Stochastic PDEs

Fri 8 – 9.30 SR 11

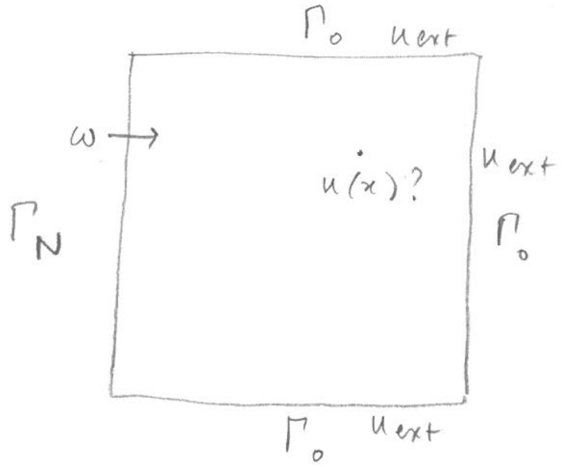
(No class: 26.10, 2.11, 28.12, 4.1)

Last class: 25.1 (13 classes in total)

Book: Prévôt – Röckner, 2007

A double syllogism

1) The world is described by PDEs

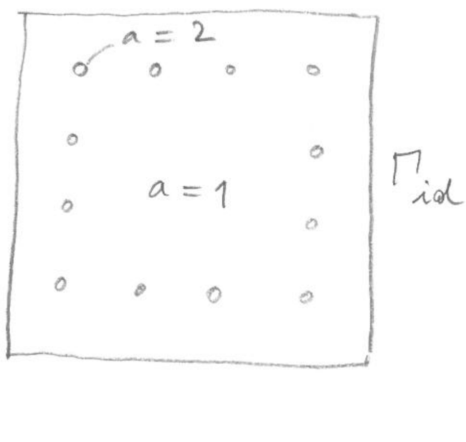


$$\begin{cases} -\nabla\cdot\big(a(x)\nabla u(x)\big) = 0 \\ a(x)\nabla u(x)\cdot n(x) = w(x) \quad \text{on } \Gamma_N \\ u(x) = u_{ext}(x) \quad \text{on } \Gamma_D \end{cases}$$

2) PDEs are based on data

$a,\ \omega,\ u_{ext}$

3) Data are often unknown / uncertain



identification:
from $a(x)\nabla u(x)\cdot n(x)$
on Γ_{id} reconstruct a

A first (ODE) example.

$a > 0$
ω, u_{ext}
constant

$$\begin{cases} -(au')' = 0 & \text{in } (0,1) \\ au'(0) = \omega \\ u(1) = u_{ext} \end{cases}$$

Solution:

- $-au'(x) + au'(0) = 0$
  $\Rightarrow \quad u'(x) = \omega / a$

- $u(x) = u(1) + \int_1^x u'(t)\,dt =$
  $= u_{ext} + (x-1)\frac{\omega}{a}$

Case 1 $\quad u_{ext} = 1, \quad a = 1, \quad \omega \in \{1, 2, 3\}$ equiprobable

$$u(x) = 1 + (x-1)\,\omega$$

$$P(\{u(1/2) \geq 0\}) = P(\{1 - \omega/2 \geq 0\}) = \frac{2}{3}$$

philosophy: $\quad u = u(x, \omega)$
in 'direction' x we have the ODE
in 'direction' ω we have the probability

<u>Case 2</u> $\quad u_{ext} = \{1, 2, 3\}$ equiprobable

$$a = \{1, 2\} \qquad \begin{cases} P(\{a=1\}) = 1/3 \\ P(\{a=2\}) = 2/3 \end{cases}$$

$$\omega = \{4, 6\} \quad \text{equiprobable}$$

Then $\quad u = u(x, \underbrace{u_{ext}, a, \omega}_{12 \text{ events}})$

$$\Omega = \{e_i\}_{i=1}^{12}$$

$$P(\{u(1/2) \geq 0\}) = \sum_{i=1}^{12} P(e_i) P(\{u(1/2, e_i) \geq 0)$$

$$= \sum_{i=1}^{12} P(u_{ext}, a, \omega)\, P(\{u_{ext} - \omega/2a \geq 0\})$$

$$= \underbrace{\frac{1}{3}\,\frac{3}{6}}_{a=1} + \underbrace{\frac{2}{3}\,\frac{5}{6}}_{a=2} = \frac{1}{6} + \frac{5}{9} = \frac{39}{54}$$

Watch out! This setting is different from what was done in class!

<u>Case 3</u> $\quad u_{ext} = 1, \; a = 1/4, \; \omega$ real-valued random variable

$\omega : \mathbb{R} \longrightarrow \mathbb{R}$ with probability density $\rho(t) = (1-|t|)^+$

$$P(\{u(1/2) \geq 0\}) = P(\{1 - 2\omega \geq 0\}) = P(\{\omega \leq 1/2\})$$

$$= \int_{-\infty}^{1/2} (1-|t|)^+ \, dt = \int_{-1}^{0} (t+1)\, dt + \int_{0}^{1/2} (1-t)\, dt$$

$$= \left[\frac{t^2}{2} + t\right]_{-1}^{0} + \left[t - \frac{t^2}{2}\right]_{0}^{1/2} = -\frac{1}{2} + 1 + \frac{1}{2} - \frac{1}{8} = \frac{7}{8}$$

<u>Expected value</u>

$$E(u(x)) = \int_{-\infty}^{+\infty} u(x,w)\,\rho(w)\,dw =$$

$$= \int_{-1}^{0} \big(1 + 2w(x-1)\big)(w+1)\,dw + \int_{0}^{1} \big(1 + 2w(x-1)\big)(1-w)\,dw =$$

$$= \int_{-1}^{0} \left(w + 2w^2(x-1) + 1 + 2w(x-1)\right) dw \quad +$$

$$+ \int_{0}^{1} \left(1 + 2w(x-1) - w - 2w^2(x-1)\right) dw =$$

$$= \left[\frac{w^2}{2} + \frac{2w^3}{3}(x-1) + w + w^2(x-1)\right]_{-1}^{0} +$$

$$+ \left[w + w^2(x-1) - \frac{w^2}{2} - \frac{2}{3}w^3(x-1)\right]_{0}^{1} =$$

$$= -\frac{1}{2} + \cancel{\frac{2}{3}(x-1)} + 1 - \cancel{(x-1)} + 1 + \cancel{(x-1)} - \frac{1}{2} - \cancel{\frac{2}{3}(x-1)} =$$

$$= 2$$

<u>Variance</u> $\quad E\left(\left(u(x) - E(u(x))\right)^2\right)$