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A New Minimum Principle for Lagrangian Mechanics

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Abstract We present a novel variational view at Lagrangian mechanics based on the minimization of *weighted inertia-energy* functionals on trajectories. In particular, we introduce a family of parameter-dependent global-in-time minimization problems whose respective minimizers converge to solutions of the system of Lagrange's equations. The interest in this approach is that of reformulating Lagrangian dynamics as a (class of) minimization problem(s) plus a limiting procedure. The theory may be extended in order to include dissipative effects thus providing a unified framework for both dissipative and nondissipative situations. In particular, it allows for a rigorous connection between these two regimes by means of Γ-convergence. Moreover, the variational principle may serve as a selection criterion in case of nonuniqueness of solutions. Finally, this variational approach can be localized on a finite time-horizon resulting in some sharper convergence statements and can be combined with time-discretization.

Keywords Lagrangian mechanics · Minimum principle · Elliptic regularization · Time discretization

Mathematics Subject Classification (2010) 70H03 · 70H30 · 65L10

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1 Introduction

Variational principles in continuum mechanics and thermodynamics have been the subject of constant attention since their early appearance more than two centuries ago. From the philosophical viewpoint, the investigation of variational principles is of a paramount importance for it corresponds to the fundamental quest for general and simple explanations of reality as we experience it. On the other hand, beside their indisputable elegance, variational principles have a clear practical impact as they originate a wealth of new perspectives and serve as unique tools for the analysis of real physical situations. Correspondingly, the mathematical literature on variational principles in mechanics is overwhelming and a number of monographs on the subject are available. Being completely beyond our purposes to attempt a comprehensive review of the development of this subject, we shall minimally refer the reader to some classical monographs (Lánczos 1970; Moiseiwitsch 2004) as well as the more recent (Basdevant 2007; Berdichevsky 2009; Ghoussoub 2008).

The focus of this note is to present a new variational principle in the context of classical Lagrangian mechanics. In particular, we shall be concerned with the evolution of a conservative dynamical system described by a set of generalized coordinates $\boldsymbol{q} \in \mathbb{R}^m$ ($m \in \mathbb{N}$) and characterized by the *Lagrangian* (Arnol'd 1989)

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) := \frac{1}{2}\dot{\boldsymbol{q}}\cdot M\dot{\boldsymbol{q}} - U(\boldsymbol{q}).$$

Here, M is the symmetric and positive definite *mass matrix*, so that $\dot{\boldsymbol{q}}\cdot M\dot{\boldsymbol{q}}/2$ is the classical *kinetic energy* term. Moreover, we assume to be given the *potential energy* $U \in \mathrm{C}^1(\mathbb{R}^m)$ which we additionally ask to be bounded from below. Lagrangians of this form naturally arise in connection with a variety of applications ranging from celestial mechanics to molecular dynamics.

We consider the minimization of the global-in-time functionals W_ε defined on entire trajectories $\boldsymbol{q} : \mathbb{R}_+ \to \mathbb{R}^m$ as

$$\mathsf{W}_\varepsilon[\boldsymbol{q}] := \int_0^\infty \mathrm{e}^{-t/\varepsilon}\left(\frac{\varepsilon^2}{2}\ddot{\boldsymbol{q}}(t)\cdot M\ddot{\boldsymbol{q}}(t) + U\big(\boldsymbol{q}(t)\big)\right)\mathrm{d}t \quad (\varepsilon > 0).$$

Note that the small parameter ε above has the physical dimension of time, so that the whole integrand in W_ε is an energy and W_ε is an action. We shall refer to the latter as *Weighted Inertia-Energy* functionals (WIE) as they feature the weighted sum of an inertial term (suitably dimensionalized) and the potential U.

The WIE functional W_ε admits minimizers $\boldsymbol{q}_\varepsilon$ in the closed and convex set

$$K_\varepsilon := \left\{\boldsymbol{q} \in \mathrm{H}^2\big(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m\big) : \boldsymbol{q}(0) = \boldsymbol{q}^0,\ \dot{\boldsymbol{q}}(0) = \boldsymbol{q}^1\right\}$$

where given initial data $\boldsymbol{q}^0 \in \mathbb{R}^m$ and $\boldsymbol{q}^1 \in \mathbb{R}^m$ are prescribed (see Lemma 2.1 below). Let us stress from the very beginning that $\boldsymbol{q} \in \mathrm{H}^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m)$ implies the integrability conditions

$$t \mapsto \mathrm{e}^{-t/\varepsilon}|\boldsymbol{q}|^2,\ t \mapsto \mathrm{e}^{-t/\varepsilon}|\dot{\boldsymbol{q}}|^2 \in \mathrm{L}^1(\mathbb{R}_+) \tag{1}$$

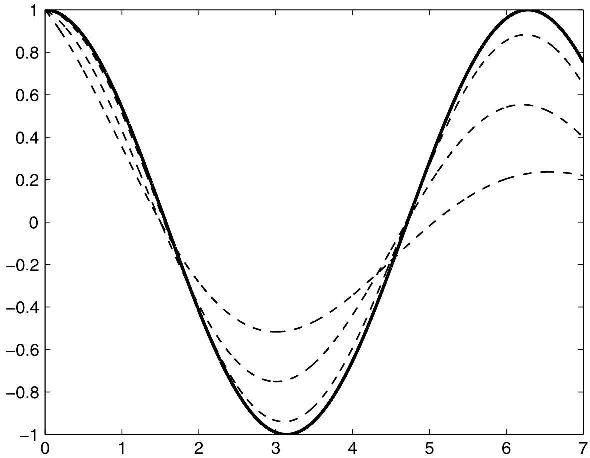

Fig. 1 Convergence for $m = 1$, $U(q) = q^2/2$, $q^0 = 1$, and $q^1 = 0$. As $\varepsilon \to 0$, the minimizers of W_ε on K_ε for $\varepsilon = 0.3, 0.1, 0.02$ (*dashed*) approach locally uniformly the solution of (2), namely $t \mapsto \cos t$ (*solid*)

by virtue of some suitable weighted Poincaré inequality form (Serra and Tilli 2012) (see (5) later on). The above integrability conditions play a crucial role in the analysis and be specifically addressed in Sect. 2.6 below.

The first result of this paper that the minimizers $\boldsymbol{q}_\varepsilon$ of the functional W_ε on K_ε admit a subsequence which converges to a solution of the system of Lagrange's equations (*Lagrangian system* for short in the sequel). Namely, we have the following.

Theorem 1.1 (WIE principle) *Let $\boldsymbol{q}_\varepsilon$ minimize W_ε in K_ε. Then, for some subsequence $\boldsymbol{q}_{\varepsilon_k}$ we have that $\boldsymbol{q}_{\varepsilon_k} \to \boldsymbol{q}$ weakly-$*$ in $\mathrm{W}^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ (hence, locally uniformly) where $\boldsymbol{q}$ is a classical solution of the Lagrangian system*

$$M\ddot{\boldsymbol{q}} + \nabla U(\boldsymbol{q}) = \boldsymbol{0} \quad \text{in } \mathbb{R}_+, \qquad \boldsymbol{q}(0) = \boldsymbol{q}^0, \qquad M\dot{\boldsymbol{q}}(0) = M\boldsymbol{q}^1. \tag{2}$$

In the easiest possible setting, namely the scalar ($m = 1$) and linear case of $U(q) = q^2/2$ with $q^0 = 1$ and $q^1 = 0$ the convergence result of Theorem 1.1 is illustrated in Fig. 1.

The WIE principle provides a new variational reformulation of the Lagrangian system (2) as a (limit of a class of) constrained minimization problem(s). Although the Cauchy problem for the Lagrangian system (2) can be analyzed directly, the WIE formulation paves the way to the treatment of the system by purely variational means. In particular, the WIE approach allows for the *direct* application of the tools of the Calculus of Variations (the Direct Method and Γ-convergence, for instance) to the evolutive differential system (2). In particular, the WIE principle provides some selection criterion in case of nonuniqueness of solutions (see Sect. 2.7 below). Note that the WIE principle concerns global minimizers only. In particular, we cannot exclude that W has extra stationary points (but see Lemma 4.1).

The WIE variational method appears to be rather general and can be readily extended at least in two relevant directions. At first, in Sect. 4, we focus on a specific finite-time horizon version of the WIE principle where the integration is confined to some finite-time interval $(0, T)$. This allows to sharpen the convergence result of

Theorem 1.1 in order to obtain error rates and turns out to be better suited for the purpose of numerical investigation.

Secondly, we extend the WIE principle to treat mixed dissipative/nondissipative situations such that of viscous dynamics, both in the infinite (Sect. 3) and finite-time horizon (Sect. 4.4). The flexibility of this variational theory is such that we can handle with the same method also the limiting purely dissipative (viscous) and purely nondissipative cases and, in particular, we can handle the case of gradient flows. We provide in Sect. 5 some Γ-convergence analysis connecting these two regimes as well as for the finite to infinite time-horizon limit.

1.1 Comparison with the Hamilton Principle

We shall now turn to the illustration of some of the specific features of the WIE formalism by focusing on its comparison with the more classical *Hamilton principle*. The latter asserts that actual trajectories of the Lagrangian system (2) on the time interval $(0, T)$ are extremizers of the *action* functional

$$S[\boldsymbol{q}] = \int_0^T \left(\frac{1}{2} \dot{\boldsymbol{q}}(t) \cdot M \dot{\boldsymbol{q}}(t) - U\big(\boldsymbol{q}(t)\big) \right) \mathrm{d}t$$

among all paths with prescribed initial and final states $\boldsymbol{q}^0$ and $\boldsymbol{q}^T$. In particular, the Lagrangian system (2) exactly corresponds to the Euler–Lagrange equation for S.

The WIE variational approach differs from the Hamilton principle in *three* crucial ways. First, Hamilton's principle is indeed a *stationarity principle* for it generally corresponds to the quest for a saddle point of the action functional (note, however, that this will be a true minimum for small T). On the contrary, the WIE principle relies on a true constrained *minimization*.

Secondly, the WIE principle is directly formulated on the whole time semiline $\mathbb{R}_+$ whereas Hamilton's approach calls for the specification of an artificial finite-time interval $(0, T)$ and a final state. In particular, the WIE functionals directly encode directionality of time by explicitly requiring the knowledge of just *initial states*.

Finally, the WIE principle is not invariant by time reversal. This is indeed crucial as the WIE perspective is naturally incorporating dissipative effects (see Sect. 3), thus qualifying it as a suitable tool in order to discuss limiting mixed dissipative/nondissipative dynamics. Note that dissipative effects cannot be directly treated via Hamilton's framework and one resorts in considering the classical Lagrange–D'Alembert principle instead.

The price to pay within the WIE functional method with respect to Hamilton's is the check of the extra limit $\varepsilon \to 0$. This is exactly the object of Theorem 1.1 and the main concern of this theory.

1.2 Review of the Literature on Weighted Functionals

Global-in-time minimization of weighted functionals has been already considered in the purely dissipative (viscous) case. In particular, this functional approach has been developed for so called *Weighted Energy-Dissipation* (WED) functionals

$$q \mapsto \int_0^T \mathrm{e}^{-t/\varepsilon} \big(\varepsilon \Psi\big(\dot{\boldsymbol{q}}(t)\big) + U\big(\boldsymbol{q}(t)\big)\big) \mathrm{d}t$$

where Ψ is a suitable nonnegative and convex dissipation potential, even in the PDE infinite-dimensional situation. In the linear case $\Psi(\dot{\boldsymbol{q}}) = |\dot{\boldsymbol{q}}|^2/2$, some results can be found in the classical monograph by Lions and Magenes (1972). As for the nonlinear case, this procedure has been followed by Ilmanen (1994) for proving existence and partial regularity of the so-called Brakke mean curvature flow of varifolds. Two examples of relaxation of gradient flows related to microstructure evolution are provided by Conti and Ortiz (2008). For λ-convex energies U, the convergence proof $\boldsymbol{q}_\varepsilon \to \boldsymbol{q}$ in Hilbert and metric spaces has been provided in Mielke and Stefanelli (2011) and Rossi et al. (2011a, 2011b), respectively. An application in the context of gradient flows driven by linear-growth functionals and, in particular, to mean curvature flow of graphs is given in Spadaro and Stefanelli (2011). On the other hand, the WED technique has been extended to rate-independent evolution $\Psi(\dot{\boldsymbol{q}}) = |\dot{\boldsymbol{q}}|$ by Mielke and Ortiz (2008), and further detailed in Mielke and Stefanelli (2008). Some application to crack propagation is given by Larsen et al. (2009). Eventually, the doubly nonlinear case $\Psi(\dot{\boldsymbol{q}}) = |\dot{\boldsymbol{q}}|^p/p$ $(p > 2)$ is addressed in Akagi and Stefanelli (2010, 2011). Some duality-based WED approach to another large class of nonlinear evolutions including the two-phase Stefan problem and the porous-media equation is presented in Akagi and Stefanelli (2012).

