SOME FIXED POINT THEOREMS

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Abstract. We investigate a few fixed point theorems from functional analysis and discuss fixed point problems in general.

1. Introduction

A *fixed point* of a relation $R \subset A \times B$ is an element $x \in A \cap B$ with xRx, i. e., $(x, x) \in R$. In case R is a function, this takes the form $x = R(x)$. In any (additive) group G, solving equations of the form $f(x) = g(x)$ can be reduced to the problem of finding a fixed point of the function $F(x) = f(x) - g(x) + x$. There are many well-known fixed point theorems, perhaps the most popular of which is Banach's theorem.

Theorem 1 (Banach)**.** *Let (X, d) be a complete metric space and $f : X \to X$ a contraction, i. e.,*

$$\sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)} < 1.$$

Then f has a unique fixed point. □

Its proof is folklore and uses telescope summation and triangle inequality. A lesser known result is the following theorem about *complete lattices*, i. e., posets where every subset has both supremum and infimum.

Theorem 2 (Tarski-Knaster)**.** *Let C be a complete lattice and $f : C \to C$ a monotonic function. Then f has a fixed point.* □

Even more holds: The set of fixed-points of f is again a complete lattice. We can actually use this theorem in everyday mathematics if we note that every non-empty compact interval $[a, b] \subset \mathbb{R}$ is a complete lattice. We formulate this result as a corollary to the previous theorem, but give a proof to illustrate the theorem's proof idea on a concrete example.

Corollary 1. *Let $f : [a, b] \to [a, b]$ be a monotonic function. Then f has a fixed point.*

Proof. Set $P = \{x \in [a, b] \mid x \leqslant f(x)\}$ and $p = \sup P$. We claim that $p = f(p)$. Since $x \leqslant p$ for all $x \in P$, it follows that $x \leqslant f(x) \leqslant f(p)$ for all $x \in P$. Thus, $f(p)$ is an upper bound of P and hence $p \leqslant f(p)$. But this also implies $f(p) \in P$ which means $f(p) \leqslant p$. □

Remark 1. The Tarski-Knaster theorem can be used to prove the Cantor-Schröder-Bernstein theorem very elegantly.

Another important result, which we will use in establishing further fixed point theorems, with a much more difficult proof is

Theorem 3 (Brouwer)**.** *Let B_n denote the closed unit ball in $\mathbb{R}^n$ and let $f : B_n \to B_n$ be continuous. Then f has a fixed point.* □

2. Schauder

In this section, we will generalize theorem 3. We will first get rid of the domain B_n in the Brouwer fixed point theorem and replace it by an arbitrary compact convex set.

Lemma 1. *Let $C \subset \mathbb{R}^n$ be a non-empty compact convex set. Then there exists a retraction $\mathbb{R}^n \to C$.*

Proof. We define the R to be the orthogonal projection onto C, i. e., $R(x)$ is the unique element in C with $\|x - R(x)\| = \inf_{y \in C} \|x - y\|$. Let $x_n \to x_\infty$ in $\mathbb{R}^n$ and $R(x_n) \not\to R(x_\infty)$. Since C is compact, we can assume that $R(x_n) \to y_\infty$ for a $y_\infty \in C$. Now,

$$\|x_\infty - y_\infty\| = \lim \|x_n - R(x_n)\| = \lim \inf_{y \in C} \|x_n - y\| = \inf_{y \in C} \|x_\infty - y\|$$

which implies $R(x_\infty) = y_\infty$. This proves that R is the desired retraction. □

Lemma 2. *Let E be a finite-dimensional normed vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $C \subset E$ be a non-empty compact convex set. Further, let $f : C \to C$ be continuous. Then f has a fixed point.*

Proof. Without loss of generality, $E = \mathbb{R}^n$ equipped with the Euclidian norm. Let $R : \mathbb{R}^n \to C$ be a retraction, B_n the closed unit ball in $\mathbb{R}^n$ and $\lambda > 0$ such that $C \subset \lambda B_n$. We define $g = f \circ R : \lambda B_n \to \lambda B_n$. Then g has a fixed point $x \in C$. It is $x = R(x)$ and thus $f(x) = x$. □

Since we will construct a sequence of "nearly fixed points" in the next proof, we formalize this notion.

Definition 1. Let (X, d) be a metric space, $f : X \to X$ a function and $\varepsilon > 0$. A point $x \in X$ is called *ε-fixed point of f* if $d(x, f(x)) \leqslant \varepsilon$.

What follows is the main theorem of this section. It is a generalization of Brouwer's theorem to infinite dimensions.

