The singularity theorems of General Relativity and their low regularity extensions

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based on past and ongoing work mainly with

Melanie Graf, James Grant,

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The singularity theorems in General Relativity

About

- rigorous results in Lorentzian differential geometry
- physically reasonable ass. (collapse, expanding univ.) $\rightsquigarrow$ singularities
- singularity $\sim$ incomplete causal geodesic
- occurrence of spacetime singularities as a generic feature of GR

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Here: Explain the structure of arguments & **low regularity** extensions

- **causality theory** for metrics down to g (loc. Lipschitz) continuous
- **analysis** of **geodesic focusing** down to $g \in C^1$, or locally Lipschitz

The structure of the singularity theorems

Pattern theorem (Senovilla, 1998)

A spacetime (M, g) is singular if it satisfies:

(I) A suitable initial condition,

(E) a condition on the curvature (energy condition),

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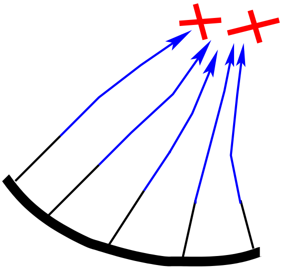
(I) $\rightsquigarrow$ causal geodesics start focusing

(E) $\rightsquigarrow$ focusing goes on (Raychaud. Riccati)

$\rightsquigarrow$ focal/conjugate point

$\rightsquigarrow$ geos. stop maximising

(C) $\rightsquigarrow$ there are maximising causal geos



Resolution: some causal geodesics stop existing before (first) conjugate point

The three classical theorems

Thm. (Penrose, 1965) A spacetime is future null incomplete if

(E) $\mathrm{Ric}\,(X,X) \geq 0$ for all X null (NEC)

(C) there is a non-compact Cauchy hypersurface Σ

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Thm. (Hawking and Penrose, 1970) Improvements:

- (E) (SEC) and genericity
- (C) chronological **(only!)**
- (I) cp. achronal set w/o. edge, trapped surface/point/submanifold

Analytic core: geodesic focusing under curvature bds.

- Geodesic curves $\gamma : I \to M$ with $\nabla_{\dot\gamma}\dot\gamma = 0$
- For data $p \in M$, $v \in T_pM$ unique maximally extended solution on $I_{p,v}$

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Message: Bounds on Ricci curvature bound length of maximisers.

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- causal geodesic completeness $\Longrightarrow$ inextendability

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- apply thms. to maximally extended spacetimes

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!! Here **regularity of extensions** becomes an issue
classical thms. only rule out C^2-extensions
but extensions are ok as long as field equations make sense

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Main challenges

(1) extend causality theory: deprived of main geometric tools

(2) extend analytic core: focusing under distributional curvature bds.

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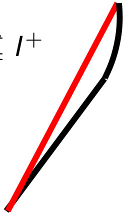
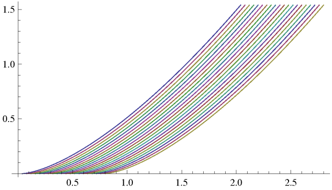
$(J^+ \circ I^+) \not\subseteq I^+$



(2) light cones bubble up



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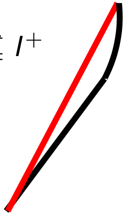

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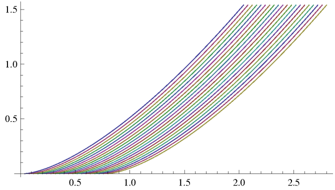

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✓ some topological features are more robust: Avez-Seifert [Sämann, 16] cone structures [Bernard & Suhr, 18], [Minguzzi, 19]

Key analytic techniques: convolution & Friedrichs lemma

Basis: chartwise regularisation of metric by convolution

$$g_\varepsilon(x) := g \star_M \rho_\varepsilon(x) := \sum \chi_i(x)\, \psi_i^* \Big(\big(\psi_{i\,*}(\zeta_i \cdot g)\big) * \rho_\varepsilon \Big)(x).$$

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Lemma (Regularisation for Lipschitz g) [CG,12], [CGHKS,24]

There are smooth Lorentzian metrics $\check{g}_\varepsilon$, $\hat{g}_\varepsilon$ with

- $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ sandwiched light cones via tweaked convolution
- $\check{g}_\varepsilon$, $\hat{g}_\varepsilon \to g$ in $W^{1,p}_{\mathrm{loc}}(M)$ $(1 \le p < \infty)$

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Non-smooth focusing: The rough guide

1. Formulate distributional (EC) for $g \in \mathrm{Lip}$
2. Derive surrogate (EC) for $\mathrm{Ric}[\check{g}_\varepsilon](X, X)$ (on K cp.) ...
3. still show smooth focusing for $\check{g}_\varepsilon$-geodesics ...extend analysis
4. show that geos of g stop maximising/existing ...extend causality