Our interest in WIE functionals has been inspired by a conjecture by De Giorgi (1996) on hyperbolic evolution. In particular, in De Giorgi (1996) it is conjectured that the minimizers of the PDE version of the functional W_ε

$$u \mapsto \int_0^T \int_{\mathbb{R}^m} \mathrm{e}^{-t/\varepsilon}\left(\frac{\varepsilon^2}{2}\left|\partial_{tt}u(x,t)\right|^2 + \frac{1}{2}\left|\nabla u(x,t)\right|^2 + \frac{1}{p}\left|u(x,t)\right|^p\right)\mathrm{d}x\,\mathrm{d}t \quad (p > 2)$$

among all space-time functions u with prescribed initial conditions, converge as $\varepsilon \to 0$ to a solution of the semilinear wave equation

$$\partial_{tt}u - \Delta u + |u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^m \times \mathbb{R}_+.$$

This conjecture has been checked positively for $T < \infty$ in Stefanelli (2011) and then for $T = \infty$ by Serra and Tilli (2012).

Already in De Giorgi (1996, Rem. 1) it is speculated that some similar result could hold for more general functionals of the Calculus of Variations as well. Our main result Theorem 1.1 provides here a positive answer to this extension of the conjecture in the finite-dimensional case.

2 The WIE Principle on $\mathbb{R}_+$

We focus here on the infinite-time horizon result of Theorem 1.1. With no loss of generality, hereafter we shall assume the potential U to be nonnegative. Note that our analysis is presently restricted to bounded-below potentials. In particular, we are not in the position of addressing blow-up phenomena. Moreover, in order to avoid cumbersome notation, we shall let $M = \rho I$ where $\rho > 0$ and I is the identity matrix. It should be, however, clear that the corresponding proofs for a general positive-definite mass matrix M can be then obtained with no particular intricacy.

A *caveat* on notation: In the remainder of the paper c stands for any positive constant, possibly depending on $\boldsymbol{q}^0$, $\boldsymbol{q}^1$, and U and changing from line to line. Note specifically that c does not depend on ρ and, later, ν and T.

2.1 Existence of Minimizers

Let us firstly record that minimizers of W_ε on K_ε actually exist.

Lemma 2.1 (Direct method) *W_ε admits a minimizer in K_ε.*

Proof Every minimizing sequence $\boldsymbol{q}_k \in K_\varepsilon$ fulfills $\rho \int_0^\infty \mathrm{e}^{-t/\varepsilon} |\ddot{\boldsymbol{q}}_k(t)|^2 \,\mathrm{d}t \le c$ and it is hence compact in $\mathrm{L}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m)$ (see (5) below). Upon extracting some subsequence, one can exploit the lower semicontinuity of U and pass to the lim inf in W_ε by Fatou's lemma. □

2.2 A Priori Estimate

The proof of Theorem 1.1 relies on an a priori estimate on the minimizers $\boldsymbol{q}_\varepsilon$ of W_ε on K_ε. We have the following.

Lemma 2.2 (A priori estimate) *Let $\boldsymbol{q}_\varepsilon$ minimize W_ε on K_ε. Then*

$$\rho \left| \dot{\boldsymbol{q}}_\varepsilon(t) \right|^2 \le c \quad \forall t > 0. \tag{3}$$

The lemma follows from the argument by Serra and Tilli (2012) where the PDE case of semilinear wave equations is treated. We hence claim no originality here. Still, we record the proof of Lemma 2.2 for the sake of later reference with respect to its extension to the mixed dissipative/nondissipative case presented in Sect. 3 below.

Proof Assume $\boldsymbol{q}_\varepsilon$ to be a minimizer and rescale time by letting $\boldsymbol{p}(t) := \boldsymbol{q}_\varepsilon(\varepsilon t)$. We define the rescaled functional G_ε as

$$G_\varepsilon[\boldsymbol{p}] := \int_0^\infty \mathrm{e}^{-t} \left(\frac{\rho}{2} \left| \ddot{\boldsymbol{p}}(t) \right|^2 + \varepsilon^2 U\big(\boldsymbol{p}(t)\big) \right) \mathrm{d}t$$

so that $\varepsilon \mathsf{W}_\varepsilon[\boldsymbol{q}_\varepsilon] = G_\varepsilon[\boldsymbol{p}]$. At first, let us check that $G_\varepsilon[\boldsymbol{p}] \le c\varepsilon^2$. Indeed, define $\widehat{\boldsymbol{p}}(t)$ componentwise as $\widehat{p}_i(t) := q_i^0 + \arctan(\varepsilon q_i^1 t)$. We have that $\widehat{\boldsymbol{p}}(0) = \boldsymbol{q}^0$ and $(\mathrm{d}/\mathrm{d}t)\widehat{\boldsymbol{p}}(0) = \varepsilon \boldsymbol{q}^1$ so that, in particular, $t \mapsto \widehat{\boldsymbol{q}}(t) := \widehat{\boldsymbol{p}}(t/\varepsilon) \in K_\varepsilon$. By exploiting the boundedness of $\widehat{\boldsymbol{p}}$ and the local boundedness of U one has

$$\begin{aligned} G_\varepsilon[\boldsymbol{p}] = \varepsilon \mathsf{W}_\varepsilon[\boldsymbol{q}_\varepsilon] \le \varepsilon \mathsf{W}_\varepsilon[\widehat{\boldsymbol{q}}] = G_\varepsilon[\widehat{\boldsymbol{p}}] \le c\varepsilon^6 \int_0^\infty t^2 \mathrm{e}^{-t} \,\mathrm{d}t + \varepsilon^2 \int_0^\infty \mathrm{e}^{-t} U\big(\widehat{\boldsymbol{p}}(t)\big)\,\mathrm{d}t \\ \le c\varepsilon^2. \end{aligned} \tag{4}$$

In the following, we shall make use of the following elementary inequality (Serra and Tilli 2012, Lemma 2.3)

$$\int_t^\infty \mathrm{e}^{-s} f^2(s)\,\mathrm{d}s \le 2\mathrm{e}^{-t} f^2(t) + 4 \int_t^\infty \mathrm{e}^{-s} \dot{f}^2(s)\,\mathrm{d}s \tag{5}$$

which follows by integration by parts and is valid for all $f \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $t \geq 0$, regardless of the finiteness of the integrals. In particular, we exploit inequality (5) in order to get that

$$\int_0^\infty \mathrm{e}^{-s}\big|\dot{\boldsymbol{p}}(s)\big|^2\,\mathrm{d}s \leq 2\varepsilon^2\big|\boldsymbol{q}^1\big|^2 + 4\int_0^\infty \mathrm{e}^{-s}\big|\ddot{\boldsymbol{p}}(s)\big|^2\,\mathrm{d}s \leq c\varepsilon^2 + \frac{c}{\rho}G_\varepsilon[\boldsymbol{p}]. \qquad (6)$$

The latter entails that $t \mapsto \mathrm{e}^{-t}|\dot{\boldsymbol{p}}(t)|^2 \in \mathrm{W}^{1,1}(\mathbb{R}_+)$ so that $\mathrm{e}^{-t}|\dot{\boldsymbol{p}}(t)|^2 \to 0$ as $t \to \infty$.

Define now, for all $t \geq 0$, the auxiliary function

$$H(t) := \int_t^\infty \mathrm{e}^{-s}\left(\frac{\rho}{2}\big|\ddot{\boldsymbol{p}}(s)\big|^2 + \varepsilon^2 U\big(\boldsymbol{p}(s)\big)\right)\mathrm{d}s$$

and note that $H \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, it is nonincreasing and nonnegative.

By considering competitors $\tilde{\boldsymbol{p}}(t) = \boldsymbol{p}(s(t))$ where s is some smooth time reparameterization, the minimality of $\boldsymbol{p}$ and the computations in Serra and Tilli (2012, Prop. 3.1) ensure that

$$\left(\frac{\rho}{2}\ddot{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}}\right)^{\boldsymbol{\cdot}} = \frac{1}{2}\big(\mathrm{e}^t H(t)\big)^{\boldsymbol{\cdot}} + \rho|\ddot{\boldsymbol{p}}|^2 + \frac{\rho}{2}\dddot{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}}. \qquad (7)$$

Let a second auxiliary function E be defined as

$$E(t) := \frac{\rho}{4}\big|\dot{\boldsymbol{p}}(t)\big|^2 - \frac{\rho}{2}\ddot{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}} + \frac{1}{2}\mathrm{e}^t H(t).$$

By virtue of relation (7), we compute that

$$\begin{aligned}\dot{E} &= \frac{\rho}{2}\ddot{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}} - \frac{\rho}{2}(\ddot{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}})^{\boldsymbol{\cdot}} + \frac{1}{2}\big(\mathrm{e}^t H(t)\big)^{\boldsymbol{\cdot}} \\ &\overset{(7)}{=} \frac{\rho}{2}\ddot{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}} - \left(\frac{1}{2}\big(\mathrm{e}^t H(t)\big)^{\boldsymbol{\cdot}} + \rho|\ddot{\boldsymbol{p}}|^2 + \frac{\rho}{2}\ddot{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}}\right) + \frac{1}{2}\big(\mathrm{e}^t H(t)\big)^{\boldsymbol{\cdot}} = -\rho|\ddot{\boldsymbol{p}}|^2, \end{aligned} \qquad (8)$$

so that $E \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ and nonincreasing. The function E is defined in such a way that

$$-\frac{\rho}{4}\big(\mathrm{e}^{-t}\big|\dot{\boldsymbol{p}}(t)\big|^2\big)^{\boldsymbol{\cdot}} + \frac{1}{2}H(t) = \mathrm{e}^{-t}E(t). \qquad (9)$$

Let us now integrate the latter on (t,T) getting

$$\begin{aligned}&\frac{\rho}{4}\mathrm{e}^{-t}\big|\dot{\boldsymbol{p}}(t)\big|^2 - \frac{\rho}{4}\mathrm{e}^{-T}\big|\dot{\boldsymbol{p}}(T)\big|^2 + \frac{1}{2}\int_t^T H(s)\,\mathrm{d}s \\ &\quad= \int_t^T \mathrm{e}^{-s}E(s)\,\mathrm{d}s \leq E(t)\int_t^T \mathrm{e}^{-s}\,\mathrm{d}s = E(t)\big(\mathrm{e}^{-t} - \mathrm{e}^{-T}\big)\end{aligned} \qquad (10)$$

where the inequality follows from the monotonicity of E. Hence, by letting $T \to \infty$ in (10) and recalling that $\mathrm{e}^{-T}|\dot{\boldsymbol{p}}(T)|^2 \to 0$, we have proved that

$$\frac{\rho}{4}\big|\dot{\boldsymbol{p}}(t)\big|^2 \leq E(t) \leq E(0). \qquad (11)$$

We now turn to the estimate of $E(0)$. At first, note that, by exploiting the bounds (4) and (6) we have that

$$\int_0^1 \left|\ddot{\boldsymbol{p}}(t)\right|^2 \mathrm{d}t \le \mathrm{e}\int_0^\infty \mathrm{e}^{-t}\left|\ddot{\boldsymbol{p}}(t)\right|^2 \mathrm{d}t \le \frac{2\mathrm{e}}{\rho}G_\varepsilon[\widehat{\boldsymbol{p}}] \overset{(4)}{\le} \frac{c}{\rho}\varepsilon^2, \tag{12}$$

$$\int_0^1 \left|\dot{\boldsymbol{p}}(t)\right|^2 \mathrm{d}t \le \mathrm{e}\int_0^\infty \mathrm{e}^{-t}\left|\dot{\boldsymbol{p}}(t)\right|^2 \mathrm{d}t \overset{(6)}{\le} c\varepsilon^2 + \frac{c}{\rho}G_\varepsilon[\widehat{\boldsymbol{p}}] \overset{(4)}{\le} c\left(1+\frac{1}{\rho}\right)\varepsilon^2. \tag{13}$$

In particular, these bounds and $H(t) \le H(0) = G_\varepsilon[\boldsymbol{p}] \le c\varepsilon^2$ suffice in order to conclude that

$$\int_0^1 E(t)\,\mathrm{d}t \le c(1+\rho)\varepsilon^2. \tag{14}$$

Eventually, by using equality (8) and integrating in time, we have

$$\begin{aligned} E(0) = \int_0^1 E(0)\,\mathrm{d}t &\overset{(8)}{=} \int_0^1 \left(E(t) + \rho\int_0^t \left|\ddot{\boldsymbol{p}}(s)\right|^2 \mathrm{d}s\right)\mathrm{d}t \\ &\le \int_0^1 E(t)\,\mathrm{d}t + \rho\int_0^1 \left|\ddot{\boldsymbol{p}}(t)\right|^2 \mathrm{d}t \overset{(14)}{\le} c(1+\rho)\varepsilon^2. \end{aligned} \tag{15}$$

Going back to (11), we have finally checked the pointwise bound $\rho|\dot{\boldsymbol{p}}(t)|^2 \le c\varepsilon^2$ and estimate (3) ensues by time rescaling. □

