Convergence of Fourier series

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May 7, 2008

Abstract

Closely following the presentation of [Wer05], we begin by introducing the (formal) Fourier series of a given function f. Using an integral representation of the n-th partial sum $s_n(f,x)$, we are going to discuss the problem of convergence for different kinds of f (e.g. $f \in C^1$ or $f \in C$). Finally, we will prove Fejér's theorem on the summability of Fourier series.

1 Preliminaries

Definition 1.1 (Fourier Series)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be integrable on $[-\pi, \pi]$ and 2π-periodic. Then we can define the Fourier series of f by

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt) \tag{1}$$

with a_k and b_k defined by the following relations:

$$a_k = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,\mathrm{d}t \qquad b_k = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,\mathrm{d}t$$

Remark 1.2 (Convergence, complex version)*.*

1. Note that the definition 1.1 is purely formal, since the limit might not exist. But if the Fourier series converges uniformly to f, then the coefficients necessarily have the above form.

2. Equation 1 can be reformulated using the complex exponential function:
$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} \qquad \text{with } c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,\mathrm{d}t \tag{2}$$

3. a_k, b_k, c_k are connected by

$$c_k = \begin{cases} \frac{1}{2}a_0 & \text{for } k = 0 \\ \frac{1}{2}(a_k - ib_k) & \text{for } k > 0 \\ \frac{1}{2}(a_{-k} + ib_{-k}) & \text{for } k < 0 \end{cases} \tag{3}$$

Definition 1.3 (The space $C_{2\pi}$)**.** Let f be continuous and 2π-periodic. Then f can be identified with an element of the space

$$C_{2\pi} := \big\{ f \in C([-\pi, \pi]) | f(\pi) = f(-\pi) \big\}.$$

$(C_{2\pi}, \|.\|_\infty)$ is a (B)-space.

Theorem 1.4 (Banach-Steinhaus)**.** *Let X be a (B)-space, Y a normed space, I an arbitrary index set and $T_i \in L(X, Y)$ $\forall i \in I$ a family of operators. Then*

$$\sup_{i \in I} \|T_i x\| < \infty \quad \forall x \in X \qquad \Longrightarrow \qquad \sup_{i \in I} \|T_i\| < \infty \tag{4}$$

Proof. For the proof we refer to [Wer05], p.141. $\square$

Definition 1.5 (Partial sums)**.** We define the n-th partial sum of the Fourier series of f by

$$s_n(f, x) := \frac{a_0}{2} + \sum_{k=1}^{n} \big[a_k \cos kx + b_k \sin kx \big] \tag{5}$$

Since this representation is somewhat impractical, we will try to derive an integral representation of (5), which is more suitable for further calculations. We start by using the above and replace a_k, b_k by their definition (cf. (1)):

$$\begin{aligned} s_n(f, t) &= \frac{a_0}{2} + \sum_{k=1}^{n} \big[a_k \cos kx + b_k \sin kx \big] \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \Big[\frac{1}{2} + \sum_{k=1}^{n} \big(\cos kt \cos ks + \sin kt \sin ks \big) \Big] \mathrm{d}s \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \Big[\frac{1}{2} + \sum_{k=1}^{n} \cos k(s - t) \Big] \mathrm{d}s \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(s + t) \Big[\underbrace{\frac{1}{2} + \sum_{k=1}^{n} \cos ks}_{\text{we denote this by (*)}} \Big] \mathrm{d}s \end{aligned}$$

Now let's take a closer look on the expression $(*)$. Using that the cosine is an even function and $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ we calculate

$$\begin{aligned}\frac{1}{2}+\sum_{k=1}^{n}\cos ks &= \frac{1}{2}\sum_{k=-n}^{n}\cos ks = \frac{1}{2}\sum_{k=-n}^{n}e^{iks}\\ &= \frac{1}{2}e^{-ins}\frac{e^{(2n+1)s}-1}{e^{is}-1} = \frac{1}{2}\frac{e^{(n+\frac{1}{2})is}-e^{-(n+\frac{1}{2})is}}{e^{\frac{1}{2}is}-e^{-\frac{1}{2}is}}\\ &= \frac{1}{2}\frac{\sin((n+\frac{1}{2})s)}{\sin\frac{s}{2}}\end{aligned}$$

