Regularisation and Curvature Bounds

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ongoing joint work with

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The wider topic

Focusing of geodesics under **curvature bounds** is the
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- rigorous results in Lorentzian differential geometry
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- singularity $\sim$ incomplete causal geodesic

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Here: bring out the analysis underneath the geometry

- classical: analysis of Riccati equation, comparison results
- low regularity: $g \in \mathrm{Lip}$, distributional curvature & regularisation

The structure of the singularity theorems

Pattern theorem [José Senovilla, 1998]

A smooth spacetime (M, g) is singular if it satisfies:

(I) A suitable initial condition,

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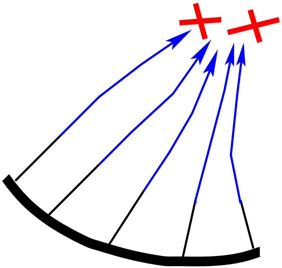
global structure of spacetime

- (I) $\rightsquigarrow$ causal geodesics start focusing
- (E) analysis of Riccati/Raychaudhuri eq.
  - $\rightsquigarrow$ focusing goes on
  - $\rightsquigarrow$ focal/conjugate point
  - $\rightsquigarrow$ geos. stop maximising
- (C) $\rightsquigarrow$ there are maximising causal geos



Resolution: some causal geodesics stop existing before (first) conjugate point

Geodesics & maximisers

- Geodesics: curves $\gamma : I \to M$ with $\nabla_{\dot\gamma}\dot\gamma = 0$, i.e.,

$$\ddot\gamma^i(s) = \Gamma^i_{jk}\big(\gamma(s)\big)\,\dot\gamma^j(s)\,\dot\gamma^k(s) \qquad \text{with } \Gamma \sim g^{-1}\,\partial g$$

- For data $p \in M$, $v \in T_pM$ unique max. extended sol.
- Locally causal geodesics ($g(\dot\gamma, \dot\gamma) \leq 0$) **maximise** Lor. distance

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- Key task

Link the curvature to the occurrence of conjugate points

Geodesic focusing

Raychaudhuri eq. for **expansion** θ along causal geodesic $\gamma : I \to M$

$$\dot{\theta} \;=\; -\mathrm{Ric}(\dot{\gamma},\dot{\gamma}) - \mathrm{tr}(\sigma^2) - \frac{\theta^2}{3}$$

$\mathrm{Ric} \sim \partial^2 g \, (\partial g)^2$

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- Assume energy condition: $\boxed{\mathrm{Ric}(\dot\gamma,\dot\gamma) \geq 0}$ (SEC)
- Assume initial condition: $\theta(0) < 0$

  $\implies \theta \to -\infty$ for some t in $[0, -3/\theta(0))$.

  $\implies \gamma$ has conjugate point

  $\implies \gamma$ stops maximising before $-3/\theta(0)$

Estimate on Ricci curvature says when geos stop maximising

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  [G,20], [KOSS,22], [SS,21]
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- Issue 1: relation between approaches ?
- Issue 2: $g \in \mathrm{Lip}$ long-term goal: $g \in H^1 \cap L^\infty$

- Main challenges: from distributional curvature get
  (1) useful curvature bounds on regularisations
  (2) omit restricting to single geodesics

Low regularity: How?

Basis: chartwise regularisation of metric by convolution

$$g_\varepsilon(x) := g \star_M \rho_\varepsilon(x) := \sum \chi_i(x)\, \psi_i^* \Big(\big(\psi_{i\,*}(\zeta_i \cdot g)\big) * \rho_\varepsilon \Big)(x).$$

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Lemma (Reg. and conv. for $g \in \mathrm{Lip}$) [CG,12], [CGHKS,24]

There are smooth Lorentzian metrics $\check{g}_\varepsilon$, $\hat{g}_\varepsilon$ with

- $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ (lightcones adjusted via tweaked convolution)
- $\check{g}_\varepsilon$, $\hat{g}_\varepsilon \to g$, and $(\check{g}_\varepsilon)^{-1}$, $(\hat{g}_\varepsilon)^{-1} \to g^{-1}$ in $W^{1,p}_{\mathrm{loc}}(M)$ $(1 \le p < \infty)$

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Lipschitz-focusing: The rough guide

1. Formulate distributional (EC) for $g \in \mathrm{Lip}$
2. Derive surrogate (EC) for $\check{g}_\varepsilon$: $\mathrm{Ric}[\check{g}_\varepsilon](X, X) \ge -\delta$ (on K cp.)
3. still show smooth focusing for $\check{g}_\varepsilon$
4. show that geodesics of g stop maximising.

