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Young-Measure Quasi-Static Damage Evolution

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Abstract

An existence result for the quasi-static evolution of incomplete damage in elastic materials is presented. The absence of gradient terms in the damage variable causes a critical lack of compactness. Therefore, the analysis is developed in the framework of Young measures, where a notion of solution is defined, presenting some improvements with respect to previous contributions. The main new feature in the proof of the existence result regards a delicate construction of the joint-recovery sequence.

1. Introduction

Damage processes are recurrent in Solid Mechanics. By undergoing loading cycles, real materials experience to a variable extent a deterioration of their respective elastic properties. This can be generally interpreted as the effect of the occurrence and growth of cracks and voids at the level of the microscopic material structure and it has a dramatic impact upon the performance of structures and materials. As such, damage modeling has been a remarkably active trend in the engineering community since the 50s, so that it is largely beyond our scope even to try to review the huge existing literature on this subject. The reader is, however, referred to [5,15,21–23] for some recent contributions.

The usual approach to damage in Continuum Mechanics is that of directly incorporating an internal variable descriptor of the state of the material into the constitutive relations. In particular, in the case of isotropic damage (that is, by assuming that deterioration has no preferential orientation), one is led to introduce a scalar damage variable z taking the value $z = 1$ at undamaged points and $z = 0$ at maximally damaged points. Hence, moving within the small-strain realm, one is generally concerned with an elastic energy functional of the form

$$\mathcal{W}(z, e(v)) := \int_{\Omega} W(z(x), e(v)(x))\, \mathrm{d}x,$$

where $e(v) := (\nabla v + \nabla v^T)/2$ is the symmetrized strain tensor and $v\colon \Omega \to \mathbb{R}^d$ denotes the displacement from the reference configuration Ω. Damage evolution is governed by the interplay of energy minimization and dissipation. In particular, damage is often very well assimilable to a quasi-static evolution process and, in this regard, the first possible choice for a dissipation mechanism from the damage state z_{old} to updated state z_{new} may be assumed to be

$$\mathcal{D}(z_{\text{old}}, z_{\text{new}}) := \int_\Omega d(z_{\text{old}}(x), z_{\text{new}}(x))\,\mathrm{d}x,$$

where $d\colon \mathbb{R}^2 \to [0, \infty]$ is the non-symmetric (pseudo-)distance defined by

$$d(\theta_1, \theta_2) := \begin{cases} \rho(\theta_1 - \theta_2) & \text{if } \theta_1 \geqq \theta_2 \\ \infty & \text{else} \end{cases}$$

for some $\rho > 0$. The asymmetry of the dissipation distance d encodes the quite natural *ansatz* of irreversibility of damage. Moreover, the 1-homogeneity of $\mathcal{D}$ is the trademark of the rate-independent nature of the damage process.

This very frame for a variational theory of rate-independent damage has attracted a good deal of attention in recent years and rigorous mathematical results are to be found, for instance, in [2,3,14,16,28,30]. The analysis of this paper resides exactly within the setting of the result by THOMAS and MIELKE [35] where the authors develop an existence theory for *incomplete* damage by directly including a gradient term of the internal variable z into the energy. By including such a gradient term, one obtains a clear compactifying effect along with the possible description of nonlocal interactions of damage in the material. On the other hand, the occurrence of damage localization often seems to be clear experimental evidence. In this respect, one is motivated in considering possibly non-regularized damage models instead.

The novelty of our contribution with respect to [35] resides, specifically, in dropping the gradient term in the damage variable from the energy, thus excluding nonlocal damage interaction. Correspondingly, we are lacking the above mentioned compact frame and we resort to considering Young measures as plausible objects for describing damage evolution. Young measures are, indeed, quite naturally suited to the treatment of non-compact problems. In particular, for rate-independent models, analyses of mechanical phenomena within the framework of Young measures have been devised in [12,20,24,26,27,29] for phase transitions, DAL MASO et al. [6,8] for plasticity with softening, and CAGNETTI and TOADER [4] for fracture mechanics. To our knowledge, no Young measure formulation has yet been proposed in the context of rate-independent damage (in the case of a gradient-flow damage model, a Young-measure analysis at the time-discrete level is reported in [33]).

The focus of this paper is on providing an existence theory for a suitable *Young-measure quasi-static evolution* of the damage model in the frame of so-called *energetic solutions à la* MIELKE and THEIL [31]. Our evolution will be represented by a family $\nu = (\nu_t)_t$ of time-parametrized Young measures which replace the pair $(z, e(v))$. According to the expected unidirectionality of the damage process, the energetic solution is required to satisfy a suitable irreversibility property. To formulate this monotonicity condition in our generalized setting, we tailor a partial order relation between Young measures (see Section 3.1), in the same spirit as in [4]. Then, the validity of a specific *global stability* condition and of the *energy balance* will be achieved by passing to the limit argument with respect to time-discretizations.

As already commented in [35], the discontinuity of the dissipation distance makes the proof of the stability condition more complicated by requiring the construction of a so-called *mutual recovery sequence*. This is exactly the point where the compactifying effect of the gradient of damage in [35] has proved to be useful in order to ensure a stronger convergence of the recovery sequence. Here, we overcome this point employing by two tools: a regularity result and a measure-reconstruction lemma. At first, we exploit the fact that some *higher integrability* of the approximating sequences can be achieved by exploiting the theory of *quasi-minima* [17]. We believe this observation (already done in [12]) to be an interesting feature of our proof which could also possibly be of some use elsewhere. Then, we provide a constructive technique to build a recovery sequence satisfying both the order constraint and the required convergence property.

The technical difficulties related to the Young measure approach force us to consider some *reduced* global stability conditions. In particular, as is quite usual in these situations, we obtain global stability for two classes of competitors: translations of ν_t by functions $(\tilde{z}, \tilde{u})$ in $L^1(\Omega; \mathbb{R}) \times H^1_0(\Omega; \mathbb{R}^d)$, and Young measures with disintegration of the form $\tilde{\mu}^x \otimes \delta_{e(\tilde{v})(x)}$, for any Young measure $\tilde{\mu}$ on $\Omega \times [0, 1]$. Minimality with respect to translations by functions coincides with the stability condition considered in [8] and [11]. Here, nevertheless, we allow milder assumptions on the energy density. The second class of tests represents, instead, a quite remarkable enlargement of the set of competitors with respect to previous contributions. These competitors, in particular, do not depend on the evolution ν_t and permit the comparison of the evolution with *all other possible damage states*.

