The focusing of geodesics under distributional curvature bounds

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based on ongoing work, jointly with

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Background: Singularity theorems of General Relativity

Focusing of geodesics under **curvature bounds** is the
main analytical ingredient in the **singularity thms. of GR**

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Contains quite some **interesting analysis**

- regularisation of distributional curvature,
  & Friedrichs lemma-type estimates

Main actors: geodesics, maximisers, conjugate points

- Geodesic curves $\gamma : I \to M$ with $\nabla_{\dot\gamma}\dot\gamma = 0$, locally

$$\ddot\gamma^i(s) = \Gamma^i_{jk}(\gamma(s))\,\dot\gamma^j(s)\,\dot\gamma^k(s) \qquad \text{with } \Gamma \sim g^{-1}\,\partial g \text{ (curvature)}$$

- For data $p \in M$, $v \in T_pM$ (here always v causal) there is a unique maximally extended solution on $I_{p,v}$

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Task: Proof occurrence of conjugate points via geodesic focusing

Geodesic focusing under curvature bounds

Expansion θ of geodesics: $\boxed{\theta \to -\infty \iff \text{conjugate point}}$

obeys **Raychaudhuri equation** along (causal) geodesic $\gamma : I \to M$

$$\dot{\theta} \;=\; -\mathrm{Ric}\,(\dot{\gamma}, \dot{\gamma}) - \mathrm{tr}(\sigma^2) - \frac{\theta^2}{3}$$

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- Assume energy condition: $\boxed{\mathrm{Ric}(\dot{\gamma},\dot{\gamma}) \geq 0}$ (SEC)
- Assume initial condition: $\theta(0) < 0$

  $\Longrightarrow \theta \to -\infty$ for some t in $[0, -3/\theta(0)) \Longrightarrow$ conjugate point

  $\Longrightarrow \gamma$ stops maximising, **hence existing** before $-3/\theta(0)$

Key argument: Estimate on Ricci curvature provides incompleteness

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- Past:
  - $g \in C^{1,1}$ ✓ [KSSV,15], [KSV,15], [GGKS,18]
  - $g \in C^1$ ✓ [Graf,20], [SS,21], [KOSS,22]
  - Synthetic setting: [GKS,19], [Cavalletti & Mondino,20], ...

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Main challenges: from distributional curvature of $g \in \mathrm{Lip}$ get

(A) useful curvature bounds on regularisations via convergence,

(B) and put this into geometric machinery (focusing)
without convergence of geodesics ($\nexists$ geos. for g)

Low regularity: How?

Basis: chartwise regularisation of metric by convolution

$$g_\varepsilon(x) := g \star_M \rho_\varepsilon(x) := \sum \chi_i(x)\, \psi_i^*\Big(\big(\psi_{i\,*}(\zeta_i \cdot g)\big) * \rho_\varepsilon\Big)(x).$$

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Lemma (Regularising Lipschitz g) [Chrusciel&Grant,12], [CGHKS,24]

There are smooth Lorentzian metrics $\check{g}_\varepsilon$, $\hat{g}_\varepsilon$ with

- $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ (sandwiched lightcones)
- $\check{g}_\varepsilon$, $\hat{g}_\varepsilon \to g$ in $W^{1,p}_{\mathrm{loc}}(M)$ $(1 \leq p < \infty)$

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Lipschitz-focusing: The rough guide

1. Formulate distributional (SEC) for $g \in \mathrm{Lip}$
2. Derive surrogate (SEC) for $\mathrm{Ric}[\check{g}_\varepsilon](X, X)$ (on K cp.) ...(A)
3. still show smooth focusing for $\check{g}_\varepsilon$-geodesics
4. show that geodesics of g stop maximising/existing ...(B)

Lipschitz-focusing: Following the rough guide

- Start with distributional condition $\mathrm{Ric}[g] \geq 0$
- Problem: $\mathrm{Ric}[\check{g}_\varepsilon] \to \mathrm{Ric}[g]$ only distributionally
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Lemma (Compatibility of Lip. regularisations [CGHKS,24]