Theorem 4. *Let E be a normed vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $C \subset E$ be a non-empty compact convex set. Further, let $f : C \to C$ be continuous. Then f has a fixed point.*

Proof. Let $\varepsilon > 0$. There exist finitely many points $x_1, \dots x_n$ such that the open balls around x_ι with radius ε cover C. The family of functions $(\varphi_\iota)_{1\leqslant\iota\leqslant n}$ defined by

$$\psi_\iota(x) = \max\left(\varepsilon - \|x - x_\iota\|, 0\right)$$

and $\varphi_\iota = \psi_\iota / \sum \psi_j$ is a continuous partition of unity in C. We define the subspace F as the linear envelope of $x_1, \dots, x_n$ and $D = C \cap F$. Further we set $\Phi(x) = \sum \varphi_\iota(x) x_\iota$. It is $\Phi(x) \in D$ for all $x \in C$ and thus $\Phi \circ f : D \to D$ a continuous function. By lemma 2, there exists a fixed point ξ_ε of $\Phi \circ f$. It is

$$\|\Phi(x) - x\| \leqslant \sum \varphi_\iota(x) \|x_\iota - x\| \leqslant \varepsilon$$

for all $x \in C$ and thus ξ_ε an ε-fixed point of f.
Set $y_n = \xi_{1/n}$. Since C is compact, there exists a convergent subsequence (y_{n_k}) of (y_n). Let $y = \lim y_{n_k}$, then

$$\|f(y_{n_k}) - y\| \leqslant \|f(y_{n_k}) - y_{n_k}\| + \|y_{n_k} - y\| \to 0$$

and by continuity of f we get $y = f(y)$. □

Another possibility is to shift compactness from the set C to the function f. This is done in the following

Corollary 2. *Let E be a Banach space over $\mathbb{R}$ or $\mathbb{C}$ and let $A \subset E$ be a non-empty closed bounded convex set. Further, let $f : A \to A$ be compact. Then f has a fixed point.*

Proof. It is $f(A)$ relatively compact and thus its closed convex envelope D is compact. Since A is closed and convex, we have $D \subset A$. This implies $f(D) \subset f(A) \subset D$ and we can apply the previous theorem to the function $f|D$. □

3. Darbo-Sadovskiĭ

Definition 2. Let X be a metric space. We define $\alpha(X)$ as the infimum of the set of all $\varepsilon > 0$ such that there exists a finite cover of X with sets of diameter $\leqslant \varepsilon$. If no such cover exists, it is $\alpha(X) = \infty$. We call α the *Kuratowski measure of non-compactness.*

In order to get used to this notion and because we will need it later on, we prove the following

Lemma 3. *If X is a compact metric space, then $\alpha(X) = 0$. A complete metric space Y is compact if and only if $\alpha(Y) = 0$.*

Proof. The first claim is trivial. Let now Y be complete and $\alpha(Y) = 0$. Suppose that Y is not compact. Then there exists an open cover $(U_\iota)_{\iota\in I}$ of Y which has no finite subcover. For every $n \in \mathbb{N}$, there exists a finite cover of Y with open balls B_j^n $(1 \leqslant j \leqslant k_n)$ of radius $\varepsilon_n = 2^{-n}$. It is (U_ι) an open cover of each B_j^n and thus, for every n, at least one of these sets, e. g., B_1^n, does not have a finite subcover of sets from (U_ι). It is no restriction to assume $B_1^n \cap B_1^m \neq \emptyset$. Let $s_n \in B_1^n$ for $n \in \mathbb{N}$. Then (s_n) is a Cauchy sequence, so let $s_n \to s$ with $s \in U_{\iota_0}$. Since the complement of U_{ι_0} is closed and $\{s\}$ is compact, the distance Δ of these two sets is > 0. Thus the open ball around s with radius Δ is a subset of U_{ι_0}. Since all B_1^n eventually are in this ball, we get a contradiction to the fact that no B_1^n has a finite subcover. □

The next lemma is stated without proof.

Lemma 4. *Let E be a Banach space over $\mathbb{R}$ or $\mathbb{C}$ and $X \subset E$. Then $\alpha(X) = \alpha(c(X))$ where $c(X)$ denotes the closed convex envelope of X.* □

In this section's main theorem, we treat functions whose images get "more and more compact". These are called *condensing* functions.

Definition 3. Let X and Y be metric spaces and $f : X \to Y$ continuous. We call f *condensing* if for all bounded $B \subset X$ with $\alpha(B) > 0$ we have $\alpha(f(B)) < \alpha(B)$.