2.3 Euler–Lagrange Equation

The proof of Theorem 1.1 follows by passing to the limit for $\varepsilon \to 0$ in the Euler–Lagrange equation for the minimizers $\boldsymbol{q}_\varepsilon$ of W_ε on K_ε. By considering internal variations, one has that

$$0 = \int_0^\infty \rho\left(\mathrm{e}^{-t/\varepsilon}\ddot{\boldsymbol{q}}_\varepsilon(t)\right)\cdot\ddot{v}\,\mathrm{d}t + \frac{1}{\varepsilon^2}\int_0^\infty \mathrm{e}^{-t/\varepsilon}\nabla U\left(\boldsymbol{q}_\varepsilon(t)\right)\cdot\boldsymbol{v}(t)\,\mathrm{d}t \tag{16}$$

for all $\boldsymbol{v} \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}_+;\mathbb{R}^m)$. Hence, minimizers of W_ε solve the Euler–Lagrange equation

$$\varepsilon^2\rho\boldsymbol{q}^{(4)} - 2\varepsilon\rho\boldsymbol{q}^{(3)} + \rho\ddot{\boldsymbol{q}} + \nabla U(\boldsymbol{q}) = \boldsymbol{0} \quad \text{in } \mathbb{R}_+ \tag{17}$$

in the distributional sense, where $\boldsymbol{q}^{(k)}$ stands for the kth derivative. In particular, the minimizers $\boldsymbol{q}_\varepsilon$ solve a fourth-order elliptic-in-time regularization of the Lagrangian system (2). Indeed, system (17) is solved in the strong sense as ∇U is continuous and the uniform bound (3) entail that

$$\varepsilon^2\rho\left(\mathrm{e}^{-t/\varepsilon}\ddot{\boldsymbol{q}}_\varepsilon(t)\right)^{\cdot\cdot} = -\mathrm{e}^{-t/\varepsilon}\nabla U\left(\boldsymbol{q}_\varepsilon(t)\right) \in \mathrm{C}\left(\mathbb{R}_+;\mathbb{R}^m\right).$$

2.4 Proof of Theorem 1.1

The pointwise estimate of Lemma 2.2 yields that, by possibly passing to not relabeled subsequences, we have that $\boldsymbol{q}_\varepsilon \to \boldsymbol{q}$ locally uniformly. Let us check that $\boldsymbol{q}$ indeed solves the Lagrangian system (2). To this aim, fix any $\boldsymbol{w} \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}_+;\mathbb{R}^m)$ and choose $\boldsymbol{v}(t) = \boldsymbol{v}_\varepsilon(t) := \mathrm{e}^{t/\varepsilon}\boldsymbol{w}(t)$ in relation (16). As one has that

$$\ddot{\boldsymbol{v}}_\varepsilon(t) = \mathrm{e}^{t/\varepsilon}\ddot{\boldsymbol{w}}(t) + (2/\varepsilon)\mathrm{e}^{t/\varepsilon}\dot{\boldsymbol{w}}(t) + \big(1/\varepsilon^2\big)\mathrm{e}^{t/\varepsilon}\boldsymbol{w}(t),$$

from (16) we get that

$$\begin{aligned} 0 &= \int_0^\infty \mathrm{e}^{-t/\varepsilon}\bigg(\rho\ddot{\boldsymbol{q}}_\varepsilon(t)\cdot\ddot{\boldsymbol{v}}_\varepsilon(t) + \frac{1}{\varepsilon^2}\nabla U\big(\boldsymbol{q}_\varepsilon(t)\big)\cdot\boldsymbol{v}_\varepsilon(t)\bigg)\,\mathrm{d}t \\ &= \int_0^\infty \bigg(\rho\ddot{\boldsymbol{q}}_\varepsilon(t)\cdot\ddot{\boldsymbol{w}}(t) + \frac{2\rho}{\varepsilon}\ddot{\boldsymbol{q}}_\varepsilon(t)\cdot\dot{\boldsymbol{w}}(t) + \frac{\rho}{\varepsilon^2}\ddot{\boldsymbol{q}}_\varepsilon(t)\cdot\boldsymbol{w}(t) + \frac{1}{\varepsilon^2}\nabla U\big(\boldsymbol{q}_\varepsilon(t)\big)\cdot\boldsymbol{w}(t)\bigg)\,\mathrm{d}t. \end{aligned}$$

In particular, one deduces from the latter that

$$\begin{aligned} &\int_0^\infty \big(\rho\boldsymbol{q}_\varepsilon(t)\cdot\ddot{\boldsymbol{w}}(t) + \nabla U\big(\boldsymbol{q}_\varepsilon(t)\big)\cdot\boldsymbol{w}(t)\big)\,\mathrm{d}t \\ &\quad = \int_0^\infty \big(\varepsilon^2\rho\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\boldsymbol{w}^{(3)}(t) + 2\varepsilon\rho\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\ddot{\boldsymbol{w}}(t)\big)\,\mathrm{d}t \\ &\quad = \int_0^T \rho\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\big(\varepsilon^2\boldsymbol{w}^{(3)}(t) + 2\varepsilon\ddot{\boldsymbol{w}}(t)\big)\,\mathrm{d}t. \end{aligned}$$

By passing to the limit in the latter as $\varepsilon \to 0$ and using the bound (3) we have that

$$\int_0^\infty \big(\rho\boldsymbol{q}(t)\cdot\ddot{\boldsymbol{w}}(t) + \nabla U\big(\boldsymbol{q}(t)\big)\cdot\boldsymbol{w}(t)\big)\,\mathrm{d}t = 0.$$

Namely, $\boldsymbol{q}$ solves $\rho\ddot{\boldsymbol{q}} = -\nabla U(\boldsymbol{q})$ in the distributional sense. By comparison in the latter we have that $\boldsymbol{q} \in \mathrm{C}^2(\mathbb{R}_+;\mathbb{R}^m)$ so that $\boldsymbol{q}$ is indeed the classical solution of (2). In case $U \in \mathrm{C}^{1,1}_{\mathrm{loc}}(\mathbb{R}^m)$, the solution of the second order Cauchy problem (2) is unique and the convergence $\boldsymbol{q}_\varepsilon \to \boldsymbol{q}$ holds for the whole sequence.

2.5 More General Potentials

By inspecting the proof of Lemmas 2.1–2.2 one realizes that the statement of Theorem 1.1 is indeed valid in some greater generality. In particular, one could require the potential U to be defined just on a non-empty open subset $D \subset \mathbb{R}^m$ and, by letting U_{ext} be its trivial extension to ∞ out of D, impose

$$0 \le U \in C^1(D) \quad \text{and} \quad U_{\mathrm{ext}} \text{ be lower semicontinuous.} \tag{18}$$

Note that the lower semicontinuity of U_{ext} expresses the fact that the potential U is actually confining the evolution to D. In particular U becomes unbounded by approaching the boundary of D. By requiring $\boldsymbol{q}^0 \in D$, under assumption (18) Theorem 1.1 still holds. The extension of the WIE principle to the latter type of potentials is not at all academical as it qualifies the WIE functional to be applicable also in some singular potential situation.

We shall also mention that, although completely neglected in this paper for the sake of simplicity, a suitably well-behaved time-dependence of the potential U (hence, in particular, a non-homogeneous flow) can be considered.

2.6 Integrability Conditions at Infinity

Before going on, we shall explicitly remark the crucial role of the two integrability conditions at infinity (1) which are fulfilled by all trajectories $\boldsymbol{q}$ in K_ε. These conditions correspond to the two *missing* boundary conditions needed in order to complement the fourth-order problem (17). In particular, conditions (1) are responsible for the *noncausality* of the problem at all levels $\varepsilon > 0$: The solution $\boldsymbol{q}$ at time t depends on *future*, i.e., its value on (t, ∞). Note, however, that by taking the limit $\varepsilon \to 0$ causality is eventually restored; see (2).

In order to illustrate this remark, let us consider once more the scalar linear situation of $U(q) = q^2/2$ and $\rho = 1$. In this case, the solution of $\varepsilon^2 q^{(4)} - 2\varepsilon q^{(3)} + \ddot{q} + q = 0$ can be computed explicitly as $q(t) = \sum_{i=1}^4 c_i \exp(\lambda_{\varepsilon,i} t)$ with

$$\lambda_{\varepsilon,1} = \frac{1-u_\varepsilon}{2\varepsilon}, \qquad \lambda_{\varepsilon,2} = \frac{1-v_\varepsilon}{2\varepsilon}, \qquad \lambda_{\varepsilon,3} = \frac{1+u_\varepsilon}{2\varepsilon}, \qquad \lambda_{\varepsilon,4} = \frac{1+v_\varepsilon}{2\varepsilon}.$$

In the latter $u_\varepsilon,\ v_\varepsilon \in \mathbb{C}$ are chosen in such a way that $u_\varepsilon^2 = 1 - 4\varepsilon i$ and $v_\varepsilon^2 = 1 + 4\varepsilon i$, respectively. By exploiting conditions (1) we readily check that, necessarily, $c_3 = c_4 = 0$. Hence, solutions to (17) in fulfilling (1) are of the form $q(t) = c_1 \exp(\lambda_{\varepsilon,1} t) + c_2 \exp(\lambda_{\varepsilon,2} t)$ and we easily check that $\lambda_{\varepsilon,1} \to i$ and $\lambda_{\varepsilon,2} \to -i$. This corresponds to the fact that the limit of minimizers of W_ε in K_ε converge to a linear combination of sin and cos, i.e., a solution of $\ddot{q} + q = 0$.

2.7 The WIE Principle as a Selection Criterion

In case the Lagrangian system (2) admits multiple solutions the WIE principle may serve as a variational selection criterion. Heuristically, this is related to the specific noncausality of the minimization process for all $\varepsilon > 0$. Indeed, differently from the solutions of the limiting differential problem (2), the minimizers of W_ε are allowed in some sense to *peek into the future* and to expend some inertia in order to exploit some possible lower-potential state.

We shall illustrate this fact by a scalar example. Fix the initial data to be $q^0 = q^1 = 0$ and choose the potential

$$U(q) = \begin{cases} -8(q^+)^{3/2} & \text{for } q \le 1, \\ 8((2-q)^+)^{3/2} - 16 & \text{for } q > 1. \end{cases}$$

Note that the potential U is C^1 but not λ-convex at $q = 0$. In particular, U is maximal for $q \le 0$ and minimal for $q \ge 2$.

The corresponding equation (2) reads $\ddot{q} = 12\sqrt{q^+}$ which, along with the prescribed initial conditions, admits the trivial solution $q(t) = 0$ as well as a continuum of solutions of the form $t \mapsto ((t-h)^+)^4$ for all $h > 0$. For all fixed $\varepsilon > 0$, the

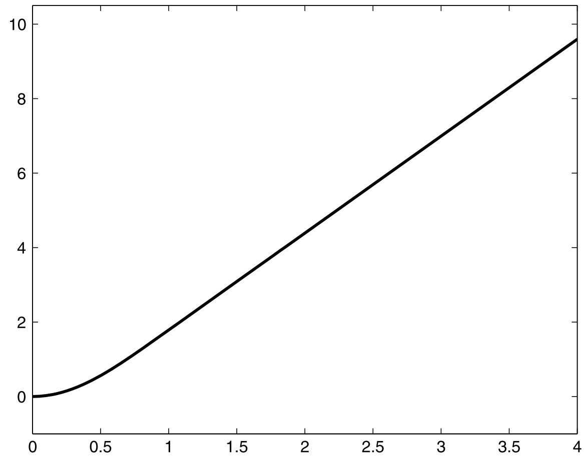

Fig. 2 The solution q_0 for $\varepsilon = 1/2$

corresponding Euler–Lagrange equation (17) (along with the initial conditions and integrability conditions at ∞) admits multiple solutions as well. At first, one has of course the trivial solution. Then, by observing that the potential U is locally Lipschitz continuous for $q > 0$, one can uniquely find the solution q_0 to (17) which vanishes just in $t = 0$; see Fig. 2. Moreover, as the Euler–Lagrange equation (17) is translation invariant, all trajectories of the form $q_h(t) = q_0(t-h)$ are solutions as well.

Note that for small times (approximately $t < 1$) we have that $\ddot{q}_0 \neq 0$ and $U(q_0)$ is negative but still not minimal. Then, at later times, the trajectory q_0 reaches the region where U is minimal and gets basically affine ($\ddot{q} \sim 0$). In particular, the integrand of the WIE functional over q_0 changes sign over time and we can (numerically) evaluate the value $\mathsf{W}_\varepsilon[q_0]$ to be negative; see Fig. 3.

As clearly $\mathsf{W}_\varepsilon = 0$ for the trivial solution and $\mathsf{W}_\varepsilon[q_h] = \mathrm{e}^{-h/\varepsilon}\mathsf{W}_\varepsilon[q_0] > \mathsf{W}_\varepsilon[q_0]$, one has that the WIE principle selects exactly the trajectory q_0. Eventually, by taking the limit $\varepsilon \to 0$, the minimizers of the WIE functional can hence be expected to converge to the particular solution $t \mapsto t^4$ of the limiting problem (2).