A further interesting feature of our result is that the specific form of the damage model allows us to prove the existence result without the help of the technical tool of *compatible systems of Young measures* developed in [7] (see also [11]). In particular, this entails a rather straightforward formulation of our notion of the solution.

Our damage model is non-brittle in the sense that partially damaged situations $z \in (0, 1)$ are actually to be expected (see Section 2.1). We shall refer to Francfort and Garroni [14], Garroni and Larsen [16], and Babadjian [2] for recent contributions on damage models for brittle materials, namely assuming $z \in \{0, 1\}$. Besides brittleness, we have to remark that the mechanical stand of the latter papers is quite different from ours. In particular, their starting point is a z-mixture of a linearly elastic *strong* and *weak* material with elasticity tensors A_s and A_w, respectively. This is to say that their energy density is assumed to be of the form

$$W(z, e) = \begin{cases} A_s\phi(e) & \text{if } z = 1 \\ A_w\phi(e) & \text{if } z = 0 \\ +\infty & \text{otherwise,} \end{cases} \tag{1.1}$$

with $\phi(e) = e^2/2$ (in the one-dimensional case) in [14,16] and a more general convex function ϕ in [2]. As no gradient terms in the damage variable are considered, evolution via time-discretization immediately calls for quasi-convexification and the passage to the limit is performed by determining the *limiting* materials via its elasticity tensor by homogenization tools. To this end, the convexity of the energy density with respect to the strain variable is needed [2, Section 1], and a price to pay is the replacement of the damage variable z by the elasticity tensor or by the damage set in the limit.

Our approach here is somewhat different as we start from an (essentially) quasi-convex energy in the first place so that no quasi-convexification is needed for the incremental step. On the one hand, this prevents us from considering linear mixture energies of the form of (1.1) in our frame. In particular, the relaxed models from [2,14,16] seem not to be directly recoverable in the present setting. On the other hand, this gives us the advantage of tracing the damage variable z into the evolution.

The paper is organized as follows. In Section 2 we present the mechanical model, and in Section 3 we recall some mathematical preliminaries. In particular, Section 3.1 presents a partial order relation between Young measures. Section 4 is devoted to the formulation of the quasi-static evolution and our main result. The existence proof is detailed in Section 5. Some technical lemmas are then collected in the Appendix.

2. The Mechanical Model

Let us specify here some notation and our general assumptions. The *reference configuration* of the body is a bounded, connected, and open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. We indicate the *displacement field* by v and the *linearized strain tensor* by $e(v) := \frac{1}{2}(\nabla v + \nabla v^{\mathrm{T}})$. The *damage variable* is $z\colon \Omega \to \mathbb{R}$ and will actually take values solely in $[0, 1]$ as an effect of our general assumptions below.

The *stored energy density* of the material is a function $W\colon \mathbb{R} \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, +\infty)$ satisfying the following hypotheses:

- $(W.1)$ W is continuous and S-cross-quasiconvex, that is satisfies property (3.4) below;
- $(W.2)$ there exist two positive constants $c_W < C_W$ such that $c_W|\varepsilon|^2 \leqq W(\theta, \varepsilon) \leqq C_W|\varepsilon|^2$ for every $\varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and every $\theta \in (-\infty, 2]$;
- $(W.3)$ for every $\theta \in \mathbb{R}$, $W(\theta, \cdot)$ is C^1 and $\left|\frac{\partial W}{\partial \varepsilon}(\theta, \varepsilon)\right| \leqq C_W(|\varepsilon| + 1)$, for every $(\theta, \varepsilon) \in (-\infty, 2] \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$;
- $(W.4)$ $\theta \mapsto W(\theta, \varepsilon)$ is non-decreasing for every $\varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$;
- $(W.5)$ $W(\theta, \varepsilon) = W(0, \varepsilon)$ for every $\theta \leqq 0$.

Hence, the *stored energy* of the material reads

$$\mathcal{W}(z, e(v)) := \int_\Omega W(z(x), e(v)(x))\,\mathrm{d}x.$$

Though the most natural assumption for the stored energy density in linearized elasticity is to be quadratic with respect to the strain variable, for sake of generality we assume here that W satisfies the weaker condition $(W.1)$. Indeed, our analysis could be retraced in the case of nonlinear elasticity as well, and in this case the quasi-convexity assumption is more desirable than the quadratic one.

The *dissipation distance* between two damage states z_{old} and z_{new} is given by

$$\mathcal{D}(z_{\mathrm{old}}, z_{\mathrm{new}}) := \int_\Omega d(z_{\mathrm{old}}(x), z_{\mathrm{new}}(x))\,\mathrm{d}x,$$

where the density d is given by

$$d(\theta_1, \theta_2) := \begin{cases} \rho|\theta_1 - \theta_2| & \text{if } \theta_1 \geqq \theta_2 \\ +\infty & \text{else,} \end{cases}$$

for every $\theta_1, \theta_2 \in \mathbb{R}$ and for a suitable $\rho > 0$.

Given two times $s < t$, the *global dissipation* of a possibly discontinuous-in-time damage evolution $z\colon [0, T] \to L^1(\Omega)$ in the interval $[s, t]$ is given by

$$\mathrm{Diss}(z; s, t) := \sup \sum_{i=1}^{k} \mathcal{D}(z(\tau_{i-1}), (z(\tau_i)),$$

where the supremum is taken among all finite partitions $s = \tau_0 < \tau_1 < \ldots < \tau_k = t$.

Note that, if $z(\tau) \geqq z(\tau')$ almost everywhere in Ω, whenever $\tau \leqq \tau'$, then

$$\mathrm{Diss}(z; s, t) = \rho \int_\Omega (z(s) - z(t))\,\mathrm{d}x.$$

For the sake of simplicity, the boundary displacement is prescribed at time t on the whole boundary $\partial\Omega$ as $u = \varphi$, where the given function $\varphi(t)$ fulfills

$$\varphi \in AC([0, T]; W^{1,p}(\Omega; \mathbb{R}^d)), \quad \text{with } 2 < p \leqq \infty.$$

Let us, however, note that other choices of the boundary conditions are indeed possible. More precisely, boundary conditions of mixed type can be considered, provided the gradient of the quasi-minima of the functional $v \mapsto \int_\Omega |e(v)|^2\,\mathrm{d}x$, with the chosen mixed boundary condition, can be shown to be higher integrable, in the spirit of Theorem 1, proved by Giaquinta and Giusti for the fully Dirichlet boundary condition case.