Lipschitz-focusing: The rough guide

Paradigm: Curvature bds. characterised by reg. [KOV,23]

distributional $\mathrm{Ric}[g] \geq 0 \quad \Longleftrightarrow \quad$ for all δ: $\mathrm{Ric}[g_\varepsilon] \geq -\delta$ for ε small

- Problem: $\mathrm{Ric}[\check{g}_\varepsilon] \to \mathrm{Ric}[g]$ only distributionally
  $\rightsquigarrow$ cannot carry $\mathrm{Ric}[g] \geq 0$ through construction
- Solution: compatibility of distinct regularisations
  $$g \in C^1: \quad \underbrace{\mathrm{Ric}[g] \star_M \rho_\varepsilon}_{\mathrm{Ric}[g]\star_M \rho_\varepsilon(X,X)\geq 0} - \mathrm{Ric}[g_\varepsilon],\ \mathrm{Ric}[g_\varepsilon] - \mathrm{Ric}[\check{g}_\varepsilon] \to 0 \text{ loc. unif.}$$

Lemma (Compatibility of reg. for $g \in \mathrm{Lip}$) [CGHKS,24]

- $\|\mathrm{Ric}[\check{g}_\varepsilon] - \mathrm{Ric}[g] \star_M \rho_\varepsilon\|_{L^p(K)} \to 0$ $(1 \leq p < \infty)$
- $\|\mathrm{Ric}[\check{g}_\varepsilon] - \mathrm{Ric}[g] \star_M \rho_\varepsilon\|_{L^\infty(K)} \leq C_K$

To be fed into the geometric machinery later on.

Compatibility of reg: Friedrichs lemma as the key step

relevant terms in $\mathrm{Ric}[\check{g}_\varepsilon] - \mathrm{Ric}[g] \star_M \rho_\varepsilon$

$$\underbrace{\left[(\psi_\beta)_* g_\varepsilon\right]^{ij}}_{=:a_\varepsilon} \Big(\underbrace{\left[\xi \partial_k ((\psi_\beta)_* g)_{lm}\right]}_{=:f} * \rho_\varepsilon \Big) - \Big(\underbrace{\left[(\psi_\beta)_* g\right]^{ij}}_{=:a} \underbrace{\xi \partial_k ((\psi_\beta)_* g)_{lm}}_{=f} \Big) * \rho_\varepsilon$$

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Prove $\boxed{a_\varepsilon f_\varepsilon - (af)_\varepsilon \to 0 \text{ in } W^{1,p}\ (1 \le p < \infty) \text{ \& bounded in } W^{1,\infty} \text{ for}}$

$$a \in \mathrm{Lip}, \quad f \in L^\infty, \quad C^\infty \ni a_\varepsilon \to a \text{ loc. unif.}, \quad f_\varepsilon := f * \rho_\varepsilon,$$

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Write relevant term as integral op. [Braverman, Milatovic, Shubin,02]

$$K_\varepsilon f(x) = \int k_\varepsilon(x,y) f(y) dy = \int \partial_{y^j} \Big(\big(a(x) - a(y)\big) \rho_\varepsilon(x-y) \Big) f(y)\, dy$$

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Properties of kernels k_ε

$$\int |k_\varepsilon(x,y)|\, dx \le C, \quad \int |k_\varepsilon(x,y)|\, dx\, dy \le C_1, \quad \int k_\varepsilon(x,y)\, dy = 0$$

give $\|K_\varepsilon f\|_{L^1(K)} \to 0$ for all $f \in C_c^\infty(\mathbb{R}^n)$ and that sufffices by UBP

Feeding the analytic result into the geometric machinery

Lemma (Curvature estimates) [CGHKS,24]

- $\mathrm{Ric}[\check{g}_\varepsilon]_-(U,U) \to 0$ in L^1 ($\forall U$ $\check{g}_\varepsilon$ causal)
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Look at set of good points in Cauchy surface
$Reg(T) \ni x$ if geo. starting at x max. up to T

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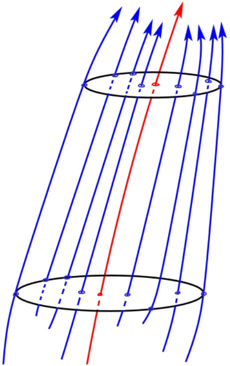

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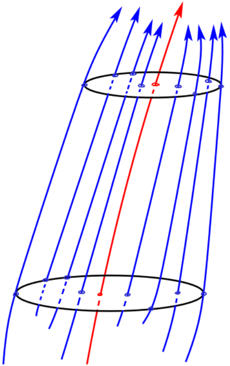

$Reg(T)$

So $\mathrm{area}_g Reg(T) = 0$; replacement for conjugate pts.

The result

Lipschitz Hawking singularity theorem [CGHKS,24]

Let (M, g) be a Lipschitz spacetime such that

(C) There is a spacelike Cauchy surface Σ with

(I) mean curvature $H_\Sigma \geq \beta > 0$, and

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Then (M, g) is causal geodesically incomplete.

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Comparison with synthetic theorem of [Cavalletti & Mondino,20] needs

- comparison of curvature conditions:

  synthetic, optimal-transport based vs. distributional

- C^1-Riemannian: $\Longleftarrow$ ✓ $\Longrightarrow$ almost [KOV,23]
- C^1-Lorentzian $\Longleftarrow$ ✓ [Braun & Calisti,23]

  So [CV,20] (almost) implies [G,20]

- Lipschitz ???

Some Literature

- [GGKS,18] M. Graf, J. Grant, M. Kunzinger, R. Steinbauer, The Hawking-Penrose Singularity Theorem for $C^{1,1}$-Lorentzian Metrics. *Commun. Math. Phys.*, 360(3): 1009–1042, 2018.
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