The main well posedness result

$V \subset H \subset V'$ triplet of separable Hilbert spaces.

W_t, $t \in [0,T]$ standard Q-Wiener process on the Hilbert space U.

$$A : V \times \Omega \longrightarrow V', \quad B : [0,T] \times \Omega \longrightarrow L^2(U,H)$$

(Hemicontinuity)

$$\lambda \in \mathbb{R} \mapsto \langle A(u+\lambda v, \omega), w \rangle \quad \text{continuous}$$
$$\forall u, v, w \in V, \ \omega \in \Omega$$

(Monotonicity)

$$\langle A(u,\omega) - A(v,\omega), u - v \rangle \geq 0$$
$$\forall u, v \in V, \ \forall \omega \in \Omega.$$

(Coercivity) $\exists \underline{c} > 0$:

$$\langle A(u,\omega), u \rangle \geq \underline{c} \, \|u\|_V^2 \quad \forall v \in V, \ \mu\text{-a.e. in } \Omega.$$

(Boundedness) $\exists \overline{c} > 0$:

$$\|A(u,\omega)\|_{V'} \leq \overline{c} \, (1 + \|u\|_V)$$
$$\forall u \in V, \ \mu\text{-a.e. in } \Omega$$

$$\|B(t,\omega)\|_{L^2(U,H)} \leq \overline{c} \quad dt \otimes \mu\text{-a.e.}$$

The equation

$$du_t + A(u_t)\,dt = B_t\,dW_t$$

The solution We say that the continuous H-valued, $\mathcal{F}_t$-adapted process u_t is a solution if $u \in L^2((0,T)\times\Omega, dt\otimes\mu; V)$ and

$$u_t - u_0 + \int_0^t A\bar{u}_s\,ds = \int_0^t B_s\,dW_s \quad \text{in } V', \text{ for all } t\in[0,T]$$

where $\bar{u}_t = u_t$ $dt\otimes\mu$-a.e. is V-valued and progressively measurable.

The result Let $u_0 \in L^2(\Omega, \mathcal{F}_0, \mu; H)$ be given. Then, there exists a unique solution u of the equation in the above sense. Moreover

$$\mathbb{E}\left(\sup_t \|u_t\|_H^2\right) < \infty.$$

namely $u \in L^2(\Omega; L^\infty(0,T; H))$.

A result in finite dimensions

$(\Omega, \mathcal{F}, \mu)$ complete probability space
$\mathcal{F}_t$ normal filtration
W_t Wiener process in $\mathbb{R}^d$

$$\left.\begin{array}{l} \sigma : [0,T] \times \Omega \longrightarrow \mathbb{R}^{d\times d} \\ b : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}^d \end{array}\right\} \text{ continuous}$$

$$\sup_{\Omega} |b(\cdot, x) - b(\cdot, y)| \leq L_b |x-y|$$

<u>Thm</u> For all $X_0 : \Omega \longrightarrow \mathbb{R}^d$ $\mathcal{F}_0$-measurable there exists a unique sol. to the SDE

$$dX_t = b(X_t)\,dt + \sigma_t\,dW_t$$

meaning a process X_t, μ-a.s.-continuous, $\mathbb{R}^d$-valued, $\mathcal{F}_t$-adapted such that, μ.a.e. and for all $t \in [0,T]$

$$X_t - X_0 + \int_0^t b(X_s)\,ds = \int_0^t \sigma_s\,dW_s .$$

Sketch of the proof

Let $\tilde{X}_t \in L^\infty(0,T; L^\infty(\Omega; \mathbb{R}^d))$ be given and define

$$X_t = X_0 - \int_0^t b(\tilde{X}_s)\,ds + \int_0^t \sigma_s\,dW$$

This defines an $\mathbb{R}^d$-valued process $X_t = S(\tilde{X}_t)$.
We would like to show that S has a fixed point.
Indeed, by letting
$X_t^1 = S(X_t^1)$, $X_t^2 = S(X_t)$ one has

$$|X_t^1 - X_t^2| = \left| -\int_0^t b(\tilde{X}_s^1) - b(\tilde{X}_s^2)\,dt \right|$$

$$\le L_b \int_0^t |\tilde{X}_s^1 - \tilde{X}_s^2|\,dt \quad \text{for a.e. } \omega \in \Omega \quad (*)$$

