The stochastic process $W : [0,T] \times (\Omega, \mathcal{F}, \mu) \to (U, \mathcal{B}(U))$ is called the Q-Wiener process if

1) $W_0 = 0$ a.e.
2) $t \mapsto W_t$ is continuous a.s.
3) $\forall\, 0 \leq t_1 < t_2 < t_3 - t_4 \leq T$ the increments $W_{t_2} - W_{t_1}$ and $W_{t_4} - W_{t_3}$ are independent
4) $\forall\, 0 \leq t_1 < t_2 \leq T$ the increment is Gaussian: $(W_{t_2} - W_{t_1}) \# \mu = N(0, (t_2 - t_1) Q)$.

The Q-Wiener process is adapted to the normal filtration

$$\mathcal{F}_t = \bigcap_{s > t} \tilde{\mathcal{F}}_s \,, \quad \tilde{\mathcal{F}}_s = \sigma\left(\{W_r : r \leq s\} \cup \{\text{null sets}\} \right).$$

<u>Martingale</u> $X_t : (\Omega, \mathcal{F}, \mu) \longrightarrow H$, $\mathcal{F}_t$ filtration, is a $\mathcal{F}_t$-martingale.

1) $E(\|X_t\|) < \infty \quad \forall t$
2) X_t is $\mathcal{F}_t$ meas $\forall t$
3) $E(X_t \mid \mathcal{F}_s) = X_s \quad \forall\, 0 \leq s \leq t$

<u>example</u>: U-valued Q-Wiener process w.r.t. its normal filtration $\mathcal{F}_t$

<u>Martingale norm</u>

$$\|X\|_{M_T^2} := \sup_{t \in [0,T]} \Big(E\left(\|X_t\|^2\right) \Big)^{1/2}$$

<u>Remark</u> $\left(M_T^2, \|\cdot\|_{M_T^2}\right)$ is Banach.

Vector-valued functions

View $u: (x,t) \mapsto \mathbb{R}$, $u \in L^1(\Omega \times (0,T))$ as a function $t \mapsto u(\cdot,t) \in L^1(\Omega)$. In particular, we would like to handle functions $u: [0,T] \longrightarrow B$ valued in a Banach space.

Simple function $\quad u(t) = \sum\limits_{i \in F} u_i \chi_{B_i}(t)$

B_i meas. and disjoint. $\quad$ finite index set

u is strongly measurable if u is the a.e. limit of simple functions.

Pettis $\quad$ u strongly measurable $\Longleftrightarrow$

$\Longleftrightarrow$ $t \mapsto \langle f, u(t) \rangle$ $\forall f \in B'$ (weakly measurable) and $\exists N$ null set such that $\{u(t) \mid t \notin N\}$ separable

Rem $\quad$ On separable spaces: strongly meas. = weakly meas.

u is Bochner integrable if there exists a sequence u_n of simple functions so that

$$t \mapsto \|u(t) - u_n(t)\|_B \in L^1(0,T) \quad \text{and} \quad \int_0^T \|u - u_n\|_B \, dt \longrightarrow 0$$

Under the latter, $\int_0^T u_n dt$ is Cauchy so that the limit is well defined.

The space of (a.e. equivalence classes of) Bochner integrable functions $L^1(0,T;B)$ is a Banach space with norm

$$\|u\|_{L^1(0,T;B)} = \int_0^T \|u\|_B \, dt$$

and the classical properties of the integral hold.

Rem if $B = \mathbb{R}^d$ then Bochner = Lebesgue

Bochner $u \in L^1(0,T;B) \iff \begin{cases} u \text{ strongly meas.} \\ t \mapsto \|u(t)\|_B \in L^1(0,T) \end{cases}$

Cor $\left.\begin{array}{l} u \text{ strongly measurable} \\ \|u(t)\| \leq g(t) \in L^1(0,T) \end{array}\right\} \Rightarrow u \in L^1(0,T;B)$

One can define $L^p(0,T;B)$ and $(L^p(0,T;B))' = L^{p'}(0,T;B')$ at least for B reflexive and $1 < p < \infty$.

We have $L^p(0,T;L^p(\Omega)) = L^p(\Omega \times (0,T))$ for all $p \in [1,\infty)$. $L^\infty(0,T;L^\infty(\Omega)) \subset L^\infty(\Omega \times (0,T))$

The construction generalizes for

$$((0,T), dx) \longrightarrow (\Omega, \mu).$$

In particular one can define

$$L^2(\Omega;B),\ L^2(\Omega \times (0,T);B),$$
$$L^2(\Omega;L^2(0,T;B)) \text{ etc.}$$

Stochastic integral in Hilbert spaces

Give meaning to $\int_0^t B_s \, dW_s \in H$, where W_t is a U-valued Q-Wiener process ($Q \in L(U)$, nonnegative, symmetric, and finite trace) and $B : [0,T] \to L(U,H)$

Step 1 Define it on elementary processes

$$\bar{B}_t = \sum_{i=0}^{N-1} B_i \, \mathbb{1}_{(t_i, t_{i+1}]}(t)$$

$B_i : \Omega \to L(U,H)$ meas ... takes a finite number of values in $L(U,H)$

Define $$\int_0^t \bar{B}_s \, dW_s = \sum_{i=0}^{N-1} B_i \left(W_{t_{i+1} \wedge t} - W_{t_i \wedge t} \right)$$

Step 2 We isometrically extend the integral from

$$(\text{simple processes}, \|\cdot\|_T) \longrightarrow (M_T^2, \|\cdot\|_{M_T^2})$$

$$\|B\|_T^2 := E\left(\int_0^T \|B_s \circ Q^{1/2}\|_{L_2}^2 \, ds \right)$$

$Q \in L(U)$ nonnegative & symmetric $\Rightarrow$
$\Rightarrow \exists! \, Q^{1/2} \in L(U)$ nonnegative & symmetric such that $Q^{1/2} \circ Q^{1/2} = Q$.

if $\operatorname{tr} Q < \infty \Rightarrow Q^{1/2} \in L_2(U)$ and
$\|Q^{1/2}\|_{L_2(U)} = \operatorname{tr} Q$, $L \circ Q^{1/2} \in L_2(U,H)$ $\forall L \in L(U,H)$

<u>Step 3</u> The construction can be extended to

all processes $B : [0,T] \longrightarrow L_2(U,H)$
such that $\mu\Big(\underbrace{\int_0^T \| B_s \circ Q^{1/2} \|^2_{L_2}}_{\to L_2^0} ds < \infty\Big) = 1$

<u>Properties</u>

1) $B : [0,T] \longrightarrow L_2^0, \quad L(H,\tilde{H}) \Rightarrow$ (linearity)

$$\Rightarrow \quad L\left(\int_0^T B_t \, dW_t\right) = \int_0^T L \circ B_t \, dW_t$$

2) $B : [0,T] \longrightarrow L_2^0, \quad f : [0,T] \to H$ cont. $\Rightarrow$

$$\Rightarrow \int_0^T (f_t, B_t \, dW_t) := \int_0^T \tilde{B}^f_t \, dW_t$$

with $(\tilde{B}^f_t, u) := (f_t, B_t u)$ is a well-def stochastic integral.