4) Passage to the limit

By extracting subsequences we have that

$$X^{(n_k)} \rightharpoonup \bar{X} \quad \text{in } K$$
$$A(X^{(n_k)}) \rightharpoonup a \quad \text{in } K^*$$

Moreover note that

$$P_{n_k} B \longrightarrow B \quad \text{in } L^2((0,T)\times\Omega, dt\otimes\mu; L_2(U,H))$$
and weakly star in $L^\infty((0,T)\times\Omega, dt\otimes\mu; L_2(U,H))$

Hence, we have that

$$\int_0^t P_{n_k} B_s \, dW_s^{(n_k)} \longrightarrow \int_0^t B_s \, dW_s$$

$$= \int_0^t P_{n_k} B_s \tilde{P}_{n_k} \, dW_s$$

projection on span $\{q_1, \dots, q_{n_k}\}$

Let us now pass to the limit in the equation. Take a test $v \in \bigcup_n H_n$, $\varphi \in L^\infty((0,T)\times\Omega)$ and compute

$$E\left(\int_0^T \langle \overline{X}_t, \varphi(t)v\rangle\, dt\right) = \lim_{k\to\infty} E\left(\int_0^T \langle X_t^{(n_k)}, \varphi(t)v\rangle\, dt\right) =$$

$$= \lim_{k\to\infty} E\left(\int_0^T \langle X_0^{(n_k)} - \int_0^t P_{n_k} A(X_s^{(n_k)})\, ds + \int_0^t P_{n_k} B_s\, dW_s^{(n_k)}, \varphi(t)v\rangle dt\right)$$

$$= E\left(\int_0^T \langle X_0 - \int_0^t a_s\, ds + \int_0^t B_s\, dW_s, \varphi(t)v\rangle\, dt\right)$$

Hence, by defining

$$X_t = X_0 - \int_0^t a_s\, ds + \int_0^t B_s\, dW_s \qquad \forall t\in[0,T]$$

we have that $X = \overline{X}$ $dt\otimes\mu$-a.e..

We just need to check that $a_s = A(\overline{X}_s)$. Let us use the Itô formula and take the expectation to get

$$E\left(2\int_0^t \langle P_{n_k} A(X_s^{(n_k)}), X_s^{(n_k)}\rangle\, ds\right) = -E\left(\|X_t^{(n_k)}\|_H^2\right) + E\left(\|X_0^{(n_k)}\|_H^2\right)$$
$$+ E\left(\int_0^t \|P_{n_k} B_s\|^2_{L_2(U,H)}\, ds\right).$$

Now take the limsup on both sides, use the fact that $\limsup\left(-\|\cdot\|^2\right) = -\liminf \|\cdot\|^2$, $P_{n_k} X_0 = X_0^{(n_k)} \to X_0$ in $L^2(\Omega, \mathcal{F}_0, \mu; H)$, and the convergence of $\int_0^\cdot P_{n_k} B_s\, dW_s^{(n_k)}$ in order to get that

$$\limsup_{k\to\infty} E\left(2\int_0^t \langle P_{n_k} A(X_s^{(n_k)}), X_s^{(n_k)}\rangle\, ds\right)$$
$$\leq -E\left(\|\overline{X}_t\|_H^2\right) + E\left(\|X_0\|_H^2\right) + E\left(\int_0^t \|B_s\|^2_{L_2(U,H)}\, ds\right)$$

On the other hand, again by Itô

$$E(\|\bar{X}_t\|_H^2) - E(\|X_0\|_H^2) + E\left(\int_0^t 2\langle a_s, \bar{X}_s\rangle ds\right) = E\left(\int_0^t \|B_s\|^2_{L^2(U,H)} ds\right) \quad \forall t \in (0,T)$$

By putting these two information together we have

$$\limsup_{k\to\infty} E\left(\int_0^t 2 \langle P_m A(X_s^{(m)}), X_s^{(m)} \rangle ds\right)$$
$$\le E\left(\int_0^t 2 \langle a_s, \bar{X}_s \rangle ds\right)$$

One can hence apply the limsup lemma and deduce that $a_s = A(X_s)$.

5) <u>Uniqueness</u> Assume that X^1 and X^2 solve the problem. Apply Itô to $X^1 - X^2$ and expectation in order to get

$$E(\|X_t^1 - X_t^2\|_H^2) + 2E\left(\int_0^t \overbrace{\langle A(X_s^1) - A(X_s^2), X_s^1 - X_s^2 \rangle}^{\ge 0} ds\right)$$
$$= E(\|X_0 - X_0\|_H^2) + E\left(\int_0^t \|B_s - B_s\|^2_{L^2(U,H)} ds\right) = 0$$

Hence $X^1 = X^2$ a.e. in $(0,T)\times\Omega$.

The latter also gives the continuous dependence in

$$X \in L^\infty((0,T); L^2(\Omega; H)) \cap L^2((0,T)\times\Omega; V)$$

with respect to data (X_0, B_t) in

$$L^2(\Omega; H) \times L^2((0,T)\times\Omega; L_2(U,H)).$$