Measure theory recap

Measurable space: $(\Omega, \mathcal{F})$ Ω set, $\mathcal{F}$ σ-algebra

$$(\Omega \in \mathcal{F}, \quad A \in \mathcal{F} \Rightarrow {}^c A \in \mathcal{F}, \quad \{A_i\}_{i \in \mathbb{N}} \in \mathcal{F} \Rightarrow \bigcup_i A_i \in \mathcal{F})$$

Measure space: $(\Omega, \mathcal{F}, \mu)$, $(\Omega, \mathcal{F})$ measurable

$\mu : \mathcal{F} \longrightarrow [0, +\infty]$ with $\mu(\emptyset) = 0$ and σ-additive

$$(\{A_i\}_{i \in \mathbb{N}} \in \mathcal{F} \text{ mutually disjoint} \Rightarrow \mu(\sqcup_i A_i) = \textstyle\sum_i \mu(A_i))$$

two base facts: $A_1 \subset A_2 \in \mathcal{F} \Rightarrow \mu(A_1) \leq \mu(A_2)$

$\{A_i\}_{i \in \mathbb{N}} \in \mathcal{F}$, $A_i \nearrow$ w.r.t. inclusion $\Rightarrow \mu(\cup_i A_i) = \lim_{i \to \infty} \mu(A_i)$

Random variable $X : (\Omega, \mathcal{F}, \mu) \longrightarrow (\Omega', \mathcal{F}')$ measurable:
$\forall A' \in \mathcal{F}'$, $X^{-1}(A') \in \mathcal{F}$.

If $\Omega' \subset \mathbb{R}$, $\mathcal{F}' = B(\Omega')$ we call X real random

$B(\Omega')$: Borelians of Ω'

probability on Ω' induced by X: $P_X(A') = \mu(X^{-1}(A'))$
$=: P(X \in A')$

If X is real, such probability can be characterized via the partition function $F(x) = P(X \leq x)$. Moreover:

- If X is discrete ($X(\Omega)$ is discrete) one has the discrete probability density $p(x) = P(X = x)$
- If X is continuous ($\# X(\Omega) = \aleph_0$) one has that
$$P(X \in A') = \nu(A') = \int_{A'} d\nu(a) \quad \forall A' \in \mathcal{F}'$$
and some measure $\nu : \mathcal{F}' \longrightarrow [0, +\infty]$ (push-forward)

Absolutely-continuous case: If $\nu \ll \mathcal{L}$ there exists $f \in L^1(\Omega')$ such that $\int f = 1$ and

$$P(a \leq X \leq b) = \int_a^b f(t)\,dt.$$

In this case $F(x) = \int_{-\infty}^{x} f(t)\,dt$, F is absolutely continuous, and $F' = f$ a.e.

Special case if $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$ for some $m \in \mathbb{R}$, $\sigma > 0$,
we say that X is Gaussian with expected value m and variance σ^2.

Moments and central moments

$$E(X^m) = \int_{-\infty}^{+\infty} x^m\,d\nu(x) = \int_\Omega X^m(\omega)\,d\mu(\omega)$$

($m = 1 \rightarrow$ expectation)

$$E((X - E(X))^m) = \int_{-\infty}^{+\infty} (x - E(X))^m\,d\nu(x) = \int_\Omega (X - E(X))^m\,d\mu$$

($m = 2 \rightarrow$ variance)

Function of a random variable X real random variable, $g: \mathbb{R} \to \mathbb{R}$ Borel. Then $Y = g(X)$ is a real random var.
If g is invertible and differentiable:

$$f_Y(y) = f_X(g^{-1}(y))\,|(g^{-1})'(y)|$$

(a.k.a. change of variables formula)

$X = Y$ in distribution iff $P(X \le x) = P(Y \le x) \ \forall x$

$\Downarrow\!\!\!\!/ \ \Uparrow$

$X = Y$ almost surely (a.s.) iff $P(X \ne Y) = 0$

$\Downarrow\!\!\!\!/ \ \Uparrow$

$X = Y$ iff $X(\omega) = Y(\omega) \ \forall \omega \in \Omega$.

Independence for real random variables

X, Y are independent if $F_{X \times Y}(x,y) = F_X(x) F_Y(y)$.

in case they admit densities: $f_{X \times Y}(x,y) = f_X(x) f_Y(y)$.

Conditional quantities

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad \text{for } P(B) > 0$$

Let X, Y be real random variables with densities f_X, f_Y and joint density $f_{X \times Y}$. Then, the conditional prob. density of Y under $X = x$ is $f_{X \times Y}(x,y) / f_X(x)$ if $f_X(x) > 0$. The conditional expectation is

$$E(Y \mid X = x) = \int y \, f_{X \times Y}(x,y) / f_X(x) \, dy$$

(this construction actually works for discrete variables...)

For $\mathcal{G} \subset \mathcal{F}$, we call conditional expectation $E(X \mid \mathcal{G})$ a random variable with properties: (1) $E(X \mid \mathcal{G})$ is $\mathcal{G}$ meas. (2) $\forall A \in \mathcal{G}$ $\int_A X(\omega)\, d\mu(\omega) = \int_A E(X \mid \mathcal{G})(\omega)\, d\mu(\omega)$.