Theorem 5. *Let E be a Banach space over $\mathbb{R}$ or $\mathbb{C}$, $A \subset E$ non-empty closed bounded convex and $f : A \to A$ condensing. Then f has a fixed point.*

Proof. Let $p \in A$. Set $\mathcal{K} = \{K \subset A \mid p \in K,\ K \text{ closed convex and } f\text{-invariant}\}$ and

$$L = \bigcap_{K \in \mathcal{K}} K.$$

It is $L \in \mathcal{K}$ and thus the minimal element of $\mathcal{K}$ since the f-invariance follows from $f(L) \subset f(K) \subset K$ for all $K \in \mathcal{K}$. We will show that L is compact. For this, we set $M = c(f(L) \cup \{p\})$ and show $M = L$. Since $f(L) \cup \{p\} \subset L$, we have $M \subset L$. It is M f-invariant since $f(M) \subset f(L) \subset M$ and thus $M \in \mathcal{K}$. This implies $L \subset M$. Now,

$$\alpha(L) = \alpha(M) = \alpha(f(L) \cup \{p\}) = \alpha(f(L))$$

and hence $\alpha(L) = 0$ because f is condensing and A is bounded. By lemma 3, it is L compact. We can now apply theorem 4 to the function $f|L$. $\square$

4. Browder-Göhde-Kirk

In this section we will generalize Banach's theorem. Namely, we relax the *strict* inequality that guarantees the Cauchy property of the constructed sequence in the original proof.

Definition 4. Let (X, d) be a metric space and let $f : X \to X$ be a function. We call f *non-expanding* if

$$\sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)} \leqslant 1.$$

Definition 5. Let E be a normed vector space over $\mathbb{R}$ or $\mathbb{C}$. We call E *uniformly convex* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in E$ with norm $\leqslant 1$ and $\|x - y\| \geqslant \varepsilon$ there holds:

$$\frac{\|x + y\|}{2} \leqslant 1 - \delta$$

The function $\delta_E : (0, \infty) \to [0, \infty]$,

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} \,\middle|\, \|x\|, \|y\| \leqslant 1, \|x - y\| \geqslant \varepsilon\right\}$$

is called the *module of convexity of E*.

Lemma 5. *Let E be a Banach space over $\mathbb{R}$ or $\mathbb{C}$, $A \subset E$ non-empty closed convex and $f : A \to A$ non-expanding such that $f(A)$ is bounded. Then, for every $\varepsilon > 0$, there exists an ε-fixed point of f.*

Proof. Let $\varepsilon > 0$ and $\Delta = \operatorname{diam} f(A)$. By assumption, there exists a $p \in f(A)$ and we define the function

$$g(x) = \lambda p + (1 - \lambda) f(x)$$

where $0 < \lambda = \min(\varepsilon/\Delta, 1)$. We can assume $\Delta > 0$ since the case $|f(A)| = 1$ is trivial. For A is convex, we have $g(A) \subset A$. It is $g : A \to A$ a contraction and since A is closed, we can apply theorem 1 and get the existence of a fixed point x of g. Now,

$$\|x - f(x)\| = \|g(x) - f(x)\| = \lambda \|p - f(x)\| \leqslant \varepsilon$$

and we proved the lemma. $\square$

Remark 2. By inspection of the proof, we find out that the lemma still holds if we replace "Banach space" by "normed vector space" and "closed" by "complete".

Theorem 6. *Let E be a uniformly convex Banach space, $A \subset E$ non-empty closed bounded convex and $f : A \to A$ non-expanding. Then f has a fixed point.*

Proof.

Lemma 6. *There exists a function $\varphi : (0, \infty) \to (0, \infty)$ such that for any two ε-fixed points $x, y \in A$, it is $z = (x + y)/2 \in A$ a $\varphi(\varepsilon)$-fixed point and $\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$.*
Further,

$$\varphi(\varepsilon) = \sqrt{\varepsilon} + \left(\frac{1}{2} \operatorname{diam} A + \varepsilon\right)\left(1 - \delta_E\left(2 - 2\sqrt{\varepsilon}\right)\right)$$

is such a function.

