INTERPOLATION OF OPERATORS ON L^p-SPACES

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Abstract. We prove the Riesz-Thorin theorem for interpolation of operators on L^p-spaces and discuss some applications. We follow in large the presentation in [D. Werner, Funktionalanalysis (Springer, 2005), p. 72-79, II.4].

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1. The theorem of Riesz-Thorin

Motivation 1.1. Let $p_0 < p_1$, $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$ $m(\Omega) < \infty$ and $T \in L(L^{p_0}(\Omega), L^{q_0}(\Omega))$. Then Tf is also defined for $f \in L^{p_1}(\Omega)$ since $L^{p_1}(\Omega) \subseteq L^{p_0}(\Omega)$. We want to suppose that $f \in L^{p_1}(\Omega)$ implies $Tf \in L^{q_1}(\Omega)$ so that we have $T \in L(L^{p_1}(\Omega), L^{q_1}(\Omega))$.
We now want to study the operator T on the $L^p(\Omega)$ spaces between $L^{p_0}(\Omega)$ and $L^{p_1}(\Omega)$. Additionally we want to get norm estimates from the norms $\|T\|_{p_0 \to q_0}$ and $\|T\|_{p_1 \to q_1}$.

Strictly speaking, if we talk about an operator $T : L^{p_0}(\Omega) \to L^{q_0}(\Omega)$ and $T : L^{p_1}(\Omega) \to L^{q_1}(\Omega)$ we mean that we have an operator $T_0 \in L(L^{p_0}(\Omega), L^{q_0}(\Omega))$ and an operator $T_1 \in L(L^{p_1}(\Omega), L^{q_1}(\Omega))$ for which

$$T_0 \mid_{L^{p_0}(\Omega) \cap L^{p_1}(\Omega)} = T_1 \mid_{L^{p_0}(\Omega) \cap L^{p_1}(\Omega)}$$

holds.

We will carry out our investigations for arbitrary open subsets Ω of $\mathbb{R}^n$ but it would be possible to deal with more general measure spaces. From now on we abbreviate $L^p(\Omega)$ with L^p. Note that we have not supposed that $m(\Omega) < \infty$, so that we have no inclusion relation between the L^p-spaces. However, we still have the following statement.

Lemma 1.2 (Lyapunov inequality)**.** *Let* $1 \le p_0, p_1 \le \infty$ *and* $0 \le \theta \le 1$. *Define* p *by* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *Then* $L^{p_0} \cap L^{p_1} \subset L^p$ *and we have*

$$\|f\|_p \le \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta} \quad \forall f \in L^{p_0} \cap L^{p_1}. \tag{1}$$

Proof. We will use Hölder's inequality to prove inequality (1). Indeed we have $\|fg\|_1 \le \|f\|_p \|g\|_q$ whenever $1 = \frac{1}{p} + \frac{1}{q}$.
Let $x := (1-\theta)p$, $y := \theta p$, $\frac{1}{z_0} := \frac{1-\theta}{p_0} p$, $\frac{1}{z_1} := \frac{\theta}{p_1} p$. With these definitions we have

$$x + y = p, \frac{1}{z_0} + \frac{1}{z_1} = 1, xz_0 = p_0 \text{ and } yz_1 = p_1.$$

Now using Hölder's inequality we obtain

$$\|f\|_p^p = \|f^p\|_1 = \|f^x f^y\|_1 \overset{\text{Hölder}}{\leq} \|f^x\|_{z_0} \|f^y\|_{z_1} = (\int |f|^{xz_0})^{\frac{1}{z_0}} (\int |f|^{yz_1})^{\frac{1}{z_1}}$$
$$= (\int |f|^{p_0})^{\frac{1-\theta}{p_0}p} (\int |f|^{p_1})^{\frac{\theta}{p_1}p} = (\|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta})^p.$$

□

In the proof of the Riesz-Thorin theorem we will need the following result from complex analysis.

Proposition 1.3 (Three line Lemma)**.** *Let $F : S \to \mathbb{C}$ be bounded and continuous, where $S := \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$. Additionally let F be analytic on S°. For $0 \leq \theta \leq 1$ let $M_\theta := \sup_{y \in \mathbb{R}} |F(\theta + iy)|$. Then we have*

$$M_\theta \leq M_0^{1-\theta} M_1^\theta.$$

To visualize the meaning of the proposition we take a look at the figure below.