3 Dissipative Evolutions

A distinctive feature of the present variational approach to Lagrangian mechanics resides in its flexibility in encompassing dissipative situations. Indeed, Theorem 1.1 can be quite straightforwardly extended to handle mixed dissipative/nondissipative situations. Let now $\rho \geq 0$ and the *viscosity coefficient* $\nu \geq 0$ be given and consider the *Weighted Inertia-Dissipation-Energy* (WIDE) functionals

$$\overline{\mathsf{W}}_\varepsilon[\boldsymbol{q}] := \int_0^\infty \mathrm{e}^{-t/\varepsilon}\left(\frac{\varepsilon^2\rho}{2}\left|\ddot{\boldsymbol{q}}(t)\right|^2 + \frac{\varepsilon\nu}{2}\left|\dot{\boldsymbol{q}}(t)\right|^2 + U\big(\boldsymbol{q}(t)\big)\right)\mathrm{d}t \quad (\varepsilon > 0).$$

Let $\boldsymbol{q}_\varepsilon$ be the minimizer of $\overline{\mathsf{W}}_\varepsilon$ on the closed and convex set

$$K_\varepsilon^\rho := \begin{cases} \{\boldsymbol{q} \in \mathrm{H}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m) : \boldsymbol{q}(0) = \boldsymbol{q}^0,\ \dot{\boldsymbol{q}}(0) = \boldsymbol{q}^1\} & \text{if } \rho > 0, \\ \{\boldsymbol{q} \in \mathrm{H}^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m) : \boldsymbol{q}(0) = \boldsymbol{q}^0\} & \text{if } \rho = 0. \end{cases}$$

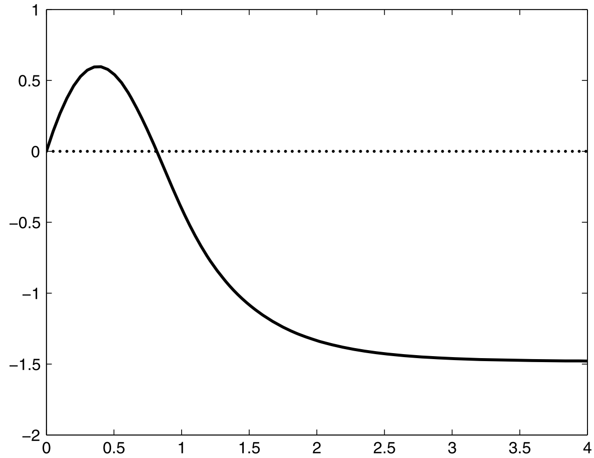

Fig. 3 The function $t \mapsto \int_0^t e^{-s/\varepsilon}(\frac{\varepsilon^2}{2}|\ddot{q}_0(s)|^2 + U(q_0(s)))\,ds$ for $\varepsilon = 1/2$

Then we have the following extension of the principle to mixed dissipative/non-dissipative situations.

Theorem 3.1 (WIDE principle) *Assume $\rho + \nu > 0$ and let $\boldsymbol{q}_\varepsilon$ minimize $\overline{\mathsf{W}}_\varepsilon$ on K_ε^ρ. Then, for some subsequence $\boldsymbol{q}_{\varepsilon_k}$ we have that $\boldsymbol{q}_{\varepsilon_k} \to \boldsymbol{q}$ weakly-$*$ in $\mathrm{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R}^m)$ if $\rho > 0$ and weakly in $\mathrm{H}^1(\mathbb{R}_+;\mathbb{R}^m)$ if $\rho = 0$ (hence, locally uniformly), where*

$$\rho\ddot{\boldsymbol{q}} + \nu\dot{\boldsymbol{q}} + \nabla U(\boldsymbol{q}) = \mathbf{0} \quad \text{in } \mathbb{R}_+, \qquad \boldsymbol{q}(0) = \boldsymbol{q}^0, \qquad \rho\dot{\boldsymbol{q}}(0) = \rho\boldsymbol{q}^1.$$

Note that the very same considerations of Sect. 2.1 can be extended to the present case in order to ensure that such minimizers exist. Let us explicitly mention that the latter result holds more generally for two symmetric and positive-definite mass and viscosity matrices M and N such that $M + N > 0$. In particular, we are in the position of treating systems resulting form the combinations of conservative and dissipative dynamics.

3.1 A Priori Estimate

As for the purely nondissipative case of Theorem 1.1, the convergence proof of Theorem 3.1 follows from an a priori estimate.

Lemma 3.2 (A priori estimate, WIDE principle) *Let $\boldsymbol{q}_\varepsilon$ minimize $\overline{\mathsf{W}}_\varepsilon$ on K_ε^ρ. Then*

$$\rho\big|\dot{\boldsymbol{q}}_\varepsilon(t)\big|^2 + \nu\int_0^t \big|\dot{\boldsymbol{q}}_\varepsilon(s)\big|^2\,ds \le c \quad \forall t > 0. \tag{19}$$

Before proceeding to the proof, let us remark that the two terms in estimate (19) are exactly the ones which are expected in the limit $\varepsilon = 0$. As such, the estimate shows a remarkable optimality with respect to possibly mixed dissipative/nondissipative dynamics. The proof of estimate (19) results by extending the one of Lemma 2.2. In particular, we extend here the argument from (Serra and Tilli 2012) in order to handle dissipative effects.

Proof We shall reconsider the proof of Lemma 2.2: By letting $\boldsymbol{q}_\varepsilon$ be a minimizer of $\overline{\mathsf{W}}_\varepsilon$ on K_ε^ρ we redefine the rescaled quantities

$$\boldsymbol{p}(t) := \boldsymbol{q}_\varepsilon(\varepsilon t), \qquad G_\varepsilon[\boldsymbol{p}] := \int_0^\infty \mathrm{e}^{-t}\left(\frac{\rho}{2}\left|\ddot{\boldsymbol{p}}(t)\right|^2 + \frac{\varepsilon\nu}{2}\left|\dot{\boldsymbol{p}}(t)\right|^2 + \varepsilon^2 U\big(\boldsymbol{p}(t)\big)\right)\mathrm{d}t$$

and, accordingly,

$$H(t) := \int_t^\infty \mathrm{e}^{-s}\left(\frac{\rho}{2}\left|\ddot{\boldsymbol{p}}(s)\right|^2 + \frac{\varepsilon\nu}{2}\left|\dot{\boldsymbol{p}}(s)\right|^2 + \varepsilon^2 U\big(\boldsymbol{p}(s)\big)\right)\mathrm{d}s.$$

By choosing again $\widehat{p}_i(t) := q_i^0 + \arctan(\varepsilon q_i^1 t)$ we have that

$$G_\varepsilon[\boldsymbol{p}] \le G_\varepsilon[\widehat{\boldsymbol{p}}] \le c\int_0^\infty \mathrm{e}^{-t}\big(\varepsilon^6\rho + \varepsilon^3\nu\big)\,\mathrm{d}t + \varepsilon^2\int_0^\infty \mathrm{e}^{-t}U\big(\widehat{\boldsymbol{p}}(t)\big)\,\mathrm{d}t \le c\varepsilon^2.$$

In particular, the bound (6) reads in this case as

$$(\rho + \varepsilon\nu)\int_0^\infty \mathrm{e}^{-s}\left|\dot{\boldsymbol{p}}(s)\right|^2\mathrm{d}s \le c\varepsilon^2 + cG_\varepsilon[\boldsymbol{p}] \le c\varepsilon^2. \tag{20}$$

On the other hand, relation (7) in this dissipative/nondissipative context reads

$$\left(\frac{\rho}{2}\ddot{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}}\right)^{\cdot} = \frac{1}{2}\big(\mathrm{e}^t H(t)\big)^{\cdot} + \rho|\ddot{\boldsymbol{p}}|^2 + \frac{\rho}{2}\dddot{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}} + \varepsilon\nu|\dot{\boldsymbol{p}}|^2. \tag{21}$$

Hence, we can redefine the function E as

$$E(t) := \frac{\rho}{4}\left|\dot{\boldsymbol{p}}(t)\right|^2 - \frac{\rho}{2}\ddot{\boldsymbol{p}}\cdot\dot{\boldsymbol{p}} + \varepsilon\nu\int_0^t\left|\dot{\boldsymbol{p}}(s)\right|^2\mathrm{d}s + \frac{1}{2}\mathrm{e}^t H(t) \quad \forall t \ge 0$$

so that, by taking the time derivative and using relation (21), we again have that

$$\dot{E} = -\rho|\ddot{\boldsymbol{p}}|^2. \tag{22}$$

Moreover, we readily check that (see (9))

$$-\frac{\rho}{4}\big(\mathrm{e}^{-t}\left|\dot{\boldsymbol{p}}(t)\right|^2\big)^{\cdot} + \frac{1}{2}H(t) + \varepsilon\nu\mathrm{e}^{-t}\int_0^t\left|\dot{\boldsymbol{p}}(s)\right|^2\mathrm{d}s = \mathrm{e}^{-t}E(t).$$

Hence, by integrating on (t,T) and using the fact that E is nonincreasing one concludes

$$\begin{aligned}&\frac{\rho}{4}\mathrm{e}^{-t}\left|\dot{\boldsymbol{p}}(t)\right|^2 - \frac{\rho}{4}\mathrm{e}^{-T}\left|\dot{\boldsymbol{p}}(T)\right|^2 + \frac{1}{2}\int_t^T H(s)\,\mathrm{d}s + \varepsilon\nu\int_t^T \mathrm{e}^{-s}\left(\int_0^s\left|\dot{\boldsymbol{p}}(r)\right|^2\mathrm{d}r\right)\mathrm{d}s\\ &\quad= \int_t^T \mathrm{e}^{-s}E(s)\,\mathrm{d}s \le \big(\mathrm{e}^{-t} - \mathrm{e}^{-T}\big)E(t) \le \big(\mathrm{e}^{-t} - \mathrm{e}^{-T}\big)E(0).\end{aligned} \tag{23}$$

Let us now take the limit for $T \to \infty$. By recalling that $\mathrm{e}^{-T}|\dot{\boldsymbol{p}}(T)|^2 \to 0$ we get

$$\frac{\rho}{4}\mathrm{e}^{-t}\left|\dot{\boldsymbol{p}}(t)\right|^2 + \varepsilon\nu\int_t^\infty \mathrm{e}^{-s}\left(\int_0^s \left|\dot{\boldsymbol{p}}(r)\right|^2 \mathrm{d}r\right)\mathrm{d}s \le \mathrm{e}^{-t}E(0).$$

In particular, $t \mapsto \mathrm{e}^{-t}\int_0^t |\dot{\boldsymbol{p}}(s)|^2\,\mathrm{d}s \in \mathrm{L}^1(\mathbb{R}_+)$ and, owing also to bound (20), it is a standard matter to compute

$$\left(\mathrm{e}^{-t}\int_0^t \left|\dot{\boldsymbol{p}}(s)\right|^2 \mathrm{d}s\right)^{\cdot} = -\mathrm{e}^{-t}\int_0^t \left|\dot{\boldsymbol{p}}(s)\right|^2 \mathrm{d}s + \mathrm{e}^{-t}\left|\dot{\boldsymbol{p}}(t)\right|^2$$

and deduce that indeed $t \mapsto \mathrm{e}^{-t}\int_0^t |\dot{\boldsymbol{p}}(s)|^2\,\mathrm{d}s \in \mathrm{W}^{1,1}(\mathbb{R}_+)$. Hence, we also have that $\mathrm{e}^{-t}\int_0^t |\dot{\boldsymbol{p}}(s)|^2\,\mathrm{d}s \to 0$ as $t \to \infty$.

We shall now go back to relation (23), handle the $\varepsilon\nu$-term by

$$\begin{aligned}\varepsilon\nu\int_t^T \mathrm{e}^{-s}\left(\int_0^s \left|\dot{\boldsymbol{p}}(r)\right|^2 \mathrm{d}r\right)\mathrm{d}s = {}& -\varepsilon\nu\mathrm{e}^{-T}\int_0^T \left|\dot{\boldsymbol{p}}(s)\right|^2 \mathrm{d}s + \varepsilon\nu\mathrm{e}^{-t}\int_0^t \left|\dot{\boldsymbol{p}}(s)\right|^2 \mathrm{d}s \\ & + \varepsilon\nu\int_t^T \mathrm{e}^{-s}\left|\dot{\boldsymbol{p}}(s)\right|^2 \mathrm{d}s,\end{aligned}$$

and take the limit $T \to \infty$ in order to get

$$\frac{\rho}{4}\left|\dot{\boldsymbol{p}}(t)\right|^2 + \varepsilon\nu\int_0^t \left|\dot{\boldsymbol{p}}(s)\right|^2 \mathrm{d}s \le E(0).$$

By arguing exactly as in (15) we check that $E(0) \le c\varepsilon^2$. Eventually, estimate (19) follows by time rescaling. □

3.2 Proof of Theorem 3.1

We aim now at passing to the limit in the Euler–Lagrange equation

$$0 = \int_0^\infty \left(\varepsilon^2\rho\left(\mathrm{e}^{-t/\varepsilon}\ddot{\boldsymbol{q}}_\varepsilon(t)\right)^{\cdot\cdot} - \varepsilon\nu\left(\mathrm{e}^{-t/\varepsilon}\dot{\boldsymbol{q}}_\varepsilon(t)\right)^{\cdot} + \mathrm{e}^{-t/\varepsilon}\nabla U\left(\boldsymbol{q}_\varepsilon(t)\right)\right)\cdot\boldsymbol{v}(t)\,\mathrm{d}t \tag{24}$$

for all $\boldsymbol{v} \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}_+;\mathbb{R}^m)$. By compactness we get that $\boldsymbol{q}_\varepsilon \to \boldsymbol{q}$ locally uniformly for some not relabeled subsequence. Fix any $\boldsymbol{w} \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}_+;\mathbb{R}^m)$ and choose $\boldsymbol{v}(t) = \boldsymbol{v}_\varepsilon(t) := \mathrm{e}^{t/\varepsilon}\boldsymbol{w}(t)$ in relation (24) getting

$$\begin{aligned}0 &= \int_0^\infty \mathrm{e}^{-t/\varepsilon}\left(\varepsilon^2\rho\ddot{\boldsymbol{q}}_\varepsilon(t)\cdot\ddot{\boldsymbol{v}}_\varepsilon(t) + \varepsilon\nu\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\dot{\boldsymbol{v}}_\varepsilon(t) + \nabla U\left(\boldsymbol{q}_\varepsilon(t)\right)\cdot\boldsymbol{v}_\varepsilon(t)\right)\mathrm{d}t \\ &= \int_0^\infty \left(\varepsilon^2\rho\ddot{\boldsymbol{q}}_\varepsilon(t)\cdot\ddot{\boldsymbol{w}}(t) + 2\varepsilon\rho\ddot{\boldsymbol{q}}_\varepsilon(t)\cdot\dot{\boldsymbol{w}}(t) + \rho\ddot{\boldsymbol{q}}_\varepsilon(t)\cdot\boldsymbol{w}(t)\right)\mathrm{d}t \\ &\quad + \int_0^\infty \left(\varepsilon\nu\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\dot{\boldsymbol{w}}(t) + \nu\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\boldsymbol{w}(t)\right)\mathrm{d}t + \int_0^\infty \nabla U\left(\boldsymbol{q}_\varepsilon(t)\right)\cdot\boldsymbol{w}(t)\,\mathrm{d}t.\end{aligned}$$