One can hence find $K \in \mathbb{N}$ such that $S^{(K)}$ is a contraction in $L^\infty(0,T; L^\infty(\Omega))$. Indeed, we have

$$\sup_{t\in[0,T]} |S^{(K)}(\tilde{X}_t^1) - S^{(K)}(\tilde{X}_t^2)| \le \frac{L_b^K T^K}{K!} \sup_{t\in[0,T]} |\tilde{X}_t^1 - \tilde{X}_t^2| \quad \text{a.e. in } \Omega \quad (**)$$

so that it suffices to take K large enough. Relation $(**)$ follows by induction: For $K=1$ it comes from $(*)$. Assume it for K and use $(*)$ in order to prove

$$|S^{(K+1)}(\tilde{X}_t^1) - S^{(K+1)}(\tilde{X}_t^2)| \le L_b \int_0^t |S^{(K)}(\tilde{X}_t^1) - S^{(K)}(\tilde{X}_t^2)|\,dt$$

$$\le L_b \int_0^t \frac{L_b^K T^K}{K!} |\tilde{X}_t^1 - \tilde{X}_t^2|\,dt \le$$

$$\le L_b \int_0^t \frac{L_b^K T^K}{K!} \sup_{[0,T]} |\tilde{X}_t^1 - \tilde{X}_t^2|\,dt = \frac{L_b^{K+1} T^{K+1}}{(K+1)!} \sup_{t\in[0,T]} |\tilde{X}_t^1 - \tilde{X}_t^2|$$

a.e. in Ω

We have hence proved that $S^{(k)}$ is a contraction in $L^\infty(0,T;L^\infty(\Omega))$.
Hence S has a unique fixed point.

All solutions to the equations are fixed points of $S^{(k)}$. Hence, the sol. is unique. □

Reminder:

S has a fixed point $\Longrightarrow$ $S^{(k)}$ has a fixed point $\forall k$

$S^{(k)}$ has a fixed point for some k $\not\Rightarrow$ S has a fixed point

$S^{(k)}$ is a contraction $\Longrightarrow$ S has a unique fixed point

Pf. Since $S^{(k)}$ is a contraction it has a unique fixed point, call it u. Then

$$S(u) = S\left(S^{(k)}(u)\right) = S^{(k+1)}(u) = \\ = S^{(k)}(S(u))$$

so that also $S(u)$ is a fixed point for $S^{(k)}$.
Since the fixed point u of $S^{(k)}$ is unique, one necessarily has that $u = S(u)$. In particular, u is a fixed point for S.

Assume now that u_1 and u_2 are two fixed points of S. Then $S^{(k)}u_1 = u_1$ and $S^{(k)}u_2 = u_2$ so that $u_1 = u_2$, for $S^{(k)}$ has a unique fixed point. □

An integration-by-parts formula

Let $V \subset H \subset V'$ be a Hilbert triplet of separable spaces, $X \in L^2(0,T;V)$, $Y \in L^2(0,T;V')$ and

$$X(t) = X(0) + \int_0^t Y(s)\,ds \qquad \forall t \in [0,T].$$

Then $X \in C([0,T];H)$ and

$$\|X(t)\|_H^2 = \|X(0)\|_H^2 + \int_0^t 2 \langle Y(s), X(s) \rangle\,ds.$$

In other words, if $X' \in L^2(0,T;V')$ then

$\langle X', X \rangle \in L^1$ and

$$\frac{d}{dt} \frac{1}{2} \|X\|_H^2 = \langle X', X \rangle.$$

An Itô formula

$X_0 \in L^2(\Omega; \mathcal{F}_0, \mu : H)$, $Y \in L^2((0,T) \times \Omega, dt \otimes \mu; V')$
$Z \in L^2((0,T) \times \Omega; dt \otimes \mu; L_2(U,H))$ progr. meas.
Define

$$X_t = X_0 + \int_0^t Y_s \, ds + \int_0^t Z_s \, dW_s \quad \forall t \in [0,T]$$

If $X \in L^2((0,T) \times \Omega, dt \otimes \mu; V)$ then X is actually H-val. continuous, $E\left(\sup_{[0,T]} \|X_t\|_H^2\right) < \infty$, and we have the Itô form.:

$$\|X_t\|_H^2 = \|X_0\|_H^2 + \int_0^t 2 \langle Y_s, \bar{X}_s \rangle \, ds + \int_0^t \|Z_s\|_{L_2(U,H)}^2 \, ds$$
$$+ 2 \int_0^t (X_s, Z_s \, dW_s)_H \quad \forall t \in [0,T] \ \mu\text{.a.e.}$$

for any progr. meas. $\bar{X} = X$ $dt \otimes \mu$ a.e.

