<u>Stochastic processes</u>: a collection $X(t),\ t \in T \subset \mathbb{R}$ where $X(t)$ is a random variable with values in $(\Omega', \mathcal{F}')$ (most times $\Omega' = \mathbb{R}$) and T is discrete or continuous.
Alternative (common) notation $X_t = X(t)$
Assume all X_t to be defined on the same $(\Omega, \mathcal{F}, \mu)$.
Then, one can see $X_t(\omega)$ as a function
$$X = X(t, \omega) : T \times \Omega \longrightarrow \Omega'$$

Fixed $w \in \Omega$, we call $t \in T \mapsto X(t, \omega)$ <u>path</u> or <u>realization</u>

<u>Example</u> Let $T = \mathbb{N}$ and X_n be real i.i.d (indep. identically distributed) with mean 0 and variance 1. Then $X(n, \omega)$ is called i.i.d <u>noise</u> and $S : \mathbb{N} \times \Omega \longrightarrow \mathbb{R}$ given by

$$S(n, \omega) = \sum_{i=1}^{n} X(i, \omega)$$

is called <u>random walk</u>.

<u>Example</u> Let $T = [0, \infty)$. The properties:

1) $W_0 = 0$ a.s.
2) $\forall\ 0 \le t_1 < t_2 < \infty$ $\quad W_{t_2} - W_{t_1}$ is gaussian with mean 0 and variance $t_2 - t_1$
3) $\forall\ 0 \le t_1 < t_2 < t_3 < t_4 < \infty$, $\ W_{t_4} - W_{t_3}$ indep. of $W_{t_2} - W_{t_1}$
4) $t \mapsto W_t$ is a.s. continuous

uniquely define a real stochastic process called <u>Brownian motion</u> or <u>Wiener process</u>

The process X_t is adapted to the filtration $\mathcal{F}_t$, $t \in T$ iff X_t is $\mathcal{F}_t$-measurable for all t, where $\mathcal{F}_t$ is an increasing family of σ-algebras

Any process is adapted to the filtration
$$\mathcal{F}_t^X := \sigma(\{X_s^{-1}(\mathcal{F}'), s \in T, s \leq t\})$$
generated by the process itself.

Markov property (for discrete processes)

$$P(X_{n+1} = y \mid X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = P(X_{n+1} = y \mid X_n = x_n)$$

Martingale property (for discrete processes)

$$E(X_{n+1} \mid X_0 X_1 \ldots X_n) = X_n$$

remark: $\{\text{markov}\} \nsubseteq \{\text{martingale}\}$

Example

The random walk $S_n = \sum_{i=1}^{n} X_i$ is markovian.
Moreover it is a martingale iff $E(X_i) = 0$
Let $E(X_i) = 0$. Then $E(S_n) = 0$ as well.
Assume that $X_i = \pm 1$. Then $E(X_i^2) = 1$ and

$$E(S_n^2) = E\left(\sum_i^n X_i \sum_j^n X_j\right) = E\left(\sum_{i,j}^n X_i X_j\right) =$$
$$= E\left(\sum_{i=1}^n X_i^2\right) = n\, E(X_i^2) = n$$

$\longrightarrow$ gambler's ruin

Markov property (for continuous processes)

$$P(X_t \in A' \mid \mathcal{F}_s^X) = P(X_t \in A' \mid X_s) \qquad (t \geq s)$$

Martingale property (for continuous processes)

$$E(X_t \mid \mathcal{F}_s^X) = X_s \qquad (t \geq s)$$

Example the Wiener process is markovian and

$$E(W_t \mid \mathcal{F}_s^W) = \underbrace{E(W_t - W_s \mid \mathcal{F}_s^W)}_{=0} + E(W_s \mid \mathcal{F}_s^W) = W_s$$

Other properties:

1) A Wiener path is nowhere monotone a.s.
2) " " " is nowhere differentiable a.s.