Hence, we have proved that

$$\begin{aligned}
&\int_0^\infty \big(\rho\ddot{\boldsymbol{q}}_\varepsilon(t)+\nu\dot{\boldsymbol{q}}_\varepsilon(t)+\nabla U\big(\boldsymbol{q}_\varepsilon(t)\big)\big)\cdot\boldsymbol{w}(t)\,\mathrm{d}t\\
&\quad=\int_0^\infty \big(\varepsilon^2\rho\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\boldsymbol{w}^{(3)}(t)+2\varepsilon\rho\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\ddot{\boldsymbol{w}}(t)-\varepsilon\nu\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\dot{\boldsymbol{w}}(t)\big)\,\mathrm{d}t\\
&\quad=\int_0^\infty \rho\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\big(\varepsilon^2\boldsymbol{w}^{(3)}(t)+2\varepsilon\ddot{\boldsymbol{w}}(t)\big)\,\mathrm{d}t-\int_0^\infty \nu\dot{\boldsymbol{q}}_\varepsilon(t)\cdot\varepsilon\dot{\boldsymbol{w}}(t)\,\mathrm{d}t.
\end{aligned}$$

Eventually, by using (19) and by passing to the lim sup as $\varepsilon\to 0$ we have that $\boldsymbol{q}$ solves

$$\rho\ddot{\boldsymbol{q}}+\nu\dot{\boldsymbol{q}}+\nabla U(\boldsymbol{q})=\boldsymbol{0}\quad \text{in } \mathbb{R}_+.$$

The check of the initial conditions $\boldsymbol{q}(0)=\boldsymbol{q}^0$ and $\rho\dot{\boldsymbol{q}}(0)=\rho\boldsymbol{q}^1$ is immediate. In case we have that $U\in \mathrm{C}^{1,1}_{\mathrm{loc}}(\mathbb{R}^m)$, the limiting problem has a unique solution and the whole sequence $\boldsymbol{q}_\varepsilon$ converges.

3.3 Gradient Flows

As a corollary of Theorem 3.1, we have checked the $\varepsilon\to 0$ limit also in the *fully dissipative* situation of *gradient flows*, namely $\rho=0$ and $\nu>0$. For the sake of definiteness, we shall record this fact in the following.

Corollary 3.3 (WED principle, gradient flows) *Let $\boldsymbol{q}_\varepsilon$ minimize the functional*

$$\boldsymbol{q}\mapsto\int_0^\infty \mathrm{e}^{-t/\varepsilon}\left(\frac{\varepsilon\nu}{2}|\dot{\boldsymbol{q}}|^2+U\big(\boldsymbol{q}(t)\big)\right)\mathrm{d}t$$

among all trajectories $t\mapsto\boldsymbol{q}(t)\in \mathrm{H}^1(\mathbb{R}_+,\mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t;\mathbb{R}^m)$ such that $\boldsymbol{q}(0)=\boldsymbol{q}^0$. Then, for some subsequence $\boldsymbol{q}_{\varepsilon_k}$ we have that $\boldsymbol{q}_{\varepsilon_k}\to\boldsymbol{q}$ weakly in $\mathrm{H}^1(\mathbb{R}_+;\mathbb{R}^m)$ where $\boldsymbol{q}$ is the unique classical solution of the gradient flow problem

$$\nu\dot{\boldsymbol{q}}+\nabla U(\boldsymbol{q})=\boldsymbol{0}\quad \textit{in } \mathbb{R}_+,\qquad \boldsymbol{q}(0)=\boldsymbol{q}^0.$$

We shall mention that the limit $\varepsilon\to 0$ in the case of gradient flows has already been tackled by a fairly different approach in Rossi et al. (2011a, 2011b). Indeed, in the latter the case of (geodesically) convex (Rossi et al. 2011b) and λ-convex (Rossi et al. 2011a) potentials in metric spaces is discussed by a Pontryagin-type argument. In particular, minimizers of the corresponding *metric* version of the functional are proved to converge, up to subsequences, to so-called *curves of maximal slope*. Let us remark that the above mentioned results do not apply to the present case as the potential is here just $\mathrm{C}^1(\mathbb{R}^m)$. This, in particular, allows us to use the variational principle as a selection criterion in case of nonuniqueness of solution of the gradient flow in the exact same spirit as in Sect. 2.7. Finally, in the case $T<\infty$, we shall directly argue on Euler–Lagrange equation. In particular, convergence will be proved starting from any sequence of stationary points.

4 The WIE Principle on $(0, T)$

Let us now move to the consideration of the finite-time horizon situation. In particular, we shall substitute in time integral on $(0,\infty)$ in the definition of W_ε (and $\overline{\mathsf{W}}_\varepsilon$, later) by an integration on $(0, T)$ for some fixed reference time $T > 0$. Namely, we consider the functionals

$$\mathsf{W}_\varepsilon^T[\boldsymbol{q}] := \int_0^T \mathrm{e}^{-t/\varepsilon}\left(\frac{\varepsilon^2\rho}{2}\left|\ddot{\boldsymbol{q}}(t)\right|^2 + U\big(\boldsymbol{q}(t)\big)\right)\mathrm{d}t \quad (\varepsilon > 0)$$

to be minimized on the convex and closed set

$$K^\rho := \begin{cases} \{\boldsymbol{q} \in \mathrm{H}^2(0,T;\mathbb{R}^m) \,:\, \boldsymbol{q}(0) = \boldsymbol{q}^0,\ \dot{\boldsymbol{q}}(0) = \boldsymbol{q}^1\} & \text{if } \rho > 0, \\ \{\boldsymbol{q} \in \mathrm{H}^1(0,T;\mathbb{R}^m) \,:\, \boldsymbol{q}(0) = \boldsymbol{q}^0\} & \text{if } \rho = 0. \end{cases}$$

This change brings the WIE approach closer to the classical formulation of the Hamilton principle where some suitable final time is prescribed. The aim of this section is that of reproducing, and in place sharpen, the convergence results of the infinite-time horizon frame of Sect. 2. Indeed, also in the finite-horizon case $T < \infty$ the limit as $\varepsilon \to 0$ of minimizers of the W_ε^T functional converge to solutions of the Lagrangian system (Theorem 4.2). Moreover, an explicit convergence rate can be exhibited (Theorem 4.3). The latter quantitative error bound is presently not available in the infinite-horizon case.

Note that the convergence proof of W_ε^T is substantially different from the corresponding one of the infinite-horizon case. In fact, the arguments of Sect. 2 heavily rely on the invariance of the time-integration interval $\mathbb{R}_+$ with respect to linear time rescalings. Additionally, the appearance of the finiteness of the time interval of integration entails the arising of two final boundary conditions at time T (see (28) below). These final boundary conditions are clearly bound to disappear in the limit $\varepsilon \to 0$. Still, they require specific attention for all $\varepsilon > 0$, exactly in the spirit of Sect. 2.6.

4.1 Well-Posedness of the Minimum Problem

From here on, we shall assume that $U \in \mathrm{C}^{1,1}_{\mathrm{loc}}(\mathbb{R}^m)$. Let us start by checking that indeed minimizers of W_ε^T on K_T exist. In the present finite-time situation the result is even stronger with respect to Lemma 2.1. Indeed, by requiring further that $U \in \mathrm{C}^{1,1}(\mathbb{R}^m)$, the functionals W_ε^T turn out to be uniformly convex (for small ε). In particular, the minimum problem is well-posed and minimizers are unique.

Lemma 4.1 (Direct method, $T < \infty$) *The functional W_ε^T admits a minimizer in K^ρ. Moreover, if $U \in \mathrm{C}^{1,1}(\mathbb{R}^m)$ and ε is small enough, the functional W_ε^T is uniformly convex in K^ρ so that the minimizer of W_ε^T on K^ρ is unique.*

Proof The existence part of the statement follows exactly as in Lemma 2.1. Let us check for uniform convexity. Recall that $U \in \mathrm{C}^{1,1}(\mathbb{R}^m)$ implies that there exists $\lambda > 0$ such that $\boldsymbol{p} \mapsto U(\boldsymbol{q}) + (\lambda/2)|\boldsymbol{q}|^2$ is convex. Given $\boldsymbol{q} \in K^\rho$, consider the function

$\boldsymbol{p}(t) := \mathrm{e}^{-t/(2\varepsilon)}\boldsymbol{q}(t)$. We rewrite $\mathsf{W}_\varepsilon^T[\boldsymbol{q}]$ via $\boldsymbol{p}$ as

$$\begin{aligned}
\mathsf{W}_\varepsilon^T[\boldsymbol{q}] &= \int_0^T \left(\frac{\varepsilon^2\rho}{2}\big|\ddot{\boldsymbol{p}}(t)\big|^2 + \frac{\rho}{2}\big|\dot{\boldsymbol{p}}(t)\big|^2 + \frac{\rho - 16\varepsilon^2\lambda}{32\varepsilon^2}\big|\boldsymbol{p}(t)\big|^2\right)\mathrm{d}t \\
&\quad + \int_0^T \biggl(\varepsilon\rho\ddot{\boldsymbol{p}}(t)\cdot\dot{\boldsymbol{p}}(t) + \frac{\rho}{4}\ddot{\boldsymbol{p}}(t)\cdot\boldsymbol{p}(t) + \frac{\rho}{4\varepsilon}\dot{\boldsymbol{p}}(t)\cdot\boldsymbol{p}(t) \\
&\quad + \mathrm{e}^{-t/\varepsilon}\left(U\big(\boldsymbol{q}(t)\big) + \frac{\lambda}{2}\big|\boldsymbol{q}(t)\big|^2\right)\biggr)\mathrm{d}t \\
&= \int_0^T \left(\frac{\varepsilon^2\rho}{2}\big|\ddot{\boldsymbol{p}}(t)\big|^2 + \frac{\rho}{4}\big|\dot{\boldsymbol{p}}(t)\big|^2 + \frac{\rho - 16\varepsilon^2\lambda}{32\varepsilon^2}\rho\big|\boldsymbol{p}(t)\big|^2\right)\mathrm{d}t \\
&\quad + \rho\biggl(\varepsilon\big|\dot{\boldsymbol{p}}(T)\big|^2 - \varepsilon\big|\dot{\boldsymbol{p}}(0)\big|^2 + \frac{1}{4}\dot{\boldsymbol{p}}(T)\cdot\boldsymbol{p}(T) - \frac{1}{4}\dot{\boldsymbol{p}}(0)\cdot\boldsymbol{p}(0) + \frac{1}{2\varepsilon}\big|\boldsymbol{p}(T)\big|^2 \\
&\quad - \frac{1}{2\varepsilon}\big|\boldsymbol{p}(0)\big|^2\biggr) + \int_0^T \mathrm{e}^{-t/\varepsilon}\left(U\big(\boldsymbol{q}(t)\big) + \frac{\lambda}{2}\big|\boldsymbol{q}(t)\big|^2\right)\mathrm{d}t \\
&=: A_\varepsilon[\boldsymbol{p}] + B_\varepsilon[\boldsymbol{p}] + C_\varepsilon[\boldsymbol{q}].
\end{aligned}$$

Here, A_ε is quadratic and uniformly convex (of constant $\alpha_\varepsilon > 0$, say) with respect to $\boldsymbol{p}$ in $\mathrm{H}^2(0, T; \mathbb{R}^m)$ for all $\varepsilon < (\rho/(16\lambda))^{1/2}$ and C_ε is clearly convex with respect to $\boldsymbol{q}$. The same holds also for the functional B_ε for it is quadratic in $\boldsymbol{p}(T)$ and $\dot{\boldsymbol{p}}(T)$. Let now $\theta \in [0, 1]$, $\boldsymbol{q}_0, \boldsymbol{q}_1 \in K^\rho$, and define accordingly $\boldsymbol{p}_0, \boldsymbol{p}_1$ as above. We have that

$$\begin{aligned}
\mathsf{W}_\varepsilon^T\big[(1-\theta)\boldsymbol{q}_0 + \theta\boldsymbol{q}_1\big] &= A_\varepsilon\big[(1-\theta)\boldsymbol{p}_0 + \theta\boldsymbol{p}_1\big] + B_\varepsilon\big[(1-\theta)\boldsymbol{p}_0 + \theta\boldsymbol{p}_1\big] \\
&\quad + C_\varepsilon\big[(1-\theta)\boldsymbol{q}_0 + \theta\boldsymbol{q}_1\big] \\
&\le -\frac{\alpha_\varepsilon}{2}\theta(1-\theta)\|\boldsymbol{p}_0 - \boldsymbol{p}_1\|_{\mathrm{H}^2}^2 + (1-\theta)\mathsf{W}_\varepsilon^T[\boldsymbol{q}_0] + \theta\mathsf{W}_\varepsilon^T[\boldsymbol{q}_1]
\end{aligned}$$

and the assertion follows as $\|\boldsymbol{p}_0 - \boldsymbol{p}_1\|_{\mathrm{H}^2}^2 \ge \varepsilon^4\mathrm{e}^{-T/\varepsilon}\|\boldsymbol{q}_0 - \boldsymbol{q}_1\|_{\mathrm{H}^2}^2$. □