2.1. A Zero-Dimensional Example

We focus here on a zero-dimensional case, that is, the case in which damage and strain are independent of x. Our aim is to show that the materials we are considering are not necessarily brittle, in the sense that the damage variable z can be expected to take intermediate values between 0 and 1.

We consider a stored energy defined by

$$W(z, e) := \frac{e^2}{2g(z)},$$

for $g(z) := \sqrt{2 - z^+}$ for every $z \in [0, 2)$. We observe that the function g is $C^2(0, 2)$ with $g' \leqq 0$ and $g'' \leqq 0$ in $(0, 2)$ and g is constant on $(-\infty, 0]$; it is now easy to see that the Hessian matrix of W is positive definite and hence W is a convex function on $[0, 2) \times \mathbb{R}$ (see [35, Lemma 5.1]).

The dissipation distance is given by

$$d(z_1, z_2) := \begin{cases} |z_1 - z_2| & \text{if } z_1 \geqq z_2 \\ +\infty & \text{otherwise,} \end{cases}$$

for every $z_1, z_2 \in \mathbb{R}$.

In this example we analyze an evolution driven by time-dependent external forces instead of time-varying boundary data; the external forces are given by $l(t) := t$.

In particular, a quasi-static evolution in the time interval $[0, 3\sqrt{2}]$ with initial datum $(z_0, e_0) := (1, 0)$ is defined *energetically* (see [31]) as a pair of time-dependent functions $(z(t), e(t))$ with $z(t) \geqq 0$, such that the following conditions are satisfied for every $\tilde{z}, \tilde{e} \in \mathbb{R}$ and every $t \in [0, 3/\sqrt{2}]$:

$$\text{initial condition: } (z(0), e(0)) = (1, 0); \tag{2.1}$$

$$\text{irreversibility: } z(s) \geqq z(t) \text{ if } s \leqq t; \tag{2.2}$$

$$\text{stability: } \frac{e^2(t)}{2g(z(t))} - te(t) \leqq \frac{\tilde{e}^2}{2g(\tilde{z})} - t\tilde{e} + d(z(t), \tilde{z}); \tag{2.3}$$

$$\text{energy equality: } \frac{e^2(t)}{2g(z(t))} - te(t) + z(0) - z(t) = -\int_0^t e(s)\,\mathrm{d}s. \tag{2.4}$$

Condition (2.3) implies that $e(t) = tg(z(t))$. Indeed, if we choose $\tilde{z} = z(t)$ in (2.3), we obtain that $e(t)$ is the unique minimizer of the convex function $e \mapsto e^2/(2g(z(t))) - te$. Therefore, it is enough to choose $z(t)$ satisfying the initial condition and the irreversibility condition, such that the energy equality (2.4) holds true for $(z(t), tg(z(t)))$, and satisfying for every $t \in [0, 3\sqrt{2}]$

$$\frac{t^2 g(z(t))}{2} - t^2 g(z(t)) \leqq \frac{t^2 g(\tilde{z})}{2} - t^2 g(\tilde{z}) + z(t) - \tilde{z},$$

for every $\tilde{z} \leqq z(t)$, that is,

$$\frac{t^2}{2}[g(\tilde{z}) - g(z(t))] \leqq z(t) - \tilde{z}, \tag{2.5}$$

for every $\tilde{z} \leqq z(t)$.

Let us first consider $z(t) \equiv 1$ for every $t \in [0, 2]$. This choice may be easily proved to fulfill (2.1)–(2.4) and hence is a quasi-static evolution for $t \in [0, 2]$. We want to show that, for $t > 2$, $z(t) = 1$ does not satisfy the stability condition (2.5) and hence $z(t)$ has to be strictly smaller than 1. We rephrase this by saying that there exists $\tilde{z} \in [0, 1]$ such that $f(\tilde{z}) > 0$ where f is given by

$$f(\tilde{z}) := \frac{t^2}{2}[g(\tilde{z}) - 1] - 1 + \tilde{z}.$$

Indeed, let us consider $\tilde{z}_t := t^2 - t^4/4 + 1 = (8 - (t^2 - 2)^2)/4$. We observe that $\tilde{z}_t \in (0, 1)$ and $f(\tilde{z}_t) = 0$ if $t \in (2, 3/\sqrt{2})$. Moreover,

$$\begin{aligned} f'(\tilde{z}_t) &= -\frac{t^2}{4\sqrt{2-\tilde{z}_t}} + 1 = -\frac{t^2}{4\sqrt{2 - \left(t^2 - \frac{t^4}{4} + 1\right)}} + 1 \\ &= -\frac{t^2}{4\left(\frac{t^2}{2} - 1\right)} + 1 = \frac{t^2 - 4}{2(t^2 - 2)} > 0, \end{aligned}$$

since $t > 2$. Therefore, there exists $\tilde{z} \in (\tilde{z}_t, 1)$ such that $f(\tilde{z}) > 0$. Hence, $z(t) = 1$ does not fulfill the stability condition (2.5) for $t \in (2, 3/\sqrt{2})$ and we will necessarily have $z(t) \in (-\infty, 1)$.

On the other hand, we cannot have $z(t) = 1$ for $t \in [0, 2]$ and $z(t) \leqq 0$ for $t \in (2, 3/\sqrt{2})$, because in this case the energy balance for $s \in (2, 3/\sqrt{2})$ would not be fulfilled as

$$\begin{aligned} &\frac{s^2 g(0)}{2} - s^2 g(0) + 1 - z(t) + \int_0^s t g(z(t))\,\mathrm{d}t \\ &= -\frac{s^2\sqrt{2}}{2} + 1 - z(t) + \int_0^2 t\,\mathrm{d}t + \int_2^s t\sqrt{2}\,\mathrm{d}t \\ &= -\frac{s^2\sqrt{2}}{2} + 1 - z(t) + 2 + \frac{s^2\sqrt{2}}{2} - 2\sqrt{2} = 1 - z(t) - 2(\sqrt{2} - 1) > 0. \end{aligned}$$

Eventually, we have proved that there exists $t \in (2, 3/\sqrt{2})$ with $z(t) \in (0, 1)$.