Proof. **Step 1:** First of all we investigate the case $M_0, M_1 \leq 1$. So we have to prove that $M_\theta \leq 1$.
Let $z_0 = x_0 + iy_0 \in S^\circ, \epsilon > 0$ and define $F_\epsilon(z) := \frac{F(z)}{1+\epsilon z}$.
This function is also bounded, continuous and analytic on S°. Moreover $\lim_{|y| \to \infty} |F_\epsilon(x + iy)| = 0$ uniformly for $x \in [0, 1]$ since $|F_\epsilon(x + iy)| \leq \frac{|F(x+iy)|}{\epsilon |y|}$ and F is bounded.
Let $r > |y_0|$ such that $|F_\epsilon(x + iy)| \leq 1$ for $0 \leq x \leq 1$ and $|y| = r$. Furthermore let R be the compact rectangle $[0, 1] \times i\,[-r, r]$. This implies that $|F_\epsilon(z)| \leq 1$ on ∂R. The maximum principle for analytic functions [K. Jähnich, Funktionentheorie (Springer, 2004), p. 30, Satz 13] now tells us that $|F_\epsilon(z)| \leq 1\ \forall z \in R$, in particular $|F_\epsilon(z_0)| \leq 1$ and thus $|F(z_0)| = \lim_{\epsilon \to 0} |F_\epsilon(z_0)| \leq 1$.

Step 2: Let M_0, M_1 be arbitrarly and $G(z) = \frac{F(z)}{\alpha^{1-z}\beta^z}$ where $\alpha > M_0$ and $\beta > M_1$. Then G is continuous, bounded and analytic on S° and $|G(z)| \leq 1$ on ∂S and by step 1 $|G(z)| \leq 1$ on S so $M_\theta \leq \alpha^{1-\theta}\beta^\theta$ and $M_\theta \leq M_0^{1-\theta} M_1^\theta$. □

Now that we have all the tools we need, we can formulate and prove the main result of this talk.

Theorem 1.4 (Interpolation theorem of Riesz - Thorin).
Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *and* $0 < \theta < 1$. *Define* p *and* q *by* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
If T *is a linear map such that*

$$T : L^{p_0} \to L^{q_0} \text{ with } \|T\|_{L^{p_0} \to L^{q_0}} = N_0$$

and

$$T : L^{p_1} \to L^{q_1} \text{ with } \|T\|_{L^{p_1} \to L^{q_1}} = N_1.$$

Then we have

$$\|Tf\|_q \leq N_0^{1-\theta} N_1^{\theta} \|f\|_p \quad \forall f \in L^{p_0} \cap L^{p_1} \tag{2}$$

if $\mathbb{K} = \mathbb{C}$ *and*

$$\|Tf\|_q \leq 2N_0^{1-\theta} N_1^{\theta} \|f\|_p \quad \forall f \in L^{p_0} \cap L^{p_1} \tag{3}$$

if $\mathbb{K} = \mathbb{R}$. *In particular, the operator* T *can be extended to a continuous linear map* $T : L^p \to L^q$ *with*

$$\|T\| \leq c N_0^{1-\theta} N_1^{\theta}$$

where $c = 1$ *if* $\mathbb{K} = \mathbb{C}$ *and* $c = 2$ *if* $\mathbb{K} = \mathbb{R}$.

Remark 1.5. Before we prove the theorem we are going to have a closer look on its assertion. If we consider the spaces L^p as functions of $\frac{1}{p}$ we may reinterpret the theorem by saying that

$$C := \left\{ (p, q) \ / \ T : L^{\frac{1}{p}} \to L^{\frac{1}{q}} \right\}$$

is a convex set. Indeed, if we take two points from this set the theorem of Riesz-Thorin shows us that their connection line is also contained in the set.

Furthermore inequality (2) tells us that the mapping

$$(\alpha, \beta) \mapsto \log \|T\|_{\frac{1}{\alpha} \to \frac{1}{\beta}}$$

is convex (which is to say the points lying on and above the graph form a convex set).

Proof. To begin with note that due to our assumptions $Tf \in L^{q_0} \cap L^{q_1}$ for $f \in L^{p_0} \cap L^{p_1}$ and by Lemma 1.2 we have that $Tf \in L^q$ for such f. We treat the case $\mathbb{K} = \mathbb{C}$ first:

Case 1: $p < \infty$ and $q > 1$

• Since the integrable step functions are dense in all L^p-spaces they are dense in $L^{p_0} \cap L^{p_1}$. So it is sufficient to show ineq. (2) for all such functions.