- $\|\mathrm{Ric}[\check{g}_\varepsilon] - \mathrm{Ric}[g] \star_M \rho_\varepsilon\|_{L^1(K)} \to 0$
- $\|\mathrm{Ric}[\check{g}_\varepsilon] - \mathrm{Ric}[g] \star_M \rho_\varepsilon\|_{L^\infty(K)} \leq C_K$ on all compact K

This is issue (A)

. . . to be fed into the geometric focusing machinery later on (B)

Lipschitz-focusing details: A Friedrichs-type lemma is key

relevant terms in $\mathrm{Ric}[\check{g}_\varepsilon] - \mathrm{Ric}[g] \star_M \rho_\varepsilon$

$$\underbrace{\left[(\psi_\beta)_* g_\varepsilon\right]^{ij}}_{=:a_\varepsilon} \Big(\underbrace{\left[\xi \partial_k ((\psi_\beta)_* g)_{lm}\right]}_{=:f} * \rho_\varepsilon \Big) - \Big(\underbrace{\left[(\psi_\beta)_* g\right]^{ij}}_{=:a} \underbrace{\xi \partial_k ((\psi_\beta)_* g)_{lm}}_{=f} \Big) * \rho_\varepsilon$$

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Prove $\boxed{a_\varepsilon f_\varepsilon - (af)_\varepsilon \to 0 \text{ in } W^{1,1} \text{ \& bounded in } W^{1,\infty} \text{ on } K \Subset M \text{ for}}$

$$a \in \mathrm{Lip}, \quad f \in L^\infty, \quad C^\infty \ni a_\varepsilon \to a \text{ loc. unif.}, \quad f_\varepsilon := f * \rho_\varepsilon,$$

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Write relevant term as integral op. [Braverman, Milatovic, Shubin,02]

$$K_\varepsilon f(x) = \int k_\varepsilon(x,y) f(y) dy = \int \partial_x \Big(\big(a(x) - a(y)\big) \rho_\varepsilon(x-y) \Big) f(y)\, dy$$

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- works since $\int_K \left| \frac{\partial a(x)}{\partial x^j} \right| \int_{\mathbb{R}^n} |f(y) - f(x)|\, \rho_\varepsilon(x-y)\, dy\, dx$

splitting the task works $\qquad \le \varepsilon\, \mathrm{Lip}(a) \|\nabla f\|_{L^\infty}$

Feeding the analytic result into the focusing machinery

Lemma (Curvature estimates) [CGHKS,24]

- $\mathrm{Ric}[\check{g}_\varepsilon]_-(\dot\gamma,\dot\gamma) \to 0$ in $L^1(K)$
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$Reg(T) \ni x$ if geo. starting at x max. up to T

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Volume estimate [Treude, Grant, 13]
& segment inequality: line-$\int$ from volume-$\int$
[Graf, Kontou, Ohanyan, Schinnerl, 24]
inspired by [Cheeger & Colding, 96]

$$\mathrm{area}_g(Reg(T)) \leq \limsup \mathrm{area}_{\check{g}_\varepsilon}(^{\check{\varepsilon}}Reg(T))$$
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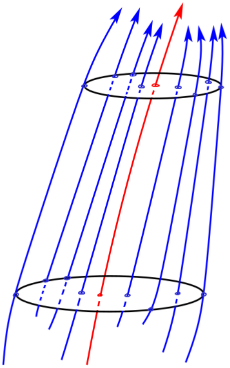

$Reg(T)$

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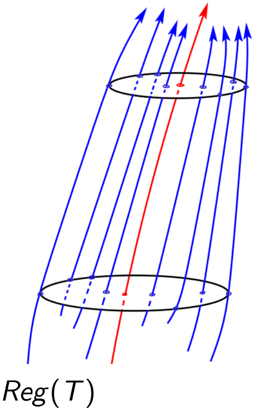
So $\mathrm{area}_g Reg(T) = 0$; as good as a conjugate point



Some related Literature

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