3. Mathematical Preliminaries

Let $\mathcal{L}^d$ denote the Lebesgue measure on $\mathbb{R}^d$, $d \geqq 1$. We sometimes use the notation $|E|$ for the Lebesgue measure of the measurable subset $E \subseteq \mathbb{R}^d$ as well. Throughout the paper Ω will be a bounded, connected, open subset of $\mathbb{R}^d$ with Lipschitz boundary. The Borel σ-algebra on Ω is denoted by $\mathcal{B}(\Omega)$. For $1 \leqq p \leqq \infty$, $\|\cdot\|_p$ stands for the usual norm on L^p, $W^{1,p}(\Omega; \mathbb{R}^d)$ denotes the usual Sobolev space, $H^1(\Omega; \mathbb{R}^d) := W^{1,2}(\Omega; \mathbb{R}^d)$, and the symbol $\langle \cdot, \cdot \rangle$ is the scalar product in H^1, if not otherwise specified. Given a function $f \in L^1(\Omega)$ and a measurable $Q \subseteq \Omega$, the mean value of f over Q is denoted by $(f)_Q$, that is,

$$(f)_Q := \frac{1}{|Q|} \int_Q f(x)\,\mathrm{d}x.$$

We indicate the positive part of a function f with $f^+ := f \vee 0$.

We recall the notion of quasi-minima of integral functionals. Given $\varphi \in H^1(\Omega;\mathbb{R}^d)$, let $\mathcal{G}$ be the functional defined by

$$\mathcal{G}(v) = \mathcal{G}(v,\Omega) := \int_\Omega G(x,\nabla v(x))\,\mathrm{d}x$$

for every $v \in \varphi + H^1_0(\Omega;\mathbb{R}^d)$, where $G\colon \Omega\times\mathbb{R}^{d\times d} \to \mathbb{R}$ is a Carathéodory function satisfying

$$G(x,F) \leqq L(|F|^2+1)$$
$$G(x,F) \geqq \tilde{G}(F) - l$$

for suitable positive constants L, l, for every $(x,F) \in \Omega\times\mathbb{R}^{d\times d}$, where $\tilde{G}\colon \mathbb{R}^{d\times d} \to \mathbb{R}$ satisfies the following estimate:

$$\exists K > 0: \quad \int_\Omega \tilde{G}(\nabla\phi(x))\,\mathrm{d}x \geqq K\|\nabla\phi\|_2^2 \quad \text{for every } \phi \in H^1_0(\Omega;\mathbb{R}^d).$$

Definition 1. (*Quasi-minimum,* [17]) Let $V \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $\lambda > 0$. A function $v \in V + H^1_0(\Omega;\mathbb{R}^d)$ is said to be a *cubic λ-quasi-minimum* for the functional $\mathcal{G}$ if for every cube of side R, $Q_R \subset \mathbb{R}^d$, and every $w \in H^1(\Omega;\mathbb{R}^d)$ such that $v - w \in H^1_0(\Omega\cap Q_R)$ we have

$$\int_{(Q_R\cap\Omega)} G(x,\nabla v(x))\,\mathrm{d}x \leqq \lambda\int_{Q_R\cap\Omega} G(x,\nabla w(x))\,\mathrm{d}x.$$

Theorem 1. (Higher integrability, [18, Ch. 6]) *Let $V \in W^{1,p}(\Omega;\mathbb{R}^d)$, for $2 < p$, and let $v \in V + H^1_0(\Omega;\mathbb{R}^d)$ be a λ-cubic quasi-minimum of the functional $\mathcal{G}$. Then, there exist constants $\gamma > 0$ and $r > 1$, depending only on λ and V, such that*

$$\int_\Omega |\nabla v|^{2r}\,\mathrm{d}x \leqq \gamma\left\{\left(\int_\Omega |\nabla v|^2\,\mathrm{d}x\right)^r + 1\right\}.$$

We recall the statement of the Korn–Poincaré inequality (see [34]): for every open, bounded, Lipschitz set $D \subset \mathbb{R}^d$, there exists a positive constant $C(D)$ such that

$$\|\nabla v\|_{H^1(D)} \leqq C(D)\|e(v)\|_{L^2(D)}, \tag{3.1}$$

for every $v \in H^1_0(D)$.

We recall the definition of cross-quasiconvexity in the form used in [13] and a related semicontinuity result [13, Theorem 4.4]. A continuous function $G\colon \mathbb{R}\times\mathbb{R}^{d\times d} \to \mathbb{R}$ is *cross-quasiconvex* if for every $\theta \in [0,1]$, $F \in \mathbb{R}^{d\times d}$ we have

$$G(\theta,F) \leqq \frac{1}{|\Omega|}\int_\Omega G(\theta+m(x), F+\nabla u(x))\,\mathrm{d}x, \tag{3.2}$$

for every $u \in H^1_0(\Omega;\mathbb{R}^d)$ and every $m \in L^\infty(\Omega)$, with $\theta + m(x) \in [0,1]$ for almost every $x \in \Omega$ and $\int_\Omega m(x)\,\mathrm{d}x = 0$.

