A short recap on (analytical) weak solutions

Elliptic problem, homogeneous Dirichlet

$$\begin{cases} -\nabla\cdot\left(a(x)\nabla u(x)\right) = f(x) & x \in D \\ u(x) = 0 & x \in \partial D \end{cases}$$

$D \subset \mathbb{R}^d$ open smooth, $a \in L^\infty(D; \mathbb{R}^{d\times d}_{sym})$,
$\exists\, \alpha > 0: \quad a\xi\cdot\xi \geq \alpha|\xi|^2 \ \forall \xi \in \mathbb{R}^d$ (ellipticity)
$f \in L^2(D)$. Define

- $V = H^1_0(D) := \{ u \in L^2(D) : \nabla u \in L^2(D; \mathbb{R}^d),$
  $u = 0 \text{ on } \partial D \}$

is a Hilbert space with the scalar product

$$((u, v)) = \int_D \nabla u \cdot \nabla v \, dx$$

- $A : V \longrightarrow V'$

$$\langle Au, v \rangle = \int_D a(x)\nabla u(x) \cdot \nabla v(x) \, dx$$

- $l \in V'$

$$\langle l, v \rangle = \int_D f(x)\, v(x) \, dx$$

Weak formulation

find $u \in V$: $\quad Au = \ell \quad$ in V' $\qquad (1)$

$$Au = \ell \iff \int_D a(x)\, \nabla u(x) \cdot \nabla v(x)\, dx =$$

$$= \int_D f(x)\, v(x)\, dx \quad \forall v \in V$$

Thm $\exists!\, u$ solving (1)

Pf This is the Lax-Milgram(-Lions) lemma. Define $a(u,v) := \langle Au, v \rangle$. Then a is bilinear, continuous

$$|a(u,v)| \leq \|a\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}$$

$$= \|a\|_{L^\infty} \|u\|_V \|v\|_V$$

and coercive

$$a(u,u) = \int_\Omega a \nabla u \cdot \nabla u\, dx \geq \alpha \int_\Omega |\nabla u|^2 dx$$

$$= \alpha \|u\|_V^2$$

Hence, the problem:

find $u \in V$: $\quad a(u,v) = \langle \ell, v \rangle \quad \forall v \in V$

has a unique solution (which depends continuously on ℓ). $\square$

This u we call weak solution of $\begin{cases} -\nabla \cdot a \nabla u = f & D \\ u = 0 & \partial D \end{cases}$

<u>Other boundary conditions</u>

- Inhomogeneous Dirichlet $\begin{cases} -\Delta u = f & D \\ u = w & \partial D \end{cases}$

Define $\tilde{u} = u - w$. Then $\tilde{u}$ solves

$$\begin{cases} -\Delta \tilde{u} = f + \Delta w \\ \tilde{u} = 0 \end{cases}$$ ← makes sense for $w \in H^1(D)$

- Homogeneous Neumann $\begin{cases} -\Delta u = f & D \\ \dfrac{\partial u}{\partial n} = 0 & \partial D \end{cases}$

$$V = H^1_{\#}(D) = H^1(D)/\mathbb{R} = \{u \in H^1(D) : \int_D u = 0\}$$

A and ℓ as before. Extra condition $\int_\Omega f = 0$

- Nonhomogeneous Neumann $\begin{cases} -\Delta u = f & D \\ \dfrac{\partial u}{\partial n} = g & \partial D \end{cases}$

$V = H^1_{\#}(D)$

A as before, $\langle \ell, v \rangle = \int_D f v + \int_{\partial D} g v$ (2)

- Mixed $\partial D = \overline{\Gamma_N \cup \Gamma_D}$, $\Gamma_N \cap \Gamma_D = \emptyset$, $\Gamma_D \neq \emptyset$

Γ_N, Γ_D open

$$\begin{cases} -\Delta u = f & D \\ u = 0 & \Gamma_D \\ \dfrac{\partial u}{\partial n} = 0 & \Gamma_N \end{cases}$$

$V = \{u \in H^1(D) : u = 0 \text{ on } \Gamma_D\}$

A and ℓ as in (2)