We are now ready for

Definition 1.6 (Dirichlet kernel)**.** We denote by

$$D_n(s) := \frac{1}{2}\frac{\sin((n+\frac{1}{2})s)}{\sin\frac{s}{2}} \tag{6}$$

with continuous extension in $D_n(0) = n + \frac{1}{2}$ the n-th Dirichlet kernel. This gives rise to the following

$$s_n(f,t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s+t)D_n(s)\,\mathrm{d}s \tag{7}$$

By $f \equiv 1$ we see that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(s)\,\mathrm{d}s = 1. \tag{8}$$

2 Continuity is not enough

Within this section, we'll see that continuity of f is not a sufficient condition for the convergence of its Fourier series. But if $f \in C^1$, everything works fine:

Theorem 2.1. *Let f be continuously differentiable and 2π-periodic. Then $s_n(f,.)$ converges uniformly to f.*

Proof. Choose $h \in \mathbb{R}$ with $0 \le h \le \pi$. Then

$$\begin{aligned}\Big|s_n(f,t)-f(t)\Big| &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(t+s)D_n(s)\,\mathrm{d}s - f(t)\right|\\ &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi}\Big[f(t+s)-f(t)\Big]D_n(s)\,\mathrm{d}s\right|\end{aligned}$$

$$\leq \frac{1}{\pi}\left[\left|\int_{-\pi}^{-h}(\dots)\right| + \underbrace{\left|\int_{-h}^{h}(\dots)\right|}_{(**)} + \underbrace{\left|\int_{h}^{\pi}(\dots)\right|}_{(***)}\right]$$

We now have to find estimates for the right hand side (independent of t!). Let now $0 < h < \pi$ with $h \leq \frac{\epsilon}{\pi\|f'\|_\infty}$. We'll start with $(**)$:

$$\left|\int_{-h}^{h}\Big[f(s+t)-f(t)\Big]D_n(s)\,\mathrm{d}s\right| \leq \int_{-h}^{h}\frac{\overbrace{|f(s+t)-f(t)|}^{\leq\|f'\|_\infty|s|}}{\underbrace{2\sin\frac{|s|}{2}}_{\geq\frac{s}{\pi}}}\underbrace{\left|\sin(n+\frac{1}{2})s\right|}_{\leq 1}\,\mathrm{d}s$$

$$\leq \int_{-h}^{h}\frac{\pi}{2}\frac{\|f'\|_\infty|s|}{|s|}\,\mathrm{d}s$$

$$\leq \frac{\pi}{2}\|f'\|_\infty 2h \quad \leq \quad \epsilon$$

Next we estimate the term $(***)$. To that end, we define

$$f_t(s) = \frac{f(s+t)-f(t)}{\sin\frac{s}{2}} \qquad \text{for } s \in [h,\pi]. \tag{9}$$

Obviously $f_t \in C^1([h,\pi])$, $\sup_{t\in\mathbb{R}}\|f_t\|_\infty < \infty$ and $\sup_{t\in\mathbb{R}}\|f_t'\|_\infty < \infty$. Set $m = n+\frac{1}{2}$. Integration by parts yields

$$\left|\int_h^\pi f_t(s)\sin ms\,\mathrm{d}s\right| = \left|f_t(s)\frac{-1}{m}\cos ms\Big|_h^\pi - \int_h^\pi f_t'(s)\frac{-1}{m}\cos ms\,\mathrm{d}s\right|$$

$$\leq \frac{2\sup_{t\in\mathbb{R}}\|f_t\|_\infty}{m} + \frac{1}{m}\sup_{t\in\mathbb{R}}\|f_t'\|_\infty(\pi-h)$$

$$\leq \frac{C}{m} = \frac{C}{n+\frac{1}{2}} \qquad \text{for some C independent of } t$$

$\Longrightarrow \exists n_0 \in \mathbb{N}$ with $\left|\int_h^\pi(\dots)\right| \leq \frac{C}{n+\frac{1}{2}} \quad \forall n > n_0$. Finally the term $\int_{-\pi}^{h}(\dots)$ can be dealt with in a similar way. Combining these results gives