• We will do so by showing that

$$\left| \int (Tf) g \right| \leq N_0^{1-\theta} N_1^{\theta} \tag{4}$$

for all integrable step functions f, g with $\|f\|_p = \|g\|_{q'} = 1$, where as usual $1 = \frac{1}{q'} + \frac{1}{q}$.

Indeed ineq. (4) tells us that the functional

$$\begin{aligned} l : L^{q'} &\to \mathbb{C} \\ g &\mapsto \int (Tf) g \end{aligned}$$

obeys $\|l\| \leq N_0^{1-\theta} N_1^{\theta}$. (Note that here we again used the fact that integrable step functions are dense in $L^{q'}$ ($q' < \infty$ since $q > 1$).)

By [FA1, 2.45] we know that $l \in (L^{q'})' \cong L^q$ is the isometrically isomorphic image of Tf so $\|Tf\|_q \leq N_0^{1-\theta} N_1^{\theta}$.

• To show ineq. (4) we define the step functions f and g by

$$f = \sum_{j=1}^{J} a_j \chi_{A_j}, \qquad g = \sum_{k=1}^{K} b_k \chi_{B_k} \tag{5}$$

and

$$\|f\|_p^p = \sum_{j=1}^{J} |a_j|^p \, \mu(A_j) = 1, \qquad \|g\|_{q'}^{q'} = \sum_{k=1}^{K} |b_k|^{q'} \, \mu(B_k) = 1$$

where μ is the Lebesgue measure on $\mathbb{R}^n$ and A_j resp. B_k are pairwise disjoint. For $z \in \mathbb{C}$ let $p(z)$ and $q'(z)$ be defined as

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}, \qquad \frac{1}{q'(z)} = \frac{1-z}{q'_0} + \frac{z}{q'_1}$$

so $p(0) = p_0$, $p(\theta) = p$ and $p(1) = p_1$ as well as $q'(0) = q'_0$, $q'(\theta) = q'$, $q'(1) = q'_1$. Using the convention $\frac{0}{0} = 0$ we set

$$f_z = |f|^{p/p(z)} \, \frac{f}{|f|} \text{ and } g_z = |g|^{q'/q'(z)} \, \frac{g}{|g|}.$$

Now f_z and g_z are integrable step functions, especially $f_z \in L^{p_1}$ which implies that Tf_z is defined since f_z is again a step function.

Finally we define $F : \mathbb{C} \to \mathbb{C}$ as $F(z) = \int (Tf_z) g_z d\nu$

By eqs. (5) we have

$$F(z) = \sum_{j=1}^{J} \sum_{k=1}^{K} |a_j|^{\frac{p}{p(z)}} \, \frac{a_j}{|a_j|} \, |b_k|^{\frac{q'}{q'(z)}} \, \frac{b_k}{|b_k|} \int_{B_k} T\chi_{A_j} d\nu.$$

This shows that F is a linear combination of terms of the form γ^z with $\gamma > 0$. So F is analytic and satisfies the assumptions of Prop. 1.3, since every function γ^z is bounded in S (see Prop 1.3) by

$$\left|\gamma^{x+iy}\right| = \gamma^x \leq \max\{1,\gamma\} \quad \forall x+iy \in S.$$

Next we estimate $|F(iy)|$ and $|F(1+iy)|$. We have

$$|F(iy)| \overset{\text{Hölder}}{\leq} \|Tf_{iy}\|_{q_0} \|g_{iy}\|_{q_0'} \leq N_0 \|f_{iy}\|_{p_0} \|g_{iy}\|_{q_0'}$$

and furthermore

$$\|f_{iy}\|_{p_0}^{p_0} = \sum_{j=1}^{J} \left| |a_j|^{\frac{p}{p(iy)}p_0} \right| \mu(A_j) \overset{eq.\ (6)}{=} \sum_{j=1}^{J} |a_j|^p \mu(A_j) = \|f\|_p^p = 1,$$

where we have used

$$||a_j|^{\frac{p}{p(iy)}p_0}| = |a_j|^{\Re((p(\frac{1-iy}{p_0})+\frac{iy}{p_1})p_0)} = |a_j|^{\frac{p}{p_0}p_0} = |a_j|^p. \tag{6}$$

Analogously we obtain $\|g_{iy}\|_{q_0'}^{q_0'} = 1$

Summing up we have

$$\sup_{y\in\mathbb{R}} |F(iy)| \leq N_0$$

and repeating the same calculation with $1 + iy$ replacing iy we obtain

$$\sup_{y\in\mathbb{R}} |F(1+iy)| \leq N_1.$$

Now finally Prop. 1.3 yields

$$\left| \int Tfg d\nu \right| = |F(\theta)| \leq \sup_{y\in\mathbb{R}} |F(\theta+iy)| \leq N_0^{1-\theta} N_1^{\theta}$$

So we have estimated ineq. (4) and we are done.