Lemma 1. (Lower semicontinuity) *If* $G\colon \mathbb{R}\times\mathbb{R}^{d\times d}\to\mathbb{R}$ *is cross-quasi-convex and fulfills*

$$0\leqq G(\theta,F)\leqq g(\theta)(1+|F|^2) \tag{3.3}$$

for every $\theta\in\mathbb{R},\ F\in\mathbb{R}^{d\times d}$*, and some* $g\in L^\infty_{\mathrm{loc}}(\mathbb{R})$*, we have that*

$$\int_\Omega G(z(x),\nabla v(x))\,\mathrm{d}x\leqq \liminf_k\int_\Omega G(z_k(x),\nabla v_k(x))\,\mathrm{d}x,$$

whenever $z_k\rightharpoonup z$ L^∞*-weakly*,* $z_k(x)\in[0,1]$ *for almost every* $x\in\Omega$*, and* $v_k\rightharpoonup v$ *weakly in* $H^1(\Omega;\mathbb{R}^d)$*.*

Note that if $H\colon\mathbb{R}\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\to\mathbb{R}$ is a continuous function satisfying

$$\begin{aligned}&0\leqq H(\theta,\varepsilon)\leqq g(\theta)(1+|\varepsilon|^2)\quad\text{for } g\in L^\infty_{\mathrm{loc}}(\mathbb{R});\\&H(\theta,\varepsilon)\leqq\frac{1}{|\Omega|}\int_\Omega H(\theta+m(x),\varepsilon+e(u)(x))\,\mathrm{d}x,\end{aligned} \tag{3.4}$$

for every $u\in H^1_0(\Omega;\mathbb{R}^d)$, $m\in L^\infty(\Omega;\mathbb{R})$ with $\int_\Omega m(x)\,\mathrm{d}x=0$ and $\theta+m(x)\in[0,1]$ for almost every $x\in\Omega$, then the function $G(\theta,F):=H(S(\theta,F))$, with $S(\theta,F):=\left(\theta,\frac{F+F^T}{2}\right)$, satisfies properties (3.2) and (3.3). We will say that a function satisfying (3.4) is *S-cross-quasiconvex*.

We define $M_b(\Omega\times\mathbb{R}^N)$ as the space of bounded Radon measures on $\Omega\times\mathbb{R}^N$. This space can be identified with the dual of the Banach space $C_0(\Omega\times\mathbb{R}^N)$ of all continuous functions $\phi\colon\Omega\times\mathbb{R}^N\to\mathbb{R}$ such that $|\phi|\geqq\varepsilon$ is compact for every $\varepsilon>0$. We will consider on $M_b(\Omega\times\mathbb{R}^N)$ the weak* topology deriving from this duality.

Let us refer to [36] for a general introduction to Young measures, and recall some definitions and fix notation. A Young measure $\mu\in Y(\Omega;\mathbb{R}^N)$ is a nonnegative measure in $M_b(\Omega\times\mathbb{R}^N)$, such that $\pi_\Omega(\mu)=\mathcal{L}^d$, where $\pi_\Omega(x,\xi):=x$. By the Disintegration Theorem, one can associate to μ a measurable family of probability measures $(\mu^x)_{x\in\Omega}$ on $\mathbb{R}^N$ in such a way that

$$\int_{\Omega\times\mathbb{R}^N} f(x,\xi)\,d\mu(x,\xi)=\int_\Omega\left(\int_{\mathbb{R}^N} f(x,\xi)\,\mathrm{d}\mu^x(\xi)\right)\mathrm{d}x,$$

for every bounded Borel function $f\colon\Omega\times\mathbb{R}^N\to\mathbb{R}$. We define the barycentre of μ as the function

$$\mathrm{bar}(\mu)(x):=\int_{\mathbb{R}^N}\xi\,\mathrm{d}\mu^x(\xi)\quad\text{for a.e. } x\in\Omega,$$

and the p-moment of μ, for $1<p\leqq\infty$, as the quantity

$$\int_{\Omega\times\mathbb{R}^N}|\xi|^p\,\mathrm{d}\mu(x,\xi).$$

We denote by $Y^p(\Omega;\mathbb{R}^N)$ the set of measures in $Y(\Omega;\mathbb{R}^N)$ with finite p-moments. Given a sequence $(\mu_k)_k$ in $Y(\Omega;\mathbb{R}^N)$, we say that $\mu_k \rightharpoonup \mu$ *p-weakly**, for $1 < p \leqq \infty$, if

$$\mu_k \rightharpoonup \mu \quad \text{in the weak* topology of } M_b(\Omega\times\mathbb{R}^N),$$
$$\int_{\Omega\times\mathbb{R}^n} |\xi|^p \, d\mu_k(x,\xi) \quad \text{are equibounded in } k.$$

Let $(D,\mathcal{F})$ be a measure space and $\mu \in Y(\Omega;\mathbb{R}^N)$. For every $\mathcal{B}(\Omega\times\mathbb{R}^N)$-$\mathcal{F}$-measurable function $f\colon \Omega\times\mathbb{R}^N \to D$, the image measure, defined by $\mu(f^{-1}(B))$ for every measurable set $B \subseteq D$, will be denoted by $f(\mu)$. In particular, if we define the translation map Tr_G associated to a function $G \in L^1(\Omega;\mathbb{R}^N)$ by

$$\mathrm{Tr}_G(x,\xi) := (x, \xi + G(x)), \quad \text{for a.e. } x\in\Omega \text{ and every } \xi \in \mathbb{R}^N,$$

for every measure $\mu \in Y(\Omega;\mathbb{R}^N)$ we can consider the translated measure $\mathrm{Tr}_G(\mu)$, defined by

$$\int_{\Omega\times\mathbb{R}^N} \phi(x,\xi)\, d\mathrm{Tr}_G(\mu)(x,\xi) := \int_{\Omega\times\mathbb{R}^N} \phi(x,\xi+G(x))\, d\mu(x,\xi),$$

for every bounded Borel function $\phi\colon \Omega\times\mathbb{R}^N \to \mathbb{R}$.

Given $\xi_0 \in \mathbb{R}^N$, the measure $\delta_{\xi_0} \in M_b(\mathbb{R}^N)$ is classically defined by

$$\int_{\mathbb{R}^N} f(\xi)\, d\delta_{\xi_0}(\xi) = f(\xi_0),$$

for every bounded Borel function $f\colon \mathbb{R}^N \to \mathbb{R}$.

For a fixed $\mathcal{B}(\Omega)$-$\mathcal{B}(\mathbb{R}^N)$-measurable function $u\colon \Omega \to \mathbb{R}^N$, the Young measure $\boldsymbol{\delta}_u \in Y(\Omega;\mathbb{R}^N)$ is defined by

$$\int_{\Omega\times\mathbb{R}^N} g(x,\xi)\, d\boldsymbol{\delta}_u(x,\xi) = \int_\Omega g(x,u(x))\, dx,$$

for every bounded Borel function $g\colon \Omega\times\mathbb{R}^N \to \mathbb{R}$.

The following lemma is a slight modification of [32, Proposition 6.5, p. 103].