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\Big[f(t+s)-f(t)\Big]D_n(s)\,\mathrm{d}s\right| \leq \frac{1}{\pi}(\epsilon+\epsilon+\epsilon) < \epsilon \tag{10}$$

$\square$

For $f \in C$ however, this doesn't hold anymore. We even have

Theorem 2.2. *There are continuous, 2π-periodic functions, whose Fourier series does not converge for some $x \in \mathbb{R}$.*

Proof. By contradiction.
Suppose that $s_n(f,t) \to f(t)$ $\forall f \in C_{2\pi}$ and $\forall t \in [-\pi, \pi]$. Then $(s_n(f,t))_{n\in\mathbb{N}}$ is bounded $\forall f \in C_{2\pi}$. Taking $s_n(.,t)$ as an element of $C'_{2\pi}$(for fixed t), this allows us to apply the theorem of Banach-Steinhaus (cf. 1.4). So

$$\sup_{n\in\mathbb{N}} \|s_n(.,t)\|_{C'_{2\pi}} < \infty \tag{11}$$

W.l.o.g. let $t = 0$. Similar to [Wer05], II.1(g) [p. 49/50], we can compute

$$l_n := \|s_n(.,0)\|_{C'_{2\pi}} = \frac{1}{\pi}\int_{-\pi}^{\pi} \Big|D_n(s)\Big| \,\mathrm{d}s \tag{12}$$

For $t = 0$ this means $\sup_{n\in\mathbb{N}} l_n < \infty$. But

$$l_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\big|\sin(n+\frac{1}{2})s\big|}{\big|2\sin\frac{s}{2}\big|}\mathrm{d}s \geq \frac{1}{\pi}\int_{-\pi}^{\pi} \big|\sin(n+\frac{1}{2})s\big|\frac{\mathrm{d}s}{|s|} \tag{13}$$

With $\sigma = (n+\frac{1}{2})s$ we get

$$\begin{aligned} l_n &\geq \frac{2}{\pi}\int_0^{(n+\frac{1}{2})\pi} \big|\sin\sigma\big|\frac{\mathrm{d}\sigma}{\sigma} \geq \frac{2}{\pi}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi} \big|\sin\sigma\big|\frac{\mathrm{d}\sigma}{\sigma} \\ &\geq \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi}\big|\sin\sigma\big|\mathrm{d}\sigma = \Big(\frac{2}{\pi}\Big)^2\sum_{k=1}^{n}\frac{1}{k} \end{aligned}$$

Therefore $l_n \to \infty$, a contradiction to $\sup_n l_n < \infty$. $\square$

The following remark and theorems that conclude this section state various important results. Because of their difficulty and length, no proofs will be given:

Remark 2.3 (DuBois-Reymond). In 1873, DuBois-Reymond gave in [dBR73] the first example of a continuous function with divergent Fourier series. It has the form $f(t) = A(t)\sin(\omega(t)t)$ (for certain $A(t) \to 0$ and $\omega(t) \to \infty$).

Theorem 2.4 (Carleson-Hunt)**.** *Let $f \in L^p([0, 2\pi])$ for $p > 1$. Then the Fourier series of f converges to f almost everywhere.*

Proof. For details we refer to [Fef73]. $\square$

Theorem 2.5 (Kolmogorov)**.** *There are functions belonging to L^1, whose Fourier series diverges almost everywhere.*

Proof. In 1923, Andrey Kolmogorov (at the age of 21!) gave the first example of such a function. Later this result was improved to divergence everywhere. See [Kol23]. $\square$

Theorem 2.6 (Kahane-Katznelson)**.** *Let $E \subseteq \mathbb{R}$ be a set of measure 0. Then there exists a continuous function f, whose Fourier series diverges $\forall x \in E$.*

Proof. For details see [KK66]. $\square$

3 Summability versus convergence

To proof Fejér's theorem, we need

Theorem 3.1 (Korovkin)**.** *For $n \in \mathbb{N}$ let $T_n \in L(C_{2\pi})$. Furthermore let T_n be a positive operator (i.e. $T_n f \geq 0$ if $f \geq 0$). Now set $x_0(t) = 1$, $x_1(t) = \cos t$ and $x_2(t) = \sin t$. Then*

$$T_n x_i \to x_i \quad i = 0, 1, 2 \qquad \Longrightarrow \qquad T_n f \to f \quad \forall f \in C_{2\pi} \tag{14}$$