Case 2: $p = \infty$

This assumption immediately implies that $p_0 = p_1 = \infty$. If $q = q_0 = q_1 = 1$ there's nothing to show. So let $q > 1$. Now f need not be integrable and we may choose $f = f_z$ $\forall z$. Analogously we can handle the case $q = 1$, $p < \infty$ (now $g_z = g$).

It remains to show ineq. (3), i.e., the case $\mathbb{K} = \mathbb{R}$. But luckily this follows from ineq. (2) and the following argument. Let $U : L^r_{\mathbb{R}} \to L^s_{\mathbb{R}}$ be a continuous linear operator between real L^p spaces. Furthermore define its canonical extension as $U_{\mathbb{C}}(f + ig) = Uf + iUg$. This map is $\mathbb{C}$-linear, $U_{\mathbb{C}} : L^r_{\mathbb{C}} \to L^s_{\mathbb{C}}$, and the following inequality holds

$$\begin{aligned}\|U_{\mathbb{C}}\| &= \sup_{\|f+ig\|=1} \|U_{\mathbb{C}}(f+ig)\| \leq \sup_{\|f+ig\|=1} (\|U(f)\| + \|U(g)\|) \\ &\leq \sup_{\|f\|=1} \|U(f)\| + \sup_{\|g\|=1} \|U(g)\| \leq 2\|U\|.\end{aligned}$$

Applying this to the assumption of the Theorem we obtain for T by using the extension $T_{\mathbb{C}}$ and ineq. (2)

$$\begin{aligned}\|Tf\|_q = \|T_{\mathbb{C}}f\|_q &\le \|T_{\mathbb{C}}\|_{p_0\to q_0}^{1-\theta}\|T_{\mathbb{C}}\|_{p_1\to q_1}^{\theta}\|f\|_p \\ &\le 2\|T\|_{p_0\to q_0}^{1-\theta}\|T\|_{p_1\to q_1}^{\theta}\|f\|_p = 2N_0^{1-\theta}N_1^{\theta}\|f\|_p\end{aligned}$$

for f real valued. □

Example 1.6. We want to give an example that eq. 2 doesn't hold in $\mathbb{R}$.
Let us have a look at $T(s,t) = (s+t, s-t)$, which has norms $\|T\|_{\infty\to 1} = 2$ and $\|T\|_{2\to 2} = \sqrt{2}$. For $\theta = \frac{1}{2}(p = 4, q = \frac{4}{3})$ we would get

$$\|(s+t, s-t)\|_{\frac{4}{3}} \le 2^{\frac{3}{4}}\|(s,t)\|_4$$

which isn't true for $s = 2$ and $t = 1$.

2. Applications

As a first relevant application we discuss properties of the convolution.

Definition 2.1. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \left\{e^{it} : 0 \le t \le 2\pi\right\}$.
The convolution of $f, g \in L^1(\mathbb{T})$ is defined by

$$(f * g)(e^{is}) = \int_0^{2\pi} f(e^{it})g(e^{i(s-t)})\frac{dt}{2\pi}.$$

Remark 2.2. For $f, g \in L^1(\mathbb{T})$ the convolution $f * g$ is measurable and the following inequalities hold

$$\begin{aligned}\int_0^{2\pi}\left|(f*g)(e^{is})\right|\frac{ds}{2\pi} &\le \int_0^{2\pi}\int_0^{2\pi}\left|f(e^{it})\right|\left|g(e^{i(s-t)})\right|\frac{dt}{2\pi}\frac{ds}{2\pi} \\ &= \int_0^{2\pi}\left|f(e^{it})\right|\int_0^{2\pi}\left|g(e^{i(s-t)})\right|\frac{ds}{2\pi}\frac{dt}{2\pi} \\ &= \int_0^{2\pi}\left|f(e^{it})\right|\frac{dt}{2\pi}\|g\|_1 \\ &= \|f\|_1\|g\|_1 . \end{aligned} \tag{7}$$

