# Synthetic curvature for GR and beyond

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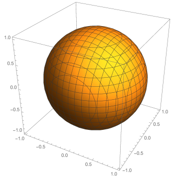
# Curvature beyond smooth spacetimes

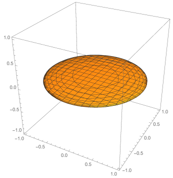
## Why at all?

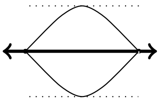
- physically relevant *models* (matched spacetimes, impulsive wave, etc.)
- *PDE* point-of-view
- *singularities* vs *curvature blow-up* — *CCH* of Penrose
- approaches to *Quantum Gravity* (no metric, e.g. causal sets)

## Why it matters?

Basic geometric properties change even if $g \in C^{1,\alpha}$







*Squeezing a sphere*:
Equator still geodesic but it's always shorter to deviate into hemispheres (Hartman-Wintner '52)

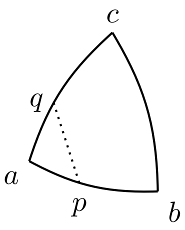
# How to detect curvature: A glimpse on Riemannian world

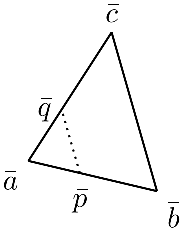
Sectional curvature $\mathrm{Sec}(X,Y) = \dfrac{\langle R(X,Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$

**Theorem (Toponogov)** $\mathrm{Sec} \geq K \iff$

For all (small) geodesic triangles $\triangle abc$ in $(M,h)$ consider a comparison triangle $\triangle \bar a\bar b\bar c$ in the $2$D space of const. curvature $K$. Then for all for all $p,q$ on its sides and corresponding comparison points $\bar p$, $\bar q$

$$d_h(p,q) \geq \bar d(\bar p, \bar q).$$




Triangle condition

- needs no manifold structure
- only distances between pts.
- works on metric spaces

# Sectional curvature bounds for metric spaces

## Definition (Length space)

A metric space $(X, d)$ is called a *length space* if $d$ is intrinsic, i.e.,

$$d(x, y) = \inf\{L(\gamma) \mid \gamma \text{ from } x \text{ to } y \text{ continuous}\}$$

geodesics $\gamma : [0, 1] \to X$ with $d(\gamma(s), \gamma(t)) = |t - s| \cdot d(\gamma(0), \gamma(1))$

## Definition (Synthetic curvature bounds)

A length space has curvature bounded below by $K$ if (locally) for all triangles $\triangle abc$ and their comparison triangles $\triangle \bar{a}\bar{b}\bar{c}$ and all points $p$, $q$ on its sides and corresponding $\bar{p}$, $\bar{q}$

$$d(p, q) \geq \bar{d}(\bar{p}, \bar{q}).$$

curvature bounded below / above: *Alexandrov spaces* / *CAT(K)-spaces*
rich theory since the 1980-ies: GH-convergence, Gromov compactness thm.

# How to detect curvature: Lorentzian world

Sectional curvature $\mathrm{Sec}(X,Y) = \dfrac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$

Kulkarni (1979): If $\mathrm{Sec}(g)$ is bounded below (above), then it is constant.

**Definition ("Correct" curvature bounds, Andersson-Howard 1998)**

A smooth Lorentzian manifold has $\mathrm{Sec} \geq K$ if *spacelike* sectional curvatures $\geq K$ and *timelike* sectional curvatures $\leq K$.

**Theorem (Alexander-Bishop 2008)**

A smooth Lorentzian manifold has $\mathrm{Sec} \geq K$ if for all (small) geodesic $\triangle abc$ and their comparison $\triangle\bar{a}\bar{b}\bar{c}$ in 2D space of const. curvature $K$ (Minkowski, (anti-)de Sitter) and all $p, q$ resp. $\bar{p}$, $\bar{q}$

$$d_{\mathrm{signed}}(p,q) \geq \bar{d}_{\mathrm{signed}}(\bar{p},\bar{q}).$$

# How to go beyond Lorentzian manifolds?

| | | |
|---|---|---|
| Riemannian manifolds | $\subsetneq$ | metric (length) spaces |
| Lorentzian mfs. / spacetimes | $\subsetneq$ | ? |

What is the analogue of metric (length) spaces in the *Lorentzian setting*?

*Serious issue:*

- natural analogue to distance: time separation function

$$\tau(p,q) = \sup\{L(\gamma) |\ \gamma \text{ future dir. causal from } p \text{ to } q\}$$

- but triangle inequality is *reversed* $\rightsquigarrow$ no metric structure

$\rightsquigarrow$ *Lorentzian (pre-)length spaces* (Kunzinger-Sämann 2018)

# Lorentzian (pre-)length spaces

*Causal space:* $X$ (metrizable) topological space with *abstract causality*

$\leq$ preorder on $X$, $\ll$ transitive relation contained in $\leq$

*Abstract time separation:* $\tau\colon X \times X \to [0, \infty]$ lower semicontinuous