Lemma 2. (Continuity) *Let* $1 < p \leqq \infty$*, and let* $(\mu_k)_k \subseteq Y^p(\Omega;\mathbb{R}^N)$ *converge p-weakly* to* $\mu \in Y^p(\Omega;\mathbb{R}^N)$*. Then, for every Carathéodory function* $f\colon \Omega\times\mathbb{R}^N \to \mathbb{R}$*, with* $|f(x,\xi)| \leqq a(x) + b(x)|\xi|^q$*, for every* $x\in\Omega$, $\xi\in\mathbb{R}^N$, $1 \leqq q < p$, $b \in L^{p/(p-q)}(\Omega)$*, and* $a \in L^1(\Omega)$*, it holds*

$$\int_{\Omega\times\mathbb{R}^N} f(x,\xi)\, d\mu_k(x,\xi) \longrightarrow \int_{\Omega\times\mathbb{R}^N} f(x,\xi)\, d\mu(x,\xi).$$

Finally, we recall that a measure $\nu \in Y^p(\Omega;\mathbb{R}^{d\times d})$ is a $W^{1,p}$*-gradient Young measure* (see, for example, [19]) for $p > 1$ if there exists a bounded sequence $(v_n)_n \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that $\boldsymbol{\delta}_{\nabla v_n} \rightharpoonup \nu$ p-weakly* as $n\to\infty$. For the characterization and the properties of such measures we refer to [32].

Thanks to Lemma 2, given a bounded sequence $(v_n)_n$ in $W^{1,p}(\Omega;\mathbb{R}^d)$ with $\boldsymbol{\delta}_{\nabla v_n} \rightharpoonup \nu$ p-weakly*, for $p > 1$, we have that $\boldsymbol{\delta}_{e(v_n)} \rightharpoonup S(\nu)$ p-weakly*, where $S\colon \Omega \times \mathbb{R}^{d\times d} \to \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is defined by $S(x, F) := (x, \frac{F+F^T}{2})$, for every $x \in \Omega$ and $F \in \mathbb{R}^{d\times d}$.

Henceforth C will stand for any positive constant, possibly depending on data and varying from line to line.

3.1. An Order Relation Between Young Measures

In this section we want to define an order relation on the set $Y(\Omega; [0, 1])$ of the Young measures on Ω with values in $\mathbb{R}$ and support contained in $\Omega \times [0, 1]$.

Definition 2. (*Order*) Given $\mu_1, \mu_2 \in Y(\Omega; [0, 1])$, we write $\mu_1 \succeq \mu_2$ if

$$\mu_1^x(\alpha, \infty) \geqq \mu_2^x(\alpha, \infty) \quad \text{for a.e. } x \in \Omega \text{ and for every } \alpha \in \mathbb{R}. \tag{3.5}$$

It is easy to see that $\succeq$ is an order and that, in the case of $\mu_1 = \boldsymbol{\delta}_{z^1}$ and $\mu_2 = \boldsymbol{\delta}_{z_2}$ for some measurable functions $z^1, z^2\colon \Omega \to [0, 1]$, we have $\boldsymbol{\delta}_{z^1} \succeq \boldsymbol{\delta}_{z^2}$ if and only if $z^1 \geqq z^2$ almost everywhere in Ω.

Now we give an equivalent characterization of this order relation.

Theorem 2. (Order characterization) *Given two Young measures* $\mu_1, \mu_2 \in Y(\Omega; [0, 1])$*, we have* $\mu_1 \succeq \mu_2$ *if and only if there exists* $\mu_{12} \in Y(\Omega; [0, 1]^2)$ *such that*

$$\pi_1(\mu_{12}) = \mu_1, \quad \pi_2(\mu_{12}) = \mu_2, \tag{3.6}$$

$$\mu_{12}^x(\{\theta_1 < \theta_2\}) = 0 \quad \textit{for a.e. } x \in \Omega, \tag{3.7}$$

where $\pi_1(x, \theta_1, \theta_2) := (x, \theta_1)$, $\pi_2(x, \theta_1, \theta_2) := (x, \theta_2)$*, for every* $(x, \theta_1, \theta_2) \in \Omega \times \mathbb{R}^2$.

Proof. Let us first prove necessity. If $\mu_1 \not\succeq \mu_2$, then there exists a measurable set $E \subseteq \Omega$ with positive measure, such that, for $x \in E$, $\mu_1^x((-\infty, \alpha_x]) > \mu_2^x((-\infty, \alpha_x])$, for a suitable $\alpha_x \in [0, 1]$. This implies that, for every μ_{12} satisfying the projection properties (3.6), we have $\mu_{12}^x(\{\theta_1 < \theta_2\}) > 0$ for $x \in E$. Indeed, for every $x \in E$ we have

$$\mu_{12}^x([0,1]\times[0,\alpha_x]) = \mu_2^x((-\infty,\alpha_x]) < \mu_1^x((-\infty,\alpha_x]) = \mu_{12}^x([0,\alpha_x]\times[0,1]),$$

and this implies

$$\begin{aligned} 0 &\leqq \mu_{12}^x((\alpha_x, 1] \times [0, \alpha_x]) = \mu_{12}^x([0, 1] \times [0, \alpha_x]) - \mu_{12}^x([0, \alpha_x] \times [0, \alpha_x]) \\ &< \mu_{12}^x([0, \alpha_x] \times [0, 1]) - \mu_{12}^x([0, \alpha_x] \times [0, \alpha_x]) = \mu_{12}^x([0, \alpha_x] \times (\alpha_x, 1]) \\ &\leqq \mu_{12}^x(\{\theta_1 < \theta_2\}), \end{aligned}$$

for every $x \in E$.

Now we prove the sufficiency of inequality (3.5). We fix $n \in \mathbb{N}$ and consider the measures $\mu_{1,n},\ \mu_{2,n}$ whose disintegration is defined by

$$\mu^x_{1,n} := \mu^x_1([0, \tfrac{1}{n}])\delta_{\frac{1}{n}} + \sum_{i=2}^{n} \mu^x_1((\tfrac{i-1}{n}, \tfrac{i}{n}])\delta_{\frac{i}{n}},$$
$$\mu^x_{2,n} := \mu^x_2([0, \tfrac{1}{n}])\delta_{\frac{1}{n}} + \sum_{i=2}^{n} \mu^x_2((\tfrac{i-1}{n}, \tfrac{i}{n}])\delta_{\frac{i}{n}}.$$

Since $(\mu^x_{1,n})_x$ and $(\mu^x_{2,n})_x$ are measurable families of probability measures on $[0, 1]$, we have that $\mu_{1,n}, \mu_{2,n} \in Y(\Omega; [0, 1])$.