Proof. Let $f \in C_{2\pi}$ and $\epsilon > 0$. We define

$$y_t(s) := \sin^2 \frac{t-s}{2} \qquad t \in [0, 2\pi] \tag{15}$$

f continuous on a cp. set $\Longrightarrow f$ uniformly continuous.

$$\exists \delta > 0 \text{ s.t. } y_t(s) \leq \delta \Longrightarrow |f(s) - f(t)| \leq \epsilon \tag{16}$$

Let $\alpha := \frac{2\|f\|_\infty}{\delta}$. Then

$$|f(s) - f(t)| \leq \epsilon + \alpha y_t(s) \qquad \forall s, t \in [0, 2\pi] \tag{17}$$

This inequality holds, because for $y_t(s) \leq \delta$ it follows directly from (16) and for $y_t(s) > \delta$, we have

$$\begin{aligned} y_t(s) > \delta \Longrightarrow \epsilon + \alpha y_t(s) = \epsilon + \frac{2\|f\|_\infty}{\delta} y_t(s) &> \epsilon + \frac{2\|f\|_\infty}{y_t(s)} y_t(s) \\ &> |f(s)| + |f(t)| \geq |f(s) - f(t)| \end{aligned}$$

Using the assumptions on T_n and $-\epsilon - \alpha y_t \leq f - f(t) \leq \epsilon + \alpha y_t$, we get

$$-\epsilon T_n x_0 - \alpha T_n y_t \leq T_n f - f(t) T_n x_0 \leq \epsilon T_n x_0 + \alpha T_n y_t \tag{18}$$

and therefore

$$|T_n f - f(t) T_n x_0| \leq \epsilon T_n x_0 + \alpha T_n y_t \tag{19}$$

Note that because of $y_t(s) = \frac{1}{2}\big[1 - \cos t \cos s - \sin t \sin s\big]$, we have that $y_t \in \operatorname{span}\{x_0, x_1, x_2\}$.

By assumption $T_n x_0 \to x_0$ and

$$\begin{aligned} T_n y_t - y_t &= \frac{1}{2}\big[T_n 1 - \cos t\, T_n \cos s - \sin t\, T_n \sin s - 1 + \cos t \cos s + \sin t \sin s\big] \\ &= \frac{1}{2}\big[(T_n x_0 - x_0) + \cos t (x_1 - T_n x_1) + \sin t (x_2 - T_n x_2)\big] \to 0 \end{aligned}$$

So apply $\epsilon T_n x_0 + \alpha T_n y_t \to \epsilon$ for $n \to \infty$ and $|T_n x_0 - x_0| \leq \epsilon$ to estimate (19) to get

$$\|T_n f - f\|_\infty \leq 2\epsilon \tag{20}$$

□

Remark 3.2 (On Korovkin's theorem). Theorem 3.1 is actually Korovkin's second theorem. His first theorem states a similar result for $f \in C([0,1])$ with $x_i(t) = t^i, \quad i = 0, 1, 2$. See [Wer05], p.143, for details.

Definition 3.3 (Fejér kernel)**.** By

$$F_n(s) = \frac{1}{2n} \frac{\sin^2 n\frac{s}{2}}{\sin^2 \frac{s}{2}} \tag{21}$$

with continuous extension $F_n(0) = \frac{1}{2}n$, we define the n-th Fejér kernel. It can also be written as

$$F_n(s) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(s) \tag{22}$$

Lemma 3.4 (Another integral representation)**.** *Let*

$$(T_n f)(x) := \frac{1}{n} \sum_{k=0}^{n-1} s_n(f, x). \tag{23}$$

Then T_n has the following integral representation using the Fejer kernel:

$$(T_n f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+s) F_n(s) \,\mathrm{d}s \tag{24}$$