Remark: it can be proved that

$$\lim_{\tau \to 0} \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})^2 = T$$

where $0 = t_0 < t_1 < \dots < t_n = T$ is a partition and $\tau = \max_i (t_i - t_{i-1})$

Stochastic integral, real case

Aim: give meaning to "$\int_0^t X_s \, dW_s$". (Riemann–Stieltjes-like)

Simple process: $S_t(\omega) = \sum_{i=1}^{n} \xi_{i-1}(\omega) \, \mathbb{1}_{[t_{i-1}, t_i)}(t)$
with ξ_{i-1} $\mathcal{F}_{t_{i-1}}$- measurable

Stochastic integral of a simple process:

$$\int_0^T S_t \, dW_t = \sum_{i=1}^{n} \xi_{i-1} \left(W_{t_i} - W_{t_{i-1}} \right)$$

Call $\mathcal{J}$ the set of simple processes with $\|S\|_{L^2(\Omega \times (0,T))} \leq C$ adapted to $\mathcal{F}_t$. Then,

$$S \in \mathcal{J} \longmapsto \int_0^T S_t \, dW_t \in L^2(\Omega)$$

$$\left\| \int_0^T S_t \, dW_t \right\|^2_{L^2(\Omega)} = \| S \|^2_{L^2(\Omega \times (0,T))}$$

Let $\overline{\mathcal{J}}$ be the completion of $\mathcal{J}$ w.r.t. the $L^2(\Omega \times (0,T))$ norm. Then, for all $S \in \overline{\mathcal{J}}$ we have $S_n \in \mathcal{J}$ such that $S_n \longrightarrow S$ in $L^2(\Omega \times (0,T))$. This allows us to define

$$\int_0^T S_t \, dW_t = \lim_n \int_0^T S_{n\,t} \, dW_t$$

<u>Remark:</u> $\int_0^T S_t \, dW_t$ is a martingale (w.r.t. T).
w.r.t. $\mathcal{F}_t$

<u>Example</u>: $\int_0^T W_t \, dW_t$?

Define $W_{nt} = \sum_{i=1}^{n} W_{\frac{T}{n}(i-1)} \mathbb{1}_{[\frac{T}{n}(i-1), \frac{T}{n} i)}(t)$

one can check that $\| W_{nt} - W_t \|_{L^2(\Omega \times (0,T))} \to 0$.

Then, $\int_0^T W_t \, dW_t = \lim_n \sum_{i=1}^{n} W_{\frac{T}{n}(i-1)} \left(W_{\frac{T}{n} i} - W_{\frac{T}{n}(i-1)} \right) =$

$$= \lim_n \left[\frac{1}{2} W_T^2 - \frac{1}{2} W_0^2 - \frac{1}{2} \sum_{i=1}^{n} \left(W_{\frac{T}{n} i} - W_{\frac{T}{n}(i-1)} \right)^2 \right]$$

$$= \frac{1}{2} W_T^2 - \frac{T}{2}$$

use the Remark on page S11

<u>Example</u>: $\int_0^T dW_t = \lim \sum_{i=1}^{n} \left(W_{\frac{T}{n} i} - W_{\frac{T}{n}(i-1)} \right)$

$$= W_T - W_0$$

This limiting construction for $\int_0^t X_s \, dW_s$ works for X_t <u>progressive</u> (X_t $\mathcal{G}_t$-adapted and $X(\omega, t)$ is $\mathcal{G}_t \times \sigma([0,t])$ measurable), <u>non-anticipative</u> (X_t is adapted to $\sigma(\mathcal{G} \cup \mathcal{F}_t^W)$ where $\mathcal{G}$ is indep. of $\mathcal{F}_t^W$), and <u>mean-square</u> integrable $\left(E\left(\int_0^T X^2(t) \, dt \right) < \infty \right)$.

($\mathcal{G}$ and $\mathcal{F}$ on (Ω, μ) are independent if $\forall G \in \mathcal{G}, F \in \mathcal{F} \Rightarrow \mu(G \cap F) = \mu(G)\mu(F)$)