Non-smooth focusing: Following the rough guide

- Start with distributional condition $\mathrm{Ric}[g] \geq 0$
- Problem: $\mathrm{Ric}[\check{g}_\varepsilon] \to \mathrm{Ric}[g]$ only distributionally
  $\rightsquigarrow$ cannot carry $\mathrm{Ric}[g] \geq 0$ through construction
- Solution: compatibility of distinct regularisations

$$\mathrm{Ric}[\check{g}_\varepsilon] \; - \; \underbrace{\mathrm{Ric}[g] \star_M \rho_\varepsilon}_{\geq 0}$$

locally uniformly only for $g \in C^1$
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Lemma (Compatibility of regularisations for $g \in \mathrm{Lip}$) [CGHKS,24]

- $\|\mathrm{Ric}[\check{g}_\varepsilon] - \mathrm{Ric}[g] \star_M \rho_\varepsilon\|_{L^p(K)} \to 0 \; (1 \leq p < \infty)$
- $\|\mathrm{Ric}[\check{g}_\varepsilon] - \mathrm{Ric}[g] \star_M \rho_\varepsilon\|_{L^\infty(K)} \leq C_K$ (advanced Friedrichs lemma)

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Feed into the geometric focusing machinery via

- volume estimates [Treude & Grant, 13]
- segment inequality [Graf, Kontou, Ohanyan, Schinnerl, 24]
  estimate line-$\int$ by volume-$\int$ inspired by [Cheeger & Colding, 96]

The Penrose and the Hawking theorems in C^1/Lipschitz

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Thm. (CGHKS, 2024) Let g be a loc. Lipschitz metric s.t.

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(I) slab mean curvature bounded below by $b > 0$.

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The Hawking-Penrose theorem in C^1

Thm. (Hawking, Penrose, 1970) A C^2-spacetime is caus. incompl. if

(E) (SEC) and the genericity cond. hold i.e.,

$$\forall\gamma \quad \exists t_0: \quad R(.,\dot\gamma)\dot\gamma : [\dot\gamma(t_0)]^\perp \to [\dot\gamma(t_0)]^\perp \quad \text{non-trivial} \qquad \text{(GC)}$$

(C) it is chronological

and it has at least one of

(I1) a cp. achronal set without edge, (I2) a future trapped surface

(I3) a closed future trapped submanifold w. $\sum_{i=1}^{n-m}\langle R(E_i,\dot\gamma)\dot\gamma, E_i\rangle \geq 0$

(I3) a future trapped point .

The Hawking-Penrose theorem in C^1

Thm. (KOSS, 2022) A C^1-spacetime is caus. incompl. if

(E) (DSEC) and the $\mathcal{D}'$-genericity cond. hold i.e.,

$$\forall\gamma \quad g\big(R(\tilde{X},\tilde{V})\tilde{V},\tilde{X}\big) \geq C > 0 \text{ in } {\mathcal{D}'}^{(1)}(U) \qquad \text{(DGC)}$$

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(B) it is maximally, causally non-branching and it has at least one of

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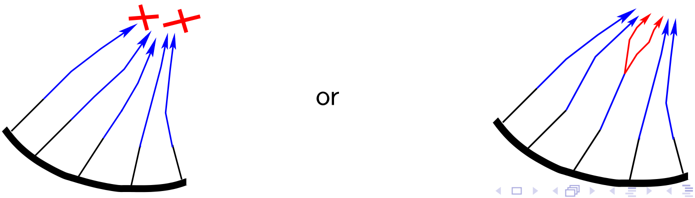

Summary, discussion, results beyond the spacetime setting

(0) spacetime results above for $g \in C^1$/Lipschitz

- close to the class. results; natural extensions of most notions
- but additional **non-branching** assumption in HP-thm.
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- segment inequality (worldvolume, QEIs): $\sim$ semi-classical thms.
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(1) causal cone structures [Minguzzi, 2019]

- upper semi-cont. distribution of cones on M (generalises light cones)
- causal core of singularity theorems may be established
- analytic parts (ECs) only to produce sets with specific causality props.

(Causal Penrose theorem, Minguzzi, 2019)

Let (M, C) be a globally hyperbolic closed cone structure admitting a non-compact stable Cauchy hypersurface Σ. Then there are no compact future trapped sets and if Σ is non-empty and compact there is an inextendible future null geodesic entirely contained in $E^+(S)$.

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(2) Synthetic approaches: Lorentzian length spaces

- causal space $(X, d, \leq \ll, \tau)$ with τ intrinsic
- (timelike) sectional curvature bounds via triangle comparison

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Let $Y = (a, b) \times_f X$ be a warped product (X metric length space, $f \in C^\infty$, non-const.) with positive timelike sectional curvature. Then $a > -\infty$ or $b < \infty$ and hence Y is past/future timelike geodesically incomplete.

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$L(\gamma = (\alpha, \beta)) = \int \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2 v_\beta^2}$ with v_β the metric derivative.

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- smooth metric measure spacetimes [McCann, 20], [Mondino, Suhr, 22]
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Let X be a timelike non-branching, globally hyperbolic LLS with a TMCP property. Let V be a Borel achronal future timelike complete subset with mean curvature bded above. Then every future timelike geodesic starting in V has a bounded maximal domain of existence.

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- synthetic (NEC) [McCann, 23], [Braun, McCann, 24]
- first-order Sobolev calculus on metric measure spacetimes (maximal weak subslope of time functions akin L-modulus of diff.) [Beran, Braun, Calisti, Gigli, McCann, Ohanyan, Rott, Sämann, 24]

Some Literature

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