4.2 Convergence of Stationary Points

We shall specify here some growth conditions on ∇U. Namely, besides $0 \le U \in \mathrm{C}^{1,1}_{\mathrm{loc}}(\mathbb{R}^m)$, we assume that

$$\forall\delta > 0\ \exists c_\delta \ge 0\ \forall\boldsymbol{q} \in \mathbb{R}^m: \quad \big|\nabla U(\boldsymbol{q})\big| \le \delta\big(U(\boldsymbol{q}) + |\boldsymbol{q}|^2\big) + c_\delta. \tag{25}$$

This follows for instance for U being the sum of a homogeneous and a subcubic potential. Let us specify the Euler–Lagrange equation for the minimizers $\boldsymbol{q}_\varepsilon$ of W_ε^T on K^ρ. In particular, one has that

$$\begin{aligned}
0 &= \rho\mathrm{e}^{-T/\varepsilon}\ddot{\boldsymbol{q}}_\varepsilon(T)\cdot\dot{\boldsymbol{v}}(T) - \rho\big(\mathrm{e}^{-t/\varepsilon}\ddot{\boldsymbol{q}}_\varepsilon\big)^{\boldsymbol{\cdot}}(T)\cdot\boldsymbol{v}(T) \\
&\quad + \int_0^T \left(\rho\big(\mathrm{e}^{-t/\varepsilon}\ddot{\boldsymbol{q}}_\varepsilon(t)\big)^{\boldsymbol{\cdot\cdot}} + \frac{1}{\varepsilon^2}\mathrm{e}^{-t/\varepsilon}\nabla U\big(\boldsymbol{q}_\varepsilon(t)\big)\right)\cdot\boldsymbol{v}(t)\,\mathrm{d}t
\end{aligned}$$

for all $\boldsymbol{v} \in \mathrm{C}^\infty(0,T;\mathbb{R}^m)$ with $\boldsymbol{v}(0) = \dot{\boldsymbol{v}}(0) = 0$, and hence

$$\varepsilon^2 \rho \boldsymbol{q}^{(4)} - 2\varepsilon\rho \boldsymbol{q}^{(3)} + \rho \ddot{\boldsymbol{q}} + \nabla U(\boldsymbol{q}) = \boldsymbol{0} \quad \text{in } (0,T), \tag{26}$$

$$\boldsymbol{q}(0) = \boldsymbol{q}^0, \quad \rho\dot{\boldsymbol{q}}(0) = \rho \boldsymbol{q}^1, \tag{27}$$

$$\rho \ddot{\boldsymbol{q}}(T) = \rho \boldsymbol{q}^{(3)}(T) = \boldsymbol{0}. \tag{28}$$

Note the occurrence of the two extra final boundary conditions (28) at time T. These conditions will disappear in the limit $\varepsilon \to 0$, see (29).

The main result of this section is the following.

Theorem 4.2 (WIE principle, $T < \infty$) *Let* $\boldsymbol{q}_\varepsilon$ *solve the Euler–Lagrange equation* (26)–(28). *Then,* $\boldsymbol{q}_\varepsilon \to \boldsymbol{q}$ *weakly in* $\mathrm{H}^1(0,T;\mathbb{R}^m)$ *where* $\boldsymbol{q}$ *solves the Lagrangian system*

$$\rho \ddot{\boldsymbol{q}} + \nabla U(\boldsymbol{q}) = \boldsymbol{0} \quad \textit{in } (0,T), \qquad \boldsymbol{q}(0) = \boldsymbol{q}^0, \qquad \rho \dot{\boldsymbol{q}}(0) = \rho \boldsymbol{q}^1. \tag{29}$$

Proof One has to start by establishing uniform estimates on $\boldsymbol{q}_\varepsilon$ in the spirit of Lemma 2.2, although necessarily by a different technique. We follow here the argument of Stefanelli (2011) and perform some modifications in order to cope with the possible nonconvexity of U (the original argument from Stefanelli (2011) works for convex potentials only). Take the scalar product of (26) and $\dot{\boldsymbol{q}}_\varepsilon - \boldsymbol{q}^1$ and integrate on $(0,t)$ getting

$$\begin{aligned}
0 &= \varepsilon^2 \rho \boldsymbol{q}_\varepsilon^{(3)}(t)\cdot\big(\dot{\boldsymbol{q}}_\varepsilon(t) - \boldsymbol{q}^1\big) - \frac{\varepsilon^2\rho}{2}\big|\ddot{\boldsymbol{q}}_\varepsilon(t)\big|^2 + \frac{\varepsilon^2\rho}{2}\big|\ddot{\boldsymbol{q}}_\varepsilon(0)\big|^2 - 2\varepsilon\rho\ddot{\boldsymbol{q}}_\varepsilon(t)\cdot\big(\dot{\boldsymbol{q}}_\varepsilon(t) - \boldsymbol{q}^1\big) \\
&\quad + 2\varepsilon\rho\int_0^t \big|\ddot{\boldsymbol{q}}_\varepsilon(s)\big|^2\,\mathrm{d}s + \frac{\rho}{2}\big|\dot{\boldsymbol{q}}_\varepsilon(t) - \boldsymbol{q}^1\big|^2 + U\big(\boldsymbol{q}_\varepsilon(t)\big) - U\big(\boldsymbol{q}^0\big) \\
&\quad + \int_0^t \nabla U\big(\boldsymbol{q}_\varepsilon(s)\big)\cdot\boldsymbol{q}^1\,\mathrm{d}s.
\end{aligned} \tag{30}$$

Now, we integrate (30) on $(0,T)$ and use the final boundary conditions (28) in order to get that

$$\begin{aligned}
0 &= \frac{\rho}{2}\int_0^T \big|\dot{\boldsymbol{q}}_\varepsilon(t) - \boldsymbol{q}^1\big|^2\,\mathrm{d}t + \int_0^T U\big(\boldsymbol{q}_\varepsilon(t)\big)\,\mathrm{d}t \\
&\quad - TU\big(\boldsymbol{q}^0\big) + \int_0^T\int_0^t \nabla U\big(\boldsymbol{q}_\varepsilon(s)\big)\cdot\boldsymbol{q}^1\,\mathrm{d}s\,\mathrm{d}t \\
&\quad - \frac{3\varepsilon^2\rho}{2}\int_0^T \big|\ddot{\boldsymbol{q}}_\varepsilon(t)\big|^2\,\mathrm{d}t + \frac{\varepsilon^2 T\rho}{2}\big|\ddot{\boldsymbol{q}}_\varepsilon(0)\big|^2 - \varepsilon\rho\big|\dot{\boldsymbol{q}}_\varepsilon(T) - \boldsymbol{q}^1\big|^2 \\
&\quad + 2\varepsilon\rho\int_0^T\int_0^t \big|\ddot{\boldsymbol{q}}_\varepsilon(s)\big|^2\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned} \tag{31}$$

Finally, add (31) to (30) with $t = T$ and use again the boundary conditions (28) getting

$$\left(2\varepsilon - \frac{3\varepsilon^2}{2}\right)\int_0^T \rho\left|\ddot{\boldsymbol{q}}_\varepsilon(t)\right|^2 \mathrm{d}t + \frac{\varepsilon^2(1+T)}{2}\rho\left|\ddot{\boldsymbol{q}}_\varepsilon(0)\right|^2 + \left(\frac{1}{2} - \varepsilon\right)\rho\left|\dot{\boldsymbol{q}}_\varepsilon(T) - \boldsymbol{q}^1\right|^2$$
$$+ 2\varepsilon\rho\int_0^T\int_0^t \left|\ddot{\boldsymbol{q}}_\varepsilon(s)\right|^2 \mathrm{d}s\,\mathrm{d}t + \frac{\rho}{2}\int_0^T \left|\dot{\boldsymbol{q}}_\varepsilon(t) - \boldsymbol{q}^1\right|^2 \mathrm{d}t + U\big(\boldsymbol{q}_\varepsilon(T)\big)$$
$$+ \int_0^T U\big(\boldsymbol{q}_\varepsilon(t)\big)\,\mathrm{d}t \le c(T) + \int_0^T \nabla U\big(\boldsymbol{q}_\varepsilon(t)\big)\cdot\boldsymbol{q}^1\,\mathrm{d}t$$
$$+ \int_0^T\int_0^t \nabla U\big(\boldsymbol{q}_\varepsilon(s)\big)\cdot\boldsymbol{q}^1\,\mathrm{d}s\,\mathrm{d}t. \tag{32}$$

The last two terms in the above right-hand side can be controlled by means of relation (25) so that we have

$$\rho\|\dot{\boldsymbol{q}}_\varepsilon\|^2_{\mathrm{L}^2} \le c(T). \tag{33}$$

Hence, by possibly passing to not relabeled subsequences, we have that $\boldsymbol{q}_\varepsilon \to \boldsymbol{q}$ uniformly. Eventually, we check that $\boldsymbol{q}$ indeed classically solves the Lagrangian system (26) by arguing along the lines of Sect. 3.2. In particular, the whole sequence converges. □

4.3 Quantitative Error Bound

As already mentioned, in the finite-time case $T < \infty$ the convergence result of Theorem 4.2 can be refined in order to yield a quantitative rate estimate.

Theorem 4.3 (Error control, $T < \infty$) *Under the assumptions of Theorem 4.2 we have that* $\rho\|\boldsymbol{q} - \boldsymbol{q}_\varepsilon\|_{\mathrm{H}^{1+\eta}} \le c(T)\varepsilon^{(1-\eta)/2}$ *for all* $\eta \in [0, 1)$.

Proof The argument relies on establishing an extra estimate. From bound (33) and the local Lipschitz continuity of ∇U, we have that $\varepsilon^2\rho\boldsymbol{q}^{(4)}_\varepsilon - 2\varepsilon\rho\boldsymbol{q}^{(3)}_\varepsilon + \rho\ddot{\boldsymbol{q}}_\varepsilon$ is uniformly bounded in $\mathrm{L}^2(0, T; \mathbb{R}^m)$, depending on T. Hence, by integrating its squared norm we have that

$$\varepsilon^4\int_0^T \rho\left|\boldsymbol{q}^{(4)}_\varepsilon(t)\right|^2 \mathrm{d}t + 4\varepsilon^2\int_0^T \rho\left|\boldsymbol{q}^{(3)}_\varepsilon(t)\right|^2 \mathrm{d}t + \int_0^T \rho\left|\ddot{\boldsymbol{q}}_\varepsilon(t)\right|^2 \mathrm{d}t$$
$$\le c(T) + 2\varepsilon^3\int_0^T \rho\boldsymbol{q}^{(4)}_\varepsilon(t)\cdot\boldsymbol{q}^{(3)}_\varepsilon(t)\,\mathrm{d}t + 2\varepsilon\int_0^T \rho\boldsymbol{q}^{(3)}_\varepsilon(t)\cdot\ddot{\boldsymbol{q}}_\varepsilon(t)\,\mathrm{d}t$$
$$- \varepsilon^2\int_0^T \rho\boldsymbol{q}^{(4)}_\varepsilon(t)\cdot\ddot{\boldsymbol{q}}_\varepsilon(t)\,\mathrm{d}t$$
$$\overset{(28)}{=} c(T) - \varepsilon^3\rho\left|\boldsymbol{q}^{(3)}_\varepsilon(0)\right|^2 - \varepsilon\rho\left|\ddot{\boldsymbol{q}}_\varepsilon(0)\right|^2 + \varepsilon^2\rho\boldsymbol{q}^{(3)}_\varepsilon(0)\cdot\ddot{\boldsymbol{q}}_\varepsilon(0)$$
$$+ 2\varepsilon^2\int_0^T \rho\left|\boldsymbol{q}^{(3)}_\varepsilon(t)\right|^2 \mathrm{d}t.$$

This entails that $\varepsilon^2\rho^{1/2}\boldsymbol{q}_\varepsilon^{(4)}$, $\varepsilon\rho^{1/2}\boldsymbol{q}_\varepsilon^{(3)}$, and $\rho^{1/2}\ddot{\boldsymbol{q}}_\varepsilon$ are bounded in $\mathrm{L}^2(0,T;\mathbb{R}^m)$. Moreover, the Gagliardo–Nirenberg inequality ensures that

$$\begin{aligned}\rho^{1/2}\big\|\boldsymbol{q}_\varepsilon^{(3)}\big\|_{\mathrm{L}^\infty} &\le c(T)\big(\rho^{1/2}\big\|\boldsymbol{q}_\varepsilon^{(3)}\big\|_{\mathrm{L}^2}+\rho^{1/2}\big\|\boldsymbol{q}_\varepsilon^{(3)}\big\|_{\mathrm{L}^2}^{1/2}\big\|\boldsymbol{q}_\varepsilon^{(4)}\big\|_{\mathrm{L}^2}^{1/2}\big)\\ &\le c(T)\left(\frac{1}{\varepsilon}+\frac{1}{\varepsilon^{3/2}}\right),\\ \rho^{1/2}\|\ddot{\boldsymbol{q}}_\varepsilon\|_{\mathrm{L}^\infty} &\le c(T)\left(1+\frac{1}{\varepsilon}\right).\end{aligned} \tag{34}$$