Proof. Similar to 1.5:

$$(T_n f)(t) = \frac{1}{n} \sum_{k=0}^{n-1} s_n(f, t)$$

$$= \frac{1}{n}\frac{1}{\pi}\int_{-\pi}^{\pi} f(s+t)\frac{\sum_{k=0}^{n-1}\sin(k+\frac{1}{2})s}{2\sin\frac{s}{2}}\,\mathrm{d}s \quad (*)$$

Using $\sum_{k=0}^{n-1} sin(k+\frac{1}{2})s = Im\sum_{k=0}^{n-1} e^{i(k+\frac{1}{2})s}$ and the formula for finite geometric series, we have

$$\begin{aligned}\sum_{k=0}^{n-1}\sin(k+\frac{1}{2})s &= \mathrm{Im}\Big(e^{\frac{is}{2}}\frac{e^{ins}-1}{e^{is}-1}\Big) = \mathrm{Im}\Big(\frac{e^{ins}e^{\frac{is}{2}}-e^{\frac{is}{2}}}{e^{is}-1}\cdot\frac{e^{-\frac{is}{2}}}{e^{-\frac{is}{2}}}\Big)\\ &= \mathrm{Im}\Big(\frac{e^{ins}-1}{e^{\frac{is}{2}}-e^{-\frac{is}{2}}}\Big) = \mathrm{Im}\,\frac{\cos ns + i\sin ns - 1}{2i\sin\frac{s}{2}}\\ &= \frac{1-\cos ns}{2\sin\frac{s}{2}} = \frac{\sin^2 n\frac{s}{2}}{\sin\frac{s}{2}}\end{aligned}$$

Combining this result with $(*)$ completes the proof. □

Theorem 3.5 (Fejér; Summability of Fourier series)**.** *Let $f \in C_{2\pi}$ and T_n be defined as in Lemma 3.4. Then $(T_n f)$ converges uniformly to f. (The Fourier series converges to f in the sense of Césaro.)*

Proof. For $-\pi \le s \le \pi$ we have that $F_n(s) \ge 0$. Therefore T_n is a positive operator on $C_{2\pi}$. Now set $x_0(t) = 1$, $x_1(t) = \cos t$ and $x_2(t) = \sin t$.

$$\begin{array}{ll} s_k(x_0,.) = x_0 & \forall k \ge 0 \\ s_0(x_1,.) = s_0(x_2,.) = 0 & \\ s_k(x_1,.) = x_1, s_k(x_2,.) = x_2 & \forall k \ge 1 \end{array}$$

With $T_n f = \frac{1}{n}\sum_{k=0}^{n-1} s_k(f,.)$ we now see that $T_n x_0 = x_0$, $T_n x_1 = \frac{n-1}{n}x_1$ and $T_n x_2 = \frac{n-1}{n}x_2$. Applying thm. 3.1 we have that

$$\|T_n f - f\|_\infty \to 0 \quad \forall f \in C_{2\pi} \tag{25}$$

This concludes the proof. □

Corollary 3.6 (Weierstrass)**.** *The space of trigonometric polynomials is dense in $(C_{2\pi}, \|.\|_\infty)$.*

Proof. $T_n f$ is a trigonometric polynomial for $f \in C_{2\pi}$. Apply thm. 3.5. □

Remark 3.7 (On the proof of thm. 3.5)*.* See [Ste07], p.121A-121C, for a direct proof that does not use any of the above identities.

References

[dBR73] Paul du Bois-Reymond, *Ueber die fourierschen reihen*, Nachr. Kön. Ges. Wiss. Göttingen **21** (1873), 571–582 (german).

[Fef73] Charles Fefferman, *Pointwise convergence of fourier series*, Ann. of. Math. **98** (1973).

[KK66] Jean-Pierre Kahane and Yitzhak Katznelson, *Sur les ensembles de divergence des series trigonometriques*, Studia math. **26** (1966), 305–306 (french).

[Kol23] Andrey Kolmogorov, *Une serie de fourier-lebesgue divergente partout*, Fundamenta math. **4** (1923), 324–328 (french).

[Ste07] Roland Steinbauer, *Funktionalanalysis 1*, 2007 (english).

[Wer05] Dirk Werner, *Funktionalanalysis*, fifth ed., Springer, 2005 (german).