Moreover, $\mu_{1,n} \rightharpoonup \mu_1$ and $\mu_{2,n} \rightharpoonup \mu_2$ weakly*, as $n \to \infty$. Indeed, let $f \in C_0(\Omega \times \mathbb{R})$; since f is uniformly continuous, there exists a modulus of continuity $\omega_f : \mathbb{R} \to \mathbb{R}$ such that for every $(x_1, \theta_1, \xi_1),\ (x_2, \theta_2, \xi_2) \in \Omega \times \mathbb{R}^2$

$$|f(x_1, \theta_1, \xi_1) - f(x_2, \theta_2, \xi_2)| \leqq \omega_f(|(x_1, \theta_1, \xi_1) - (x_2, \theta_2, \xi_2)|),$$
$$\lim_{\delta \to 0} \omega_f(\delta) = 0.$$

Therefore, we have for $h = 1, 2$

$$\begin{aligned}
&\Big|\int_{\Omega\times\mathbb{R}} f(x, \theta)\ \mathrm{d}\mu_{h,n}(x, \theta) - \int_{\Omega\times\mathbb{R}} f(x, \theta)\ \mathrm{d}\mu_h(x, \theta)\Big| \\
&= \Big|\int_\Omega \Big(\int_\mathbb{R} f(x, \xi)\,\mathrm{d}\mu^x_{h,n}(\theta) - \int_\mathbb{R} f(x, \theta)\ \mathrm{d}\mu^x_h(\theta)\Big)\ \mathrm{d}x\Big| \\
&\leqq \int_\Omega \Big|\mu^x_h([0, \tfrac{1}{n}]) f(x, \tfrac{1}{n}) + \sum_{i=2}^{n} \mu^x_h\left((\tfrac{i-1}{n}, \tfrac{i}{n}]\right) f\left(x, \tfrac{i}{n}\right) \\
&\qquad - \Big(\int_\mathbb{R} f(x, \theta)\ \mathrm{d}\mu^x_h(\theta)\Big)\Big|\ \mathrm{d}x \\
&\leqq \int_\Omega \Big(\int_{[0,\frac{1}{n}]} |f(x, \tfrac{1}{n}) - f(x, \theta)|\ \mathrm{d}\mu^x_h(\theta)\Big)\ \mathrm{d}x \\
&\qquad + \int_\Omega \Big(\sum_{i=2}^{n} \int_{(\frac{i-1}{n},\frac{i}{n}]} |f(x, \tfrac{i}{n}) - f(x, \theta)|\ \mathrm{d}\mu^x_h(\theta)\Big)\,\mathrm{d}x \\
&\leqq \omega_f(1/n) \int_\Omega \mu^x_h([0, 1])\ \mathrm{d}x = \omega_f(1/n)|\Omega| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

For almost every $x \in \Omega$, we set

$$A^x_1 := \mu^x_{1,n}([0, \tfrac{1}{n}]), \quad A^x_i := \mu^x_{1,n}((\tfrac{i}{n}, \tfrac{i+1}{n}]) \quad \text{for every } i = 2, \ldots, n,$$
$$B^x_1 := \mu^x_{2,n}([0, \tfrac{1}{n}]), \quad B^x_j := \mu^x_{2,n}((\tfrac{j}{n}, \tfrac{j+1}{n}]) \quad \text{for every } j = 2, \ldots, n.$$

Since $\mu_1 \succeq \mu_2$, we deduce that

$$\sum_{i=1}^{k} A^x_i \leqq \sum_{j=1}^{k} B^x_j \quad \text{for every } k = 1, \ldots, n,$$

$$\sum_{i=1}^{n} A_i^x = \mu_1^x([0,1]) = 1 = \mu_2^x([0,1]) = \sum_{j=1}^{n} B_j^x,$$
$$0 \leqq A_i^x \leqq 1, \quad 0 \leqq B_j^x \leqq 1, \quad \text{for every } i \text{ and } j,$$

for almost every $x \in \Omega$. Hence $(A_i^x)_i$ and $(B_j^x)_j$ satisfy the hypotheses of Theorem 5 in Appendix A, and we can find a matrix $(C_{ij}^x)_{ij}$ with measurable entries in [0, 1] such that

$$\sum_{i=1}^{n} C_{ij}^x = B_j^x, \tag{3.8}$$
$$\sum_{j=1}^{n} C_{ij}^x = A_i^x, \tag{3.9}$$
$$C_{ij}^x = 0 \quad \text{if } i < j. \tag{3.10}$$

Let us define

$$\mu_{12,n}^x := \sum_{i,j=1}^{n} C_{ij}^x \delta_{\left(\frac{i}{n},\frac{j}{n}\right)},$$

for almost every $x \in \Omega$. We have, therefore, that $\mu_{12,n}^x([0,1]^2) = \sum_{ij} C_{ij}^x = \sum_i A_i^x = \sum_j B_j^x = 1$, and $x \mapsto \mu_{12,n}^x(E)$ is measurable for every Borel set E. Hence, $(\mu_{12,n}^x)_x$ represents the disintegration of a Young measure on Ω with values in $[0,1]^2$. Thanks to conditions (3.8), (3.9), and (3.10), we have

$$\mu_{12,n}^x(\{\theta_1 < \theta_2\}) = \sum_{i<j} C_{ij} = 0, \tag{3.11}$$

$$[\pi_1(\mu_{12,n})]^x = \sum_{ij} C_{ij}^x \delta_{\frac{i}{n}} = \sum_i \left(\sum_j C_{ij}^x\right)\delta_{\frac{i}{n}} = \sum_i A_i^x \delta_{\frac{i}{n}} = \mu_{1,n}^x, \tag{3.12}$$