Take now the difference between (29) and (26), test it on $\dot{\boldsymbol{p}}_\varepsilon := \dot{\boldsymbol{q}}-\dot{\boldsymbol{q}}_\varepsilon$, and integrate on $(0,t)$ getting

$$\begin{aligned}\frac{\rho}{2}\big|\dot{\boldsymbol{p}}_\varepsilon(t)\big|^2 &= -\varepsilon^2\int_0^t \rho\boldsymbol{q}_\varepsilon^{(4)}(s)\cdot\dot{\boldsymbol{p}}_\varepsilon(s)\,\mathrm{d}s+2\varepsilon\int_0^t\rho\boldsymbol{q}_\varepsilon^{(3)}(s)\cdot\dot{\boldsymbol{p}}_\varepsilon(s)\,\mathrm{d}s\\ &\quad-\int_0^t\big(\nabla U\big(\boldsymbol{q}(s)\big)-\nabla U\big(\boldsymbol{q}_\varepsilon(s)\big)\big)\cdot\dot{\boldsymbol{p}}_\varepsilon(s)\,\mathrm{d}s\\ &\le -\varepsilon^2\rho\boldsymbol{q}_\varepsilon^{(3)}(t)\cdot\dot{\boldsymbol{p}}_\varepsilon(t)+\varepsilon^2\int_0^t\rho\boldsymbol{q}_\varepsilon^{(3)}(s)\cdot\ddot{\boldsymbol{p}}_\varepsilon(s)\,\mathrm{d}s+2\varepsilon\rho\ddot{\boldsymbol{q}}_\varepsilon(t)\cdot\dot{\boldsymbol{p}}_\varepsilon(t)\\ &\quad-2\varepsilon\int_0^t\rho\ddot{\boldsymbol{q}}_\varepsilon(s)\cdot\ddot{\boldsymbol{p}}_\varepsilon(s)\,\mathrm{d}s+c\int_0^t\rho\big|\boldsymbol{p}_\varepsilon(s)\big|\big|\dot{\boldsymbol{p}}_\varepsilon(s)\big|\,\mathrm{d}s\\ &\overset{(34)}{\le} c(T)\varepsilon+\frac{\rho}{4}\big|\dot{\boldsymbol{p}}_\varepsilon(t)\big|^2+c(T)\int_0^t\rho\big|\dot{\boldsymbol{p}}_\varepsilon(s)\big|^2\,\mathrm{d}s\end{aligned}$$

so that by means of the Gronwall lemma we get that $\rho\|\dot{\boldsymbol{q}}-\dot{\boldsymbol{q}}_\varepsilon\|_{\mathrm{L}^\infty}^2 \le c(T)\varepsilon$. By interpolation (Bergh and Löfström 1976), for all $\eta\in(0,1)$ we have

$$\rho\|\boldsymbol{q}-\boldsymbol{q}_\varepsilon\|_{(\mathrm{W}^{1,\infty},\mathrm{H}^2)_{\eta,1}} \le c(T)\|\dot{\boldsymbol{q}}-\dot{\boldsymbol{q}}_\varepsilon\|_{\mathrm{L}^\infty}^{1-\eta}\|\boldsymbol{q}-\boldsymbol{q}_\varepsilon\|_{\mathrm{H}^2}^{\eta} \le c(T)\varepsilon^{(1-\eta)/2}$$

(which is stronger than the statement). Eventually, we conclude by noting that

$$\big(\mathrm{W}^{1,\infty},\mathrm{H}^2\big)_{\eta,1}\subset\big(\mathrm{W}^{1,\infty},\mathrm{H}^2\big)_{\eta,2}\subset\big(\mathrm{H}^1,\mathrm{H}^2\big)_{\eta,2}=\mathrm{H}^{1+\eta}$$

with continuous injections. □

4.4 Dissipative Evolutions

Also in the finite-time case, the convergence result of Theorem 4.2 can be extended to mixed dissipative/nondissipative situations. In particular, by letting $\rho,\nu\ge 0$ one considers the minimization of the WIDE functionals

$$\overline{\mathsf{W}}_\varepsilon^T[\boldsymbol{q}] := \int_0^T \mathrm{e}^{-t/\varepsilon}\left(\frac{\varepsilon^2\rho}{2}\big|\ddot{\boldsymbol{q}}(t)\big|^2+\frac{\varepsilon\nu}{2}\big|\dot{\boldsymbol{q}}(t)\big|^2+U\big(\boldsymbol{q}(t)\big)\right)\mathrm{d}t \quad (\varepsilon>0)$$

over the convex set K^ρ. By assuming $\rho+\nu>0$ and letting ε be small enough the same results of Lemma 4.1 hold. In particular, for $U\in\mathrm{C}^{1,1}(\mathbb{R}^m)$ the functional $\overline{\mathsf{W}}_\varepsilon^T$

is uniformly convex hence admitting a unique minimizer on K^ρ. Moreover, we have the following.

Theorem 4.4 (WIDE principle, $T < \infty$) *Let* $\rho + \nu > 0$, $\boldsymbol{q}_\varepsilon$ *minimize* $\overline{\mathsf{W}}_\varepsilon^T$ *in* K^ρ, *and* (25) *hold. Then*, $\boldsymbol{q}_\varepsilon \to \boldsymbol{q}$ *weakly-*$*$ *in* $\mathrm{W}^{1,\infty}(0,T;\mathbb{R}^m)$ *if* $\rho > 0$ *and weakly in* $\mathrm{H}^1(0,T;\mathbb{R}^m)$ *if* $\rho = 0$ (*hence*, *locally uniformly*), *where*

$$\rho\ddot{\boldsymbol{q}} + \nu\dot{\boldsymbol{q}} + \nabla U(\boldsymbol{q}) = \boldsymbol{0} \quad \textit{in } (0,T), \qquad \boldsymbol{q}(0) = \boldsymbol{q}^0, \qquad \rho\dot{\boldsymbol{q}}(0) = \rho\boldsymbol{q}^1.$$

We shall not present here the detailed proof of the latter as it can be obtained along the very same lines (and some additional technicalities) of the proof of Theorem 3.1. Some detail in this direction is however provided in the forthcoming (Liero and Stefanelli 2012) where some infinite-dimensional PDE situation is discussed. The conclusions of Theorem 4.3 hold unchanged as long as $\rho > 0$ and the proof is indeed an extension of the proposed one. For $\rho = 0$, one resorts in the (necessarily weaker) quantitative convergence result $\nu\|\boldsymbol{q} - \boldsymbol{q}_\varepsilon\|_{\mathrm{H}^\eta} \le c(T)\varepsilon^{(1-\eta)/2}$ for $\eta \in [0,1)$.

5 Γ-Convergence

The present variational formalism is well-suited in order to describe limiting behaviors. In particular, starting from the mixed dissipative/nondissipative situation of Sect. 3, we shall here comment on the possibility of considering from a variational viewpoint the limits $\rho \to 0$ and $\nu \to 0$. This will be done within the classical frame of Γ-convergence (Dal Maso 1993; De Giorgi and Franzoni 1979). Additionally, we will prove that, under suitable specifications, the finite-horizon problem Γ-converges to the infinite-horizon problem as $T \to \infty$.

Let us mention that all the Γ-limits are taken for constant ε as combined Γ-convergence analyses for both parameters and $\varepsilon \to 0$ are currently not available. The reader is, however, referred to Mielke and Ortiz (2008) and Akagi and Stefanelli (2011), Mielke and Stefanelli (2008) for some Γ-convergence result on WED functionals in the doubly nonlinear parabolic setting.

5.1 Viscous Γ-Limit $\rho \to 0$

We start by defining the functionals F^ρ over the common space $\mathrm{H}^1(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t;\mathbb{R}^m)$ for $\rho \ge 0$, $\nu > 0$ as

$$F^\rho[\boldsymbol{q}] = \begin{cases} \int_0^\infty \mathrm{e}^{-t/\varepsilon}\left(\frac{\varepsilon^2\rho}{2}|\ddot{\boldsymbol{q}}(t)|^2 + \frac{\varepsilon\nu}{2}|\dot{\boldsymbol{q}}(t)|^2 + U(\boldsymbol{q}(t))\right)\mathrm{d}t & \text{if } \boldsymbol{q} \in K_\varepsilon^\rho, \\ \infty & \text{else.} \end{cases}$$

Our result reads as follows.

Lemma 5.1 (Γ-limit $\rho \to 0$) *We have that* $F^\rho \xrightarrow{\Gamma} F^0$ *weakly in* $\mathrm{L}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t;\mathbb{R}^m)$.

Proof Given $\boldsymbol{q} \in K_\varepsilon^0$ one can use singular perturbations in order to find a sequence $\boldsymbol{q}^\rho \in K_\varepsilon^\rho$ with $\boldsymbol{q}^\rho \to \boldsymbol{q}$ strongly in K_ε^0 such that $\rho \int_0^\infty \mathrm{e}^{-t/\varepsilon} |\ddot{\boldsymbol{q}}^\rho(t)|^2 \,\mathrm{d}t \to 0$. On the other hand, let $\boldsymbol{q}^\rho \to \boldsymbol{q}$ weakly in $\mathrm{L}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m)$. As $F^0 \le F^\rho$, we readily check that $F^0[\boldsymbol{q}] \le \liminf_{\rho\to 0} F^0[\boldsymbol{q}^\rho] \le \liminf_{\rho \to 0} F^\rho[\boldsymbol{q}^\rho]$. □

Let us now check that the latter Γ-convergence result is sufficient in order to prove that, as $\rho \to 0$, (subsequences of) minimizers converge to a minimizer. To this aim, we just need to check for the precompactness of the minimizers of F^ρ with respect to the weak $\mathrm{L}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m)$ topology. Although for minimizers, this follows from estimate (19), we could also argue directly by

$$\frac{\rho}{2}\int_0^\infty \mathrm{e}^{-t/\varepsilon}\big|\ddot{\boldsymbol{q}}^\rho\big|^2\,\mathrm{d}t \le F^\rho\big(\boldsymbol{q}^\rho\big) \le F^\rho(\widehat{\boldsymbol{q}}) < \infty$$

where $\widehat{q}_i(t) = q_i^0 + \arctan(q_i^1 t)$. Hence, again by (5), the required precompactness follows.

5.2 Nondissipative Γ-Limit $\nu \to 0$

In order to formalize our Γ-convergence result, we introduce the functionals G^ν for $\rho > 0$ and $\nu \ge 0$ as

$$G^\nu[\boldsymbol{q}] = \begin{cases} \int_0^\infty \mathrm{e}^{-t/\varepsilon}\big(\frac{\varepsilon^2\rho}{2}|\ddot{\boldsymbol{q}}(t)|^2 + \frac{\varepsilon\nu}{2}|\dot{\boldsymbol{q}}(t)|^2 + U(\boldsymbol{q}(t))\big)\,\mathrm{d}t & \text{if } \boldsymbol{q} \in K_\varepsilon, \\ \infty & \text{else.} \end{cases}$$

We have the following.

Lemma 5.2 (Γ-limit $\nu \to 0$) *We have that* $G^\nu \xrightarrow{\Gamma} G^0$ *weakly in* $\mathrm{L}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m)$.

Proof The existence of a recovery sequence is immediate by pointwise convergence since $G^\nu[\boldsymbol{q}] \to G^0[\boldsymbol{q}]$ for all $\boldsymbol{q} \in K_\varepsilon$. Moreover, we have that $G^0 \le G^\nu$ and we readily check that $G^0[\boldsymbol{q}] \le \liminf_{\nu\to 0} G^0[\boldsymbol{q}^\nu] \le \liminf_{\nu\to 0} G^\nu[\boldsymbol{q}^\nu]$ and the assertion follows. □

Arguing exactly as in Sect. 5.1 we can check that the minimizers of G^ν are weakly precompact in $\mathrm{L}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m)$. In particular, the latter Γ-convergence result entails the convergence of (subsequences of) minimizers of G^ν to a minimizer of G^0.

5.3 Infinite-Horizon Γ-Limit $T \to \infty$

We shall be considering all functionals to be defined on the common space $\mathrm{H}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m)$ and specify, for all $t \in (0, \infty]$,

$$F^t[\boldsymbol{q}] := \mathsf{W}_\varepsilon^t[\boldsymbol{q}] \quad \text{if } \boldsymbol{q} \in K_\varepsilon^\rho \text{ and } \boldsymbol{q} \text{ is affine on } [T, \infty) \quad \text{and} \quad F^t[\boldsymbol{q}] = \infty \quad \text{else.}$$

Hence, our result reads as follows.

Lemma 5.3 (Γ-limit $T \to \infty$) *Assume that U is quadratically bounded. Then we have that $F^T \xrightarrow{\Gamma} F^\infty$ weakly in* $\mathrm{L}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m)$.