Proof. We first show the fixed point property. Set

$$\rho = \frac{\|x-y\|}{2} + \varepsilon.$$

It follows that $\|f(z)-x\| \leqslant \rho$ and $\|f(z)-y\| \leqslant \rho$. Thus, both $(f(z)-x)/\rho$ and $(f(z)-y)/\rho$ lie in the closed unit ball and have distance $\|x-y\|/\rho$. Since E is uniformly convex, we get

$$\frac{1}{\rho}\|f(z)-z\| = \frac{1}{2}\left\|\frac{f(z)-x}{\rho} - \frac{f(z)-y}{\rho}\right\| \leqslant 1-\delta_E\left(\frac{\|x-y\|}{\rho}\right) = 1-\delta_E\left(2-2\frac{\varepsilon}{\rho}\right).$$

We distinguish two cases: $\rho \leqslant \sqrt{\varepsilon}$ and $\rho > \sqrt{\varepsilon}$. If $\rho \leqslant \sqrt{\varepsilon}$, then $\|f(z)-z\| \leqslant \sqrt{\varepsilon} \leqslant \varphi(\varepsilon)$. If $\rho > \sqrt{\varepsilon}$, the claim follows because δ_E is non-decreasing and $\rho \leqslant 1/2 \cdot \operatorname{diam} A + \varepsilon$. This proves the fixed point property of φ. It remains to show that $\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$.
It suffices to show that $\lim_{\xi\to 2}\delta_E(\xi) = 1$. We note that $\lim_{\xi\to 2}\delta_E(\xi) = \sup_{\xi<2}\delta_E(\xi)$ since δ_E is non-decreasing. Suppose this supremum is < 1. Then there exists a $\beta > 0$ such $\delta_E(\xi) < 1-\beta$ for all $\xi < 2$. It follows that for every $\gamma > 0$, there exist x_γ, y_γ with norm $\leqslant 1$ and distance $\geqslant 2-\gamma$ such that

$$1 - \frac{\|x_\gamma + y_\gamma\|}{2} \leqslant 1-\beta.$$

If we now set $y_\gamma^* = -y_\gamma$, we get $\left\|x_\gamma - y_\gamma^*\right\| \geqslant 2\beta > 0$ and

$$1 - \frac{\left\|x_\gamma + y_\gamma^*\right\|}{2} \leqslant \frac{\gamma}{2}.$$

But from the definition of δ_E, we obtain

$$1 - \frac{\left\|x_\gamma + y_\gamma^*\right\|}{2} \geqslant \delta_E(2\beta)$$

which is a contradiction for $\gamma = \delta_E(2\beta) > 0$. (It is $\delta_E > 0$ since E is uniformly convex.) □

We now continue with the proof of the theorem. We set

$$\eta(r) = \inf\{\|f(x)-x\| \mid x \in A, \|x\| \leqslant r\}$$

and

$$s_0 = \inf\{r \geqslant 0 \mid \eta(r) = 0\}.$$

Since A is bounded, so is $f(A)$ and we can apply lemma 5 to get the existence of an $r \geqslant 0$ with $\eta(r) = 0$. This implies $s_0 < \infty$. There exists a sequence (x_n) in A with $\|x_n\| \to s_0$ and $\|f(x_n)-x_n\| \to 0$. We will show that (x_n) converges. For f is continuous (even Lipschitz), it then follows that with $x = \lim x_n$, we have $\|f(x)-x\| = 0$. It is $x \in A$ since A is closed.
Suppose that (x_n) does not converge. Then $s_0 > 0$, because otherwise $x_n \to 0$. Since (x_n) is not Cauchy, there exists an $\varepsilon > 0$ and a subsequence (x_{n_k}) of (x_n) such that $\left\|x_{n_{k+1}} - x_{n_k}\right\| \geqslant \varepsilon$ for all k.
Choose any $s_0 < s_1 < 2s_0$ with

$$s_2 = s_1\left(1-\delta_E\left(\frac{\varepsilon}{2s_0}\right)\right) < s_0$$

and $N \in \mathbb{N}$ such that for all $k > N$, we have $\|x_{n_k}\| \leqslant s_1$. Then $x_{n_{k+1}}/s_1$ and x_{n_k}/s_1 are in the closed unit ball and their distance is $\geqslant \varepsilon/s_1$. We get that

$$\frac{\left\|x_{n_{k+1}} + x_{n_k}\right\|}{2} \leqslant s_1\left(1-\delta_E\left(\frac{\varepsilon}{s_1}\right)\right) \leqslant s_2 < s_0$$

for $k > N$. For $\|f(x_{n_k}) - x_{n_k}\| \to 0$, we get $\left\|f((x_{n_{k+1}} + x_{n_k})/2) - (x_{n_{k+1}} + x_{n_k})/2\right\| \to 0$ by lemma 6 and thus $\eta(s_2) = 0$ which is a contradiction since $s_2 < s_0$. □

To make this theorem practicable, we state the following result (without proof).

Lemma 7. *For every $1 < p < \infty$, it is L^p uniformly convex.* □

References

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