## Definition (Kunzinger-Sämann 2018)

$(X, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if for $p \leq q \leq r$

$$\tau(p,r) \geq \tau(p,q) + \tau(q,r) \quad \text{and} \quad \tau(p,q) \begin{cases} = 0 & \text{if } x \not\leq y \\ > 0 & \Leftrightarrow x \ll y \end{cases}$$

Examples

- *smooth spacetimes* $(M, g)$ with usual time separation function $\tau$
- *Lorentz-Finsler* spacetimes, spacetimes of *low regularity* $(g \in C^0 + ...)$
- *finite directed graphs* (causal sets)

Lorentzian *causality theory* $\tau$ intrinsic. . . *Lorentzian length space*

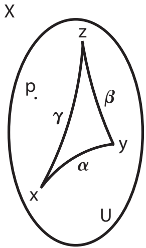
# Timelike curvature via triangle comparison

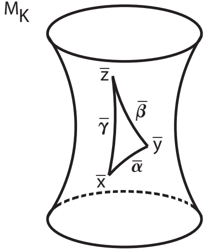
## Definition (Synthetic curvature bounds)

$(X, \ll, \leq, \tau)$ has *timelike curvature* $\geq K$ if

1. some technical conditions hold
2. for all *small timelike triangles* $\Delta abc$ and their comparison $\Delta\bar{a}\bar{b}\bar{c}$ in $M_K$ and all $p, q$ resp. $\bar{p}$, $\bar{q}$

$$\tau(p,q) \leq \bar{\tau}(\bar{p},\bar{q}).$$




Faithful extension of sectional curvature bounds to "metric" Lorentzian setting

# Selected results

## Theorem (Kunzinger-Sämann 2018, Beran-Sämann 2022)

In a strongly causal Lorentzian pre-length space with *timelike curvature bounded below* timelike geodesics do *not branch*.

## Theorem (Grant-Kunzinger-Sämann 2019)

A timelike geodesically complete spacetime (or LLS) is *inextendible as a regular LLS*, i.e., any LLS-extension necessarily has unbounded curvature.

Extends (Beem-Ehrlich) and $C^0$-result (Galloway-Ling-Sbierski 2018).

## Splitting theorem (Beran-Ohanyan-Rott-Solis 2023)

Let $(X, \ll, \leq, \tau)$ be a globally hyperbolic LLS with global timelike $K \geq 0$. If $X$ contains a complete timelike line (+ some technical conditions) then it splits into a product $\mathbb{R} \times S$ with $S$ an Alexandrov space with $K \geq 0$.

Generalises smooth Lorentzian as well as synthetic Riemannian results.

# More on Lorentzian (pre-)length spaces

- *causal ladder* (Kunzinger-Sämann 2018, Aké Hau-Cabrera-Solis 2020)
- *Generalized cones*, i.e., Lorentzian warped products of length spaces with $1$-dim base and singularity theorems (Alexander-Graf-Kunzinger-Sämann 2021)
- *null distance* & Lorentzian length spaces (Kunzinger-S. 2022)
- *Gluing* of Lorentzian length spaces (Beran-Rott 2022)
- Hyperbolic *angles* (Barrera-de Oca-Solis 2022, Beran-Sämann 2022)
- *time functions* on Lorentzian (pre-)length spaces (Burtscher-García-Heveling 2021)
- Lorentzian Hausdorff *dimension*, *measure* (McCann-Sämann 2021)
- *Causal boundaries* (Ake Hau-Burgos-Solis 2023, Burgos-Flores-Herrera 2023)
- *Machine learning* in spacetimes (Law-Lucas 2023)

# Ricci bounds via optimal transport: the basic idea

- *Optimal Transport*: Monge, Kantorovich, move matter in the cheapest / optimal way from $X$ to $Y$
- *Minimize*

$$\int_{X\times Y} c(x,y)\,\mathrm{d}\pi(x,y)$$

over couplings $\pi \in \mathcal{P}(X \times Y)$ w. given marginals $(\mathrm{pr}_X)_\sharp\pi = \mu_1$, $(\mathrm{pr}_Y)_\sharp\pi = \mu_2$

- What is *optimal* depends on *distances* and *geometry* !
- Turn this on its head: define *curvature* by requiring that OT behaves as in model spaces
  - Riemannian case: cost $c = d$
  - Lorentzian case: cost $c = \tau$

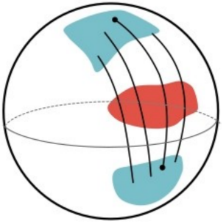


Figure: Transporting *clouds* of points on the sphere

# Ricci Bounds via Optimal Transport: Riemannian case

**Thm. (Ric. bds. & displacement convexity, Lott-Villani, Sturm 2006-09)**

$(M, g)$ complete Riemannian manifold

$$\mathrm{Ric}_g \geq 0 \Longleftrightarrow (M, d_g, \mathrm{vol}_g) \text{ is an } \mathrm{RCD}(0, \infty)\text{-space}$$

