2}) =: T_n(\Omega)$ hence by the change of variables formula (1) we obtain

$$\begin{aligned} |B(0,1)| &= |B^\circ(0,1)| = \int\limits_{B^\circ(0,1)} dx = \int\limits_{\Omega} |\det DT_n(x)|\, dx \\ &= \int\limits_0^1 \int\limits_0^{2\pi} \int\limits_0^{\pi} \dots \int\limits_0^{\pi} r^{n-1} \sin\theta_1 \dots \sin(\theta_{n-2})^{n-2} d\varphi d\theta_1 \dots d\theta_{n-1} dr \\ &= \int\limits_0^1 r^{n-1} \int\limits_{\partial B(0,1)} dS(y)\, dr = \frac{r^n}{n} \alpha(n)\, n \mid_0^1 = \alpha(n). \end{aligned}$$

Finally, we want to integrate some continuous function u over **general balls and spheres**. For $\partial B(x,r)$ with arbitrary $x \in \mathbb{R}^n$ and $r > 0$ we choose the parametrization

$$S:\ y = (\varphi, \theta_1, \dots \theta_{n-2}) \mapsto x + T_n(r, \varphi, \theta_1, \dots \theta_{n-2}) = x + r\, T_n(1, \varphi, \theta_1, \dots \theta_{n-2}).$$

Now we have in analogy to the above that

$$dS(y) = |\det DT_n|_r = r^{n-1} \sin\theta_1 \dots \sin(\theta_{n-2})^{n-2} d\varphi d\theta_1 \dots d\theta_{n-2},$$

and so we obtain

$$
\begin{aligned}
&\int\limits_{\partial B(x,r)} u(y) dS(y) \\
&= \int\limits_0^{2\pi}\int\limits_0^{\pi}\dots\int\limits_0^{\pi} u(x + r\, T_n(1,\varphi,\theta_1,\dots\theta_{n-2})\, r^{n-1}\sin\theta_1\dots\sin(\theta_{n-2})^{n-2} d\varphi d\theta_1\dots d\theta_{n-2} \\
&= r^{n-1}\int\limits_{\partial B(0,1)} u(x+ry)\, dS(y).
\end{aligned}
$$

Similarly we may write $B^\circ(x,r) = x + T_n((0,r)\times(0,2\pi)\times(0,\pi)^{n-2}) =: T(\Omega)$, where we have set $T(s,\varphi,\theta_1,\dots,\theta_{n-2}) \mapsto x + T_n(s,\varphi,\theta_1,\dots,\theta_{n-2})$. This immediately gives

$$
\begin{aligned}
\int\limits_{B(x,r)} u(x)\, dx &= \int\limits_{\Omega} u(T(s,\varphi,\theta_1,\dots,\theta_{n-2})|\det DT|\, d(s,\varphi,\theta_1,\dots,\theta_{n-2}) \\
&= \int\limits_0^r\int\limits_0^{2\pi}\int\limits_0^{\pi}\dots\int\limits_0^{\pi} u(T(s,\varphi,\dots,\theta_i,\dots) s^{n-1}\sin\theta_1\dots\sin(\theta_{n-2})^{n-2} d\varphi d\theta_1\dots d\theta_{n-2} ds \\
&= \int\limits_0^r s^{n-1}\int\limits_{\partial B(0,1)} u(x+sy) dS(y) ds = \int\limits_0^r\int\limits_{\partial B(x,s)} u(y) dS(y) ds.
\end{aligned}
$$

Finally, if we set $u \equiv 1$ we obtain the area of the spheres as well as the volume of the balls of radius r

$$
\begin{aligned}
|\partial B(x,r)| &= r^{n-1}|\partial B(0,1)| = r^{n-1} n\alpha(n) \\
|B(x,r)| &= \int\limits_0^r s^{n-1} n\alpha(n) ds = r^n\alpha(n).
\end{aligned}
$$

Some integral theorems and formulas

To finish this note we recall the Gaussian divergence theorem and some of its consequences which are essential in several places of the lecture course.

Let $U \subseteq \mathbb{R}^n$ open and bounded with $\mathcal{C}^1$-boundary ∂U and outer unit normal vector $\nu : \partial U \to \mathbb{R}^n$. Moreover, let $F : \overline{U} \to \mathbb{R}^n$ be a $\mathcal{C}^1$-function. Then the **divergence theorem** (see e.g. [1, §15, Satz 3]) says that

$$\int\limits_U \mathrm{div} F\, dx = \int\limits_{\partial U} F\cdot\nu\, dS.$$

In particular, for F of the form $F = (0,\dots,0,u,0,\dots,0)$ where the i-th component of F is given by some $\mathcal{C}^1$-function $u : \overline{U} \to \mathbb{R}$ we obtain the **Gauss-Green theorem**

$$\int\limits_U u_{x_i}\, dx = \int\limits_{\partial U} u\nu^i\, dS \qquad (1 \le i \le n).$$

Applying this formula with u replaced by the product uv of some $\mathcal{C}^1(\overline{U})$-functions u, v we obtain the **integration by parts formula**

$$\int\limits_u u_{x_i} v\, dx = -\int\limits_U u v_{x_i}\, dx + \int\limits_{\partial U} uv\,\nu^i\, dS.$$

The following so-called **Gaussian formulas** for $u, v \in \mathcal{C}^2(\overline{U})$-functions are again easy consequences of the integration by parts formula:

(i) $\displaystyle\int_U \triangle u \, dx \;=\; \int_{\partial U} \frac{\partial u}{\partial \nu} \; dS$

(ii) $\displaystyle\int_U Du \cdot Dv \, dx \;=\; -\int_U u \triangle v \, dx + \int_{\partial U} u \, \frac{\partial v}{\partial \nu} \; dS$

(iii) $\displaystyle\int_U \Big(u \triangle v - v \triangle u \Big) \, dx \;=\; \int_{\partial U} \Big(u \, \frac{\partial v}{\partial \nu} - v \, \frac{\partial u}{\partial \nu} \Big) \, dS$

References

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[4] G. Fischer, Lineare Algebra (17. Auflage, Vieweg+Teubner, Braunschweig, 2009).