Corollary

$$E\left(\|X_t\|_H^2\right) = E\left(\|X_0\|_H^2\right) + \int_0^t E\left(2 \langle Y_s, X_s \rangle + \|Z\|_{L_2(U,H)}^2\right) ds$$

Strategy of the proof of the main existence thm:

1) Galerkin method : reduce to finite dimensions, n.
2) solve the finite-dimensional SDE - System and find $X^{(n)} : [0,T] \times \Omega \longrightarrow \mathbb{R}^n$
3) prove estimates on $X^{(n)}$ independently of n
4) use these estimates to pass to the limit as $n \to \infty$

1) Approximation

Let $e_i \in V$ be an orthonormal basis of H, call $H_n = \text{span}\{e_1, \dots, e_n\}$ and assume that $\overline{\bigcup_n H_n}^V = V$. Define the projection $P_n : V' \longrightarrow H_n$ as

$$P_n y = \sum_{i=1}^{n} \langle y, e_i \rangle e_i$$

Of course $P_n\big|_H$ is the orthogonal projection on H_n.
Let $g_i \in U$ be an orthonormal basis and define

$$W_t^{(n)} = \sum_{i=1}^{n} \underbrace{(W_t, g_i)_U}_{B_t^i} g_i$$

2) Finite-dimensional existence

By means of these notations we can define the finite-dimensional problem

$$\begin{cases} dX_t^{(n)} + P_n A(X_t^{(n)})\,dt = P_n B_t \, dW_t^{(n)} \\ X_0^{(n)} = P_n X_0 \end{cases}$$

This is (almost) of the same type of that already discussed. In particular, it has a unique solution

$X^{(n)} \in L^2((0,T)\times\Omega, dt\otimes\mu; V) =: K$

Recall that

$K' = L^2((0,T)\times\Omega, dt\otimes\mu; V')$.

In fact, $P_n A$ is not Lipschitz and an extra approx. would be needed...

3) A-priori estimate

We have that

$$\|X^{(n)}\|_K + \|A(X^{(n)})\|_{K'} + \sup_{[0,T]} E\left(\|X_t^{(n)}\|_H^2\right) < C$$

independently of n

motivation: apply Itô's formula (in the fin.-dimens. setting) getting

$$\|X_t^{(n)}\|_H^2 = \|P_n X_0\|_H^2 - \int_0^t 2 < P_n A(X_s^{(n)}), X_s^{(n)} > ds$$
$$\int_0^t \|P_n B_s\|_{L_2(U,H)}^2 ds + 2\int_0^t (Z_s^{(n)} dW_s^{(n)}, X_s^{(n)})_H$$

$\forall t \in (0,T)$, μ.a.s.

By taking the expectation we have

$$(*) \quad E\left(\|X_t^{(n)}\|_H\right) + E\left(\int_0^t 2 < P_m A(X_s^{(n)}), X_s^{(n)} > ds\right)$$

$$= E\left(\|P_m X_0\|_H^2\right) + E\left(\int_0^t \|P_m B_s\|^2_{L_2(U,H)} ds\right) < \hat{C}$$

for all $t \in (0,T)$

here we use
$\|X_0\|_H < \infty$, $T < \infty$
$\|B\|_{L^\infty(L_2(U,H))} < \infty$

We use the coercivity of A in order to get that

$$\hat{C} > E\left(\int_0^t 2 < P_m A(X_s^{(n)}), X_s^{(n)} > ds\right)$$

$$\geq E\left(\int_0^t 2\underline{c}\, \|X_s^{(m)}\|_V^2\, ds\right) =$$

$$= \int_0^t \int_\Omega 2\underline{c}\, \|X_s^{(n)}\|_V^2\, ds\, d\mu = \|X^{(n)}\|_K^2$$

Now use the bound on A to get that

$$\|A(X^{(m)})\|_{K'}^2 = \int_0^t \int_\Omega \|A(X_s^{(m)})\|_{V'}^2\, ds\, d\mu$$

$$\leq \int_0^t \int_\Omega \bar{c}^2 \left(1 + \|X_s^{(m)}\|_V\right)^2 ds\, d\mu$$

$$\leq C(\bar{c}, |\Omega|, T)\left(1 + \|X^{(m)}\|_K^2\right) < C$$

Eventually, estimate $(*)$ gives that

$$\sup_{[0,T]} E\left(\|X_t^{(m)}\|_H^2\right) < \hat{C}.$$