**Definitions.** On a metric measure space $(X, d, \mathfrak{m})$ we define

- *Wasserstein distance:* $W_2(\mu_0, \mu_1) = \left( \inf_{\pi \in \Pi} \int_{X \times X} d(x, y)^2 \, d\pi(x, y) \right)^{\frac{1}{2}}$
- *Wasserstein geodesic:* continuous curve $(\mu_t)_{0 \leq t \leq 1}$ in $P_2(X)$ with

$$W_2(\mu_s, \mu_t) = |t - s| \cdot W_2(\mu_1, \mu_2)$$

- *Entropy functional:* $\mathrm{Ent}(\mu|\mathfrak{m}) = -\int \rho \log(\rho) d\mathfrak{m}$ for $\mu = \rho\mathfrak{m}$
- $\mathrm{RCD}(0, )$*-space:* $\mathrm{Ent}(\mu|\mathfrak{m})$ convex along Wasserstein geodesics

Again turn this into definition of synthetic curvature bounds.

$\rightsquigarrow$ *theory of CD-spaces:* stability under measured GH-convergence

# Ricci Bounds via Optimal Transport: Lorentzian case

**Thm. (Ric. bds. & displacement conv., McCann, Mondino-Suhr 2020)**

$(M, g)$ globally hyperbolic spacetime

$\mathrm{Ric}(X, X) \geq 0$ for $X$ timelike $\Longleftrightarrow$ $(M, d_g, \mathrm{vol}_g)$ is $\mathrm{TCD}(K, N)$-space

**Definitions.** Measured Lorentzian pre-length space $(X, d, \mathfrak{m} \ll, \leq, \tau)$

- OT & causality (Eckstein-Miller 2017) $\mu_1, \mu_2 \in \Pi_{\ll}$
- *$p$-Lorentz Wasserstein distance:* $(0 < p < 1)$

$$l_p(\mu_1, \mu_2) = \Big(\sup_{\pi \in \Pi_{\ll}} \int_{X \times X} \tau(x, y)^p \, d\pi(x, y)\Big)^{1/p}$$

- *Entropy functional:* $\mathrm{Ent}(\mu|\mathfrak{m}) = -\int \rho \log(\rho) d\mathfrak{m}$ for $\mu = \rho \mathfrak{m}$
- *$\mathrm{TCD}(K, N)$:* along $l_p$-geos $\mu_t$ we have for $e(t) := \mathrm{Ent}(\mu_t|\mathfrak{m})$

$$e''(t) - \frac{1}{N} e'(t)^2 \geq K \int_{X \times X} \tau(x, y)^2 \pi(dxdx)$$

# Selected results

**Hawking's singularity theorem in TMCP (Cavaletti-Mondino 2024)**

Let $(X, d, \mathfrak{m} \ll, \leq, \tau)$ be a globally hyperbolic measured LLS such that

1. $\mathrm{TCD}(0, N)$ (replaces (SEC)), with
2. a Borel achronal FTC set $V$ w. synthetic mean curvature $\leq H_0 < 0$.

Then $\tau_V \leq D_{H_0,0,N}$ on $I^+(V)$.

Complements low regularity spacetime singularity theorems
(Graf 2020, Kunzinger-Ohanyan-Schinnerl-S. 2022, see S. 2023)

- Synthetic vacuum Einstein equations (Mondino-Suhr 2023)
- Differential calculus for time functions on LLS:
  (Beran-Braun-Calisti-Gigli-McCann-Ohanyan-Rott-Sämann 2024)
- Lorentzian splitting (new proof for class. result, synthetic in progress)
  (Braun-Gigli-McCann-Ohanyan-Sämann 2024)

# Outlook

**(Measured) Measured Lorentzian Length Spaces $(X, d, \mathfrak{m} \ll, \leq, \tau)$**

- provide a general mathematical setting for
  - *sectional* curvature and
  - *Ricci* curvature (bounds)
- that contains
  - *low regularity spacetimes* but also
  - *discrete spaces*

Gives framework for

- approaches to non-smooth spacetime geometry

  $g \in C^{0,1}$, $g \in C^0$ + causally plain
- fundamentally discrete approaches to QG

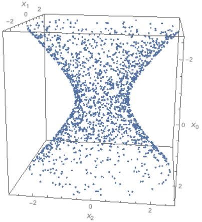
  Causal set theory, Causal Fermion systems

# Outlook: Causal set theory

- ingredients: $(X, \leq)$, partial order; called *causal set* that is
  *locally finite:* $J(x,y) = \{z : x \leq z \leq y\}$ finite
- CS hypothesis: QT of causal sets $X$; $(M,g)$ approximation of $X$

$$\mathcal{C}(M, \rho_C) \;\ni\; X \longleftrightarrow (M,g)$$

sprinkling




**Hauptvermutung of CST**

$X$ can be embedded at density $\rho_C$ into two distinct spacetimes iff they are "close".

- terminology:
  - chain: $C := (x_i)_{i=1}^n$: $x_i < x_{i+1}$
  - length: $L(C) = n$
  - $\tau(x,y) := \sup\{L(C) : C$ chain from $x$ to $y\}$

$(X, \ll, \leq, \tau)$ is a Lorentzian pre-length space

Hauptvermutung translates into statement on convergence of LLS.

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