Proof For all $\boldsymbol{q} \in K_\varepsilon^\rho$ define $\boldsymbol{q}^T = \boldsymbol{q}$ on $[0, T]$ and $\boldsymbol{q}^T$ affine on $[T, \infty)$. Then, it is easy to check that $F^T[\boldsymbol{q}^T] \to F^\infty[\boldsymbol{q}]$. Assume now to be given $\boldsymbol{q}^T \to \boldsymbol{q}^\infty$ weakly in $\mathrm{L}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m)$. By taking with no loss of generality $\liminf_{T\to\infty} F^T[\boldsymbol{q}^T] < \infty$, we have that

$$\liminf_{T\to\infty} \int_0^T \mathrm{e}^{-t/\varepsilon} \big|\ddot{\boldsymbol{q}}^T(t)\big|^2 \,\mathrm{d}t \ge \int_0^\infty \mathrm{e}^{-t/\varepsilon} \big|\ddot{\boldsymbol{q}}^\infty(t)\big|^2 \,\mathrm{d}t$$

and $\boldsymbol{q}^T \to \boldsymbol{q}^\infty$ pointwise almost everywhere. Eventually, $F^T[\boldsymbol{q}^T] \to F^\infty[\boldsymbol{q}^\infty]$ by dominated convergence as $U(\boldsymbol{q}^T) \le c(1 + |\boldsymbol{q}^T|^2)$. □

Let now $\tilde{\boldsymbol{q}}(t) = \boldsymbol{q}^0 + t\boldsymbol{q}^1$. Then all minimizers $\boldsymbol{q}^T$ of F^T fulfill

$$\frac{\rho}{2} \int_0^\infty e^{-t/\varepsilon} \big|\ddot{\boldsymbol{q}}^T\big|^2 \,\mathrm{d}t = \frac{\rho}{2} \int_0^T e^{-t/\varepsilon} \big|\ddot{\boldsymbol{q}}^T\big|^2 \,\mathrm{d}t \le F^T\big[\boldsymbol{q}^T\big] \le F^T[\tilde{\boldsymbol{q}}] < \infty$$

independently of T. In particular, $\boldsymbol{q}^T$ are weakly precompact in $\mathrm{L}^2(\mathbb{R}_+, \mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t; \mathbb{R}^m)$. Hence, by Lemma 5.3, it converges up to subsequences to a minimizer of F^∞.

6 Time Discretization

We collect in this section some remark on suitable time-discrete versions of the WIE principle. Let us focus first on the finite-time case of Sect. 4. Setting the time step $\tau := T/n$ ($n \in \mathbb{N}$), we consider the time-discrete functionals

$$\mathsf{W}_{\varepsilon\tau}[\boldsymbol{q}_0, \dots, \boldsymbol{q}_n] = \sum_{j=2}^{m} \tau \mathrm{e}_{\varepsilon\tau,j} \frac{\varepsilon^2 \rho}{2} \big|\delta^2 \boldsymbol{q}_j\big|^2 + \sum_{j=2}^{n-1} \tau \mathrm{e}_{\varepsilon\tau,j+2} U(\boldsymbol{q}_j).$$

Given $(\boldsymbol{q}_0, \dots, \boldsymbol{q}_n)$, in the latter we have indicated with $\delta\boldsymbol{q}$ its *discrete derivative*, namely $\delta\boldsymbol{q}_j := (\boldsymbol{q}_j - \boldsymbol{q}_{j-1})/\tau$, $\delta^2\boldsymbol{q} = \delta(\delta\boldsymbol{q})$, and so on. The weights $\mathrm{e}_{\varepsilon\tau,j}$ are given by $\mathrm{e}_{\varepsilon\tau,j} := (\varepsilon/(\varepsilon+\tau))^j$ and play the role of the decaying weight $t \mapsto \mathrm{e}^{-t/\varepsilon}$ in the discrete setting. In particular, note that $\mathrm{e}_{\varepsilon\tau,0} = 1$ and $\delta \mathrm{e}_{\varepsilon\tau,j} + \mathrm{e}_{\varepsilon\tau,j}/\varepsilon = 0$. Namely, $\mathrm{e}_{\varepsilon\tau,j}$ is the implicit Euler discretization of the Cauchy problem $\dot{w} + w/\varepsilon = 0$ and $w(0) = 1$.

We shall be concerned with minimizing $\mathsf{W}_{\varepsilon\tau}$ over the discrete analog of K_ε that is

$$K_{\varepsilon\tau} := \begin{cases} \{(\boldsymbol{q}_0, \dots, \boldsymbol{q}_n) \,:\, \boldsymbol{q}_0 = \boldsymbol{q}^0,\ \delta\boldsymbol{q}_1 = \boldsymbol{q}^1\} & \text{if } \rho > 0, \\ \{(\boldsymbol{q}_0, \dots, \boldsymbol{q}_n) \,:\, \boldsymbol{q}_0 = \boldsymbol{q}^0\} & \text{if } \rho = 0. \end{cases}$$

In case $U \in \mathrm{C}^{1,1}(\mathbb{R}^m)$ it can be proved that, at least for small ε, this minimization problem has a unique solution. Moreover, the minimizer fulfills the discrete Euler–Lagrange system

$$\varepsilon^2 \rho \delta^4 \boldsymbol{q}_{j+2} - 2\varepsilon\rho\delta^3 \boldsymbol{q}_{j+1} + \rho\delta^2 \boldsymbol{q}_j + \nabla U(\boldsymbol{q}_j) = 0, \quad j = 2, \dots, n-2, \quad (35)$$

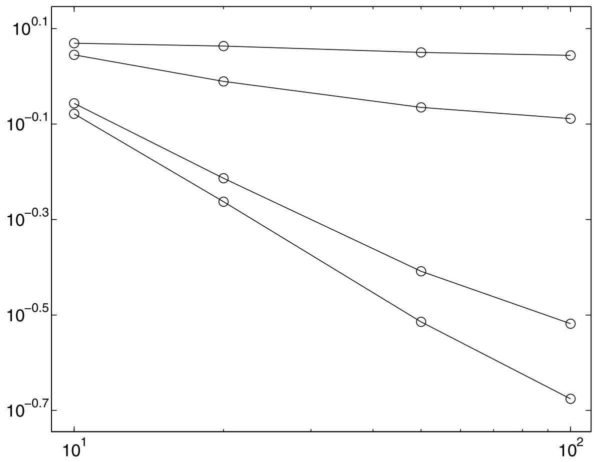

Fig. 4 Convergence in the nonlinear case of $U(q) = q^4/4$, $q^0 = 1$, $q^1 = 0$, $T = 10$. The figure plots in log-log scale the error in the uniform norm $\sup_{[0,T]} |\boldsymbol{q} - \boldsymbol{q}_{\varepsilon\tau}|$ against $1/\tau$. The different error curves correspond to the different choices $\varepsilon = 0.2, 0.1, 0.05, 0.02$ (*top to bottom*)

$$\boldsymbol{q}_0 = \boldsymbol{q}^0, \qquad \rho\delta\boldsymbol{q}_1 = \rho\boldsymbol{q}^1, \tag{36}$$

$$\rho\delta^2\boldsymbol{q}_n = \rho\delta^3\boldsymbol{q}_n = \boldsymbol{0}. \tag{37}$$

This scheme is proved to be convergent in Stefanelli (2011) and can be extended in order to cope with the dissipative case of Sect. 3 (see Liero and Stefanelli 2012).

The system (35)–(37) can be regarded as the *variational integrator* (Hairer et al. 2006) corresponding to the WIE principle. We shall stress that the scheme (35)–(37) is computationally more expensive (a system of $n \times m$ nonlinear equations) with respect to the classical implicit Euler scheme (corresponding to $\varepsilon = 0$ in (35), n systems of m nonlinear equations), not speaking of explicit or symplectic Euler (direct substitution) (Hairer et al. 2006). Indeed, for all $\varepsilon > 0$ the time-discrete WIE principle is noncausal and a full system over the time indices has to be solved. This is particularly critical for final conditions (37) are crucially entering the picture. An illustration of the convergence of the scheme is given in Fig. 4.

A remarkable trait of the scheme (35)–(37) is, however, that of showing some additional stability for $\varepsilon > 0$. In particular, some *explicit* version of the scheme (35)–(37) (i.e., replacing $\nabla U(\boldsymbol{q}_j)$ with $\nabla U(\boldsymbol{q}_{j-1})$ in (35)) shows conditional stability. This contrasts with the instability of the explicit Euler scheme.

Let us mention that the infinite-horizon situation $T = \infty$ seems less amenable from the numerical viewpoint. This is due to the fact that the final conditions (37) have to be replaced with specific summability conditions at infinity as commented in Sect. 2.6. In order to avoid solving an infinite system of equations, one might consider imposing two extra initial conditions such that the above mentioned summability is met in a sort of a *shooting* strategy. As the linear case of Sect. 2.6 shows, this turns, however, out to be a tricky task.

Before closing this section, let us mention that the same drawback is of course exhibited also by the modifications of (35) given by

$$\varepsilon^2\rho\delta^4\boldsymbol{q}_j - 2\varepsilon\rho\delta^3\boldsymbol{q}_j + \rho\delta^2\boldsymbol{q}_j + \nabla U(\boldsymbol{q}_i) = 0 \quad \text{for } i = j,\ j-1,\ j-2.$$

Note that the latter schemes cannot be obtained as Euler–Lagrange equation of (variants of) the functionals $\mathsf{W}_{\varepsilon\tau}$.

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References

Akagi, G., Stefanelli, U.: A variational principle for doubly nonlinear evolution. Appl. Math. Lett. **23**(9), 1120–1124 (2010)

Akagi, G., Stefanelli, U.: Weighted energy-dissipation functionals for doubly nonlinear evolution. J. Funct. Anal. **260**(9), 2541–2578 (2011)

Akagi, G., Stefanelli, U.: Doubly nonlinear evolution equations as convex minimization problems (2012, in preparation)

Arnol'd, V.I.: Mathematical Methods of Classical Mechanics, 2nd edn. Graduate Texts in Mathematics, vol. 60. Springer, New York (1989). Translated from the Russian by K. Vogtmann and A. Weinstein

Basdevant, J.-L.: Variational Principles in Physics. Springer, New York (2007)

Berdichevsky, V.L.: Variational Principles of Continuum Mechanics. I. Interaction of Mechanics and Mathematics. Springer, Berlin (2009). Fundamentals

Bergh, J., Löfström, J.: Interpolation Spaces. An Introduction. Grundlehren der Mathematischen Wissenschaften, vol. 223. Springer, Berlin (1976)

Conti, S., Ortiz, M.: Minimum principles for the trajectories of systems governed by rate problems. J. Mech. Phys. Solids **56**, 1885–1904 (2008)

Dal Maso, G.: An Introduction to Γ-Convergence. Progress in Nonlinear Differential Equations and Their Applications, vol. 8. Birkhäuser Boston Inc., Boston (1993)

De Giorgi, E.: Conjectures concerning some evolution problems. Duke Math. J. **81**(2), 255–268 (1996)

De Giorgi, E., Franzoni, T.: On a type of variational convergence. In: Proceedings of the Brescia Mathematical Seminar, Italian, vol. 3, pp. 63–101. Univ. Cattolica Sacro Cuore, Milan (1979)

Ghoussoub, N.: Selfdual Partial Differential Systems and Their Variational Principles. Universitext. Springer (2008, in press)

Hairer, E., Lubich, Ch., Wanner, G.: Geometric Numerical Integration, 2nd edn. Springer Series in Computational Mathematics, vol. 31. Springer, Berlin (2006). Structure-preserving algorithms for ordinary differential equations

Ilmanen, T.: Elliptic regularization and partial regularity for motion by mean curvature. Mem. Am. Math. Soc. **108**(520), x+90 (1994)

Lánczos, C.: The Variational Principles of Mechanics, 4th edn. Mathematical Expositions, vol. 4. University of Toronto Press, Toronto (1970)

Larsen, C.J., Ortiz, M., Richardson, C.L.: Fracture paths from front kinetics: relaxation and rate independence. Arch. Ration. Mech. Anal. **193**(3), 539–583 (2009)

Liero, M., Stefanelli, U.: The weighted inertia-dissipation-energy variational approach to hyperbolic-parabolic semilinear systems (2012, in preparation)

Lions, J.-L., Magenes, E.: Non-homogeneus Boundary Value Problems and Applications, vol. 1. Springer, New York (1972)

Mielke, A., Ortiz, M.: A class of minimum principles for characterizing the trajectories and the relaxation of dissipative systems. ESAIM Control Optim. Calc. Var. **14**(3), 494–516 (2008)

Mielke, A., Stefanelli, U.: A discrete variational principle for rate-independent evolution. Adv. Calc. Var. **1**(4), 399–431 (2008)

Mielke, A., Stefanelli, U.: Weighted energy-dissipation functionals for gradient flows. ESAIM Control Optim. Calc. Var. **17**(1), 52–85 (2011)

Moiseiwitsch, B.L.: Variational Principles. Dover Publications, Mineola (2004). Corrected reprint of the 1966 original

Rossi, R., Savaré, G., Segatti, A., Stefanelli, U.: Weighted energy-dissipation functionals for gradient flows in metric spaces (2011a, in preparation)

Rossi, R., Savaré, G., Segatti, A., Stefanelli, U.: A variational principle for gradient flows in metric spaces. C. R. Math. Acad. Sci. Paris **349**, 1224–1228 (2011b)

Serra, E., Tilli, P.: Nonlinear wave equations as limits of convex minimization problems: proof of a conjecture by De Giorgi. Ann. Math. (2012, to appear)

Spadaro, E.N., Stefanelli, U.: A variational view at mean curvature evolution for linear growth functionals. J. Evol. Equ. (2011, to appear)

Stefanelli, U.: The De Giorgi conjecture on elliptic regularization. Math. Models Methods Appl. Sci. **21**(6), 1377–1394 (2011)