In particular, $f * g \in L^1(\mathbb{T})$ and fixing $f \in L^1$ and writing $T_f g = f * g$ we have $T_f : L^1 \to L^1$ with $\|T_f\|_{L^1\to L^1} \le \|f\|_1$. Similarly we have

$$\begin{aligned}\sup_{s\in[0,2\pi]}\left|(f*g)(e^{is})\right| &\le \sup_{s\in[0,2\pi]}\int_0^{2\pi}\left|f(e^{it})\right|\left|g(e^{i(s-t)})\right|\frac{dt}{2\pi} \\ &= \int_0^{2\pi}\left|f(e^{it})\right|\sup_{s\in[0,2\pi]}\left|g(e^{i(s-t)})\right|\frac{dt}{2\pi} \\ &= \int_0^{2\pi}\left|f(e^{it})\right|\frac{dt}{2\pi}\|g\|_\infty \\ &= \|f\|_1\|g\|_\infty . \end{aligned} \tag{8}$$

This yields $f * g \in L^\infty(\mathbb{T})$ for $g \in L^\infty(\mathbb{T})$ and with the notation as above
$T_f : L^\infty \to L^\infty$ again with $\|T_f\|_{L^\infty\to L^\infty} \le \|f\|_1$.

Proposition 2.3 (Young's inequality)**.** *Let* $1 \le p, q \le \infty$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \ge 0$. *If* $f \in L^p$ *and* $g \in L^q$ *then* $f * g \in L^r$ *and we have*

$$\|f * g\|_r \le \|f\|_p \|g\|_q .$$

Proof. $\mathbb{T}$ is finite, so $f * g$ is defined since $L^p[\mathbb{T}], L^q[\mathbb{T}] \subset L^1[\mathbb{T}]$.

Step 1: Let $f \in L^1$ be fixed. Due to ineq. (7) in Remark 2.2 we have

$$\|f * g\|_1 = \|f\|_1 \|g\|_1 \text{ resp. } \|T_f\|_{1\to 1} \le \|f\|_1 .$$

Furthermore ineq. (8) in the Remark above yields

$$\|T_f\|_{\infty\to\infty} \le \|f\|_1 .$$

Using theorem 1.4 with $\theta = \frac{1}{q'}$ where q' is the conjugated exponent of q we get

$$\|T\|_{q\to q} \le \|f\|_1 .$$

That means

$$\|f * g\|_q \le \|T_f\|_{q\to q} \|g\|_q \le \|f\|_1 \|g\|_q \quad \forall f \in L^1, g \in L^q. \tag{9}$$

Step 2: Let now $g \in L^q$ be fixed.

The Hölder inequality for $f \in L^{q'}$ and $e^{is} \in \mathbb{T}$ leads us to

$$\begin{aligned} \left|(f * g)(e^{is})\right| &\le \int_0^{2\pi} \left|f(e^{it})g(e^{i(s-t)})\right| \frac{dt}{2\pi} \\ &\overset{\text{Hölder}}{\le} \left(\int_0^{2\pi} \left|f(e^{it})\right|^{q'} \frac{dt}{2\pi}\right)^{\frac{1}{q'}} \left(\int_0^{2\pi} \left|g(e^{it})\right|^{q} \frac{dt}{2\pi}\right)^{\frac{1}{q}} \\ &= \|f\|_{q'} \|g\|_q , \end{aligned}$$

$$\text{which implies } \|f * g\|_\infty \le \|f\|_{q'} \|g\|_q . \tag{10}$$

Using ineq. (9) and ineq. (10) for the operator $T_g f = f * g$ we have

$$\begin{aligned} &\|T_g\|_{1\to q} \le \|g\|_q \text{ and} \\ &\|T_g\|_{q'\to\infty} \le \|g\|_q . \end{aligned}$$

We now choose θ such that $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{q'}$, then $\theta = \frac{q}{p'}$. Moreover $0 \le \theta \le 1$ since $\frac{1}{p} + \frac{1}{q} \ge 1$. And with this choice of θ we have

$$\frac{1-\theta}{q} + \frac{\theta}{\infty} = \frac{1}{p} - \frac{1}{q'} = \frac{1}{q} - 1 + \frac{1}{p} = \frac{1}{r}.$$

So applying Thm. 1.4 we obtain

$$\|T_g\|_{p\to r} \le \|g\|_q ,$$

which means that

$$\|f * g\|_r \le \|f\|_p \|g\|_q \quad \forall f \in L^p[\mathbb{T}], g \in L^q[\mathbb{T}].$$

□

Another application is the following.

Proposition 2.4. *Using the same assumptions as in Theorem* 1.4 *and supposing that* $T : L^{p_0} \to L^{q_0}$ *is compact, we get that* $T : L^p \to L^q$ *is compact.*

Proof. See [D. Werner, Funktionalanalysis (Sprinter, 2005), p. 79, Satz II.4.5]. □