We hence have that $Au = \ell$ admits a solution, namely there exists a <u>weak</u> solution of

$$\begin{cases} -\Delta u + \beta(u) = f & \text{in } D \\ u = 0 & \text{in } \partial D \end{cases}$$

<u>Passing to limits into maximal monotone relations</u>

<u>Easy case</u> (very rare!)

$u_n \to u$ in V, $Au_n \rightharpoonup \xi$ in V' $\Longrightarrow$ $\xi = Au$

Pf Fix $v \in D(A)$ and compute

$$\langle \xi - Av, u - v \rangle =$$

$$\langle \xi, u \rangle - \langle Av, u \rangle - \langle \xi, v \rangle + \langle Av, v \rangle \overset{(*)}{=}$$

$$= \limsup_n \langle Au_n, u_n \rangle - \lim_n \langle Av, u_n \rangle - \langle \xi, v \rangle + \langle Av, v \rangle =$$

$$= \lim_n \langle Au_n - Av, u_n - v \rangle \geq 0.$$

Then ξ must be Au, otherwise one could "extend" A by keeping monotonicity by letting

$$D(\tilde{A}) = D(A) \cup \{u\}$$

$$\tilde{A}v = \begin{cases} Av & \text{if } v \neq u \\ \xi & \text{if } v = u \end{cases}$$

this contradicts maximality. $\square$

<u>Tricky case</u> (very often!)

$u_n \to u$ in V, $Au_n \rightharpoonup \xi$ in V', $\langle \xi, u \rangle \overset{(**)}{\geq} \limsup_n \langle Au_n, u_n \rangle \Longrightarrow \xi = Au$

Pf Use $(**)$ in $(*)$ above. $\square$

A sufficient condition for maximality:

$A : H \to H$ monotone & hemicontinuous
$(\lambda \in \mathbb{R} \mapsto \langle A(x+\lambda u), v \rangle$ continuous
for all $x, u, v \in H) \Rightarrow A$ maximal

Pf Let ξ be such that $\langle \xi - Ay, x - y \rangle \geq 0$
$\forall y \in H$. We need to prove that $\xi = Ax$.
Choose $y = tx + (1-t)z$ for any fixed $z \in H$.
Then:

$$\langle \xi - A(tx+(1-t)z), (1-t)(x-z) \rangle \geq 0$$

Divide by $1-t$ and take $t \to 1^-$. Hemicontinuity ensures that

$$\langle \xi - Ax, x - z \rangle \geq 0$$

Chose now $z = x \pm w$ and get that

$$(\xi - Ax, w) = 0 \quad \forall w \in H$$

Hence $\xi = Ax$. $\square$

<u>Passing to the limit</u> in $Au_n = l_n$ where A is monotone and hemicontinuous and $u_n \rightharpoonup u$ in V. and $l_n \to l$ in V'.

As $Au_n = l_n$ we have that

$$\langle Au_n - l_n, u_n - v \rangle = 0 \quad \forall v \in V$$

Since A is monotone, one finds that

$$\langle Av - l_n, u_n - v \rangle \leq 0 \quad \forall v \in V$$

By passing to the limit we deduce

$$\langle Av - l, u - v \rangle \leq 0 \quad \forall v \in V$$

Choose now $v = tu + (1-t) w$ for any $w \in V$ fixed. Then

$$\langle A(tu + (1-t)w) - l, \cancel{(1-t)}(u - w) \rangle \leq 0 \quad \forall w \in V$$

Pass to the limit as $t \to 1^-$ and get

$$\langle Au - l, u - w \rangle \leq 0 \quad \forall w \in V$$

Choose $w = u \pm z$ to get

$$\langle Au - l, z \rangle = 0 \quad \forall z \in V$$

Hence $Au = l$. $\square$

<u>Operators</u> Let $(U, (\cdot,\cdot)_U)$ and $(H, (\cdot,\cdot)_H)$ be real separable Hilbert spaces.

$L(U,H) = \{ T : U \to H$ linear & continuous$\}$
$L(U) = L(U,U)$

- $T \in L(U)$ is symmetric iff $(Tu,v)_U = (u,Tv)_U \ \forall u,v \in U$.
  is nonnegative iff $(Tu,u)_U \geq 0 \ \forall u \in U$

- $T \in L(U,H)$ is nuclear if $\exists \{u_i\} \in U$, $\{h_i\} \in H$ s.t.

$$T v = \sum_{i \in \mathbb{N}} h_i \, (v, u_i)_U \quad \forall v \in U \qquad (*)$$

with the property that $\sum_i \|u_i\|_U \|h_i\|_H < \infty$.

$L_1(U,H) = \{ T \in L(U,H)$ nuclear$\}$ is a Banach space with the norm

$$\|T\|_{L_1(U,H)} := \inf \sum_i \|u_i\|_U \|h_i\|_H$$

where $\{u_i\}$ and $\{h_i\}$ realize $(*)$.

- $T \in L_1(U)$ nonnegative & symmetric is called trace-class.

- $T \in L(U)$ and $\{e_k\}$ orthonormal basis in U.
  We define

$$tr(T) = \sum_k (Te_k, e_k)_U$$

whenever convergent.
This definition does not depend on $\{e_k\}$ if T is nuclear

- $T \in L(U,H)$ is Hilbert-Schmidt if

$$\sum_k \|Te_k\|_H^2 < \infty$$

for some (hence all) orthonormal basis $\{e_k\}$ of U.

$$\left(\sum_k \|Te_k\|_H^2 = \sum_k \sum_j |(Te_k, f_j)| = \sum_j \|T^* f_j\|^2 \right)$$

$$L_2(U,H) = \{ T \in L(U,H) \text{ Hilbert-Schmidt} \}$$

- $T_1, T_2 \in L_2(U,H)$. By setting

$$(T_1, T_2)_{L_2(U,H)} := \sum_i \langle T_1 e_i, T_2 e_i \rangle_H$$

we get that $\left(L_2(U,H), (\cdot,\cdot)_{L_2(U,H)}\right)$ is a separable Hilbert space.

Stochastic processes with values in U sep. Hilbert

The measure p on $(U, \sigma(U))$ is the Gaussian $N(m, Q)$, with mean $m \in U$ and covariance operator $Q \in L(U)$ nonnegative, symmetric, with finite trace if

$$\hat{p}(u) = \int_U e^{i(u,v)_U} \, dp(v) \equiv e^{i(m,u)_U - \frac{1}{2}(Qu,u)_U}.$$

The random variable $X : (\Omega, \mathcal{F}, \mu) \to (U, \sigma(U))$ is Gaussian if $X_{\#}\mu = N(m, Q)$. In this case

1) $\forall u$ the real random var $(X,u)_U$ is normally distributed

2) $E((X,u)_U) = (m,u)_U$

3) $E((X-m,u)_U (X-m,v)_U) = (Qu,v)_U$

4) $E(\|X-m\|^2_U) = \operatorname{tr} Q$.

All Gaussian random variables can be represented as

$$X = \sum_{i=1}^{\infty} \sqrt{\lambda_k} \, \beta_k \, e_k + m \quad \text{in } L^2(\Omega, \mathcal{F}, \mu; U)$$

where e_k is an orthonormal basis made of eigenvectors of Q, λ_k are the eigenvalues (decreasing order), β_k are independent Gaussian rand. variables with $\beta_k \# \mu = N(0,1)$.