$$[\pi_2(\mu_{12,n})]^x = \sum_{ij} C_{ij}^x \delta_{\frac{j}{n}} = \sum_j \left(\sum_i C_{ij}^x\right)\delta_{\frac{j}{n}} = \sum_j B_j^x \delta_{\frac{j}{n}} = \mu_{2,n}^x, \tag{3.13}$$

for almost every $x \in \Omega$. Since $(\mu_{12,n})_n$ are Young measures with compact support and hence have equibounded moments of every order, we can always find a subsequence $(\mu_{12,n_k})_k$ and a Young measure $\mu_{12} \in Y(\Omega; [0,1]^2)$ such that $\mu_{12,n} \rightharpoonup \mu_{12}$ weakly*. Since $\mu_{1,n} \rightharpoonup \mu_1$ and $\mu_{2,m} \rightharpoonup \mu_2$ weakly*, thanks to the projections properties (3.12) and (3.13), we deduce that

$$\pi_1(\mu_{12}) = \mu_1 \quad \pi_2(\mu_{12}) = \mu_2,$$

and hence μ_{12} satisfies the projection property (3.6). Eventually, we observe that for every open subset E of Ω, $E \times \{\theta_1 < \theta_2\}$ is open, and hence $\mu_{12}(E \times \{\theta_1 < \theta_2\}) \leqq \liminf_k \mu_{12,n_k}(E \times \{\theta_1 < \theta_2\}) = 0$, thanks to identity (3.11). This implies that $\mu_{12}^x(\{\theta_1 < \theta_2\}) = 0$ for almost every $x \in \Omega$, that is, (3.7). $\square$

Remark 1. (Order of the barycentres) Note that $\mu_1 \succeq \mu_2$ implies $\mathrm{bar}(\mu_1) \geqq \mathrm{bar}(\mu_2)$ almost everywhere in Ω, whereas the opposite implication is false. Indeed, if $\mu_1 \succeq \mu_2$, by Lemma 2 there exists $\mu_{12} \in Y(\Omega; [0,1]^2)$ with $\pi_i(\mu_{12}) = \mu_i$, $i = 1, 2$ and $\mu_{12}^x(\{\theta_1 < \theta_2\}) = 0$, for almost every $x \in \Omega$; in particular we have

$$\int_E [\mathrm{bar}(\mu_1) - \mathrm{bar}(\mu_2)]\, \mathrm{d}x = \int_{E\times[0,1]} \theta_1\, \mathrm{d}\mu_1(x,\theta_1) - \int_{E\times[0,1]} \theta_2\, \mathrm{d}\mu(x,\theta_2)$$
$$= \int_{E\times[0,1]^2} (\theta_1 - \theta_2)\, \mathrm{d}\mu_{12}(x,\theta_1,\theta_2) = \int_{E\times\{\theta_1 \geqq \theta_2\}} (\theta_1 - \theta_2)\, \mathrm{d}\mu_{12}(x,\theta_1,\theta_2) \geqq 0,$$

for every measurable subset E of Ω. This implies $\mathrm{bar}(\mu_1) \geqq \mathrm{bar}(\mu_2)$ almost everywhere in Ω. On the other hand, let us consider

$$\mu_1^x := \frac{1}{2}\delta_{1/4} + \frac{1}{2}\delta_{3/4} \quad \text{for a.e. } x \in \Omega$$
$$\mu_2^x := \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \quad \text{for a.e. } x \in \Omega.$$

We have $\mathrm{bar}(\mu_1) = \mathrm{bar}(\mu_2) \equiv \frac{1}{2}$ almost everywhere in Ω, but $\mu_1^x(0,1] = 1 > \mu_2^x(0,1] = 1/2$ and $\mu_1^x(3/4,1] = 0 < \mu_2^x(3/4,1] = 1/2$, for almost every $x \in \Omega$, therefore $\mu_1 \not\succeq \mu_2 \not\succeq \mu_1$.

3.2. Sequences of Functions Generating a Young Measure

Let us recall (see [32, Theorem 7.7]) that any Young measure $\mu \in Y^p(\Omega; \mathbb{R}^N)$ can be generated by a suitable sequence of functions $(z_n)_n \subset L^p(\Omega; \mathbb{R}^N)$, in the sense that $\boldsymbol{\delta}_{z_n} \rightharpoonup \mu$ p-weakly*, as $n \to \infty$.

In particular, given a measure $\mu_{12} \in Y(\Omega; [a,b] \times [c,d])$, for $-\infty < a < b < \infty$, $-\infty < c < d < \infty$, there exists a sequence $(z_n^1, z_n^2)_n$ of pairs of functions in $L^1(\Omega; [a,b] \times [c,d])$, such that $\boldsymbol{\delta}_{(z_n^1(x), z_n^2(x))} \rightharpoonup \mu_{12}$ weakly*. The question we want to consider in this section is the following: assume that we have already fixed a sequence $(\bar{z}_n^1)_n$ generating the projection of μ_{12} over $\Omega \times [a,b]$. Is it possible to construct a sequence $(z_n^2)_n$ such that $\boldsymbol{\delta}_{(\bar{z}_n^1, z_n^2)} \rightharpoonup \mu_{12}$ weakly* as $n \to \infty$? An affirmative answer to this question is given by the following.

Theorem 3. (Measure reconstruction) *Let Ω be a bounded open subset of $\mathbb{R}^d$, and μ a measure in $Y(\Omega; \mathbb{R}^2)$ with support contained in $\Omega \times [a,b] \times [c,d]$, for $-\infty < a < b < \infty$, $-\infty < c < d < \infty$. We write μ_1 for $\pi_1(\mu)$ and μ_2 for $\pi_2(\mu)$, where $\pi_1(x,\theta,\xi) := (x,\theta)$ and $\pi_2(x,\theta,\xi) := (x,\xi)$, for every $(x,\theta,\xi) \in \Omega \times \mathbb{R}^2$.*

Given a sequence $(z_n^1)_n$ in $L^\infty(\Omega; [a,b])$ such that

$$\boldsymbol{\delta}_{z_n^1} \rightharpoonup \mu_1 \quad \textit{weakly*}, \tag{3.14}$$

there exists a sequence $(z_n^2)_n$ in $L^\infty(\Omega; [c,d])$ such that

$$\boldsymbol{\delta}_{(z_n^1, z_n^2)} \rightharpoonup \mu \quad \textit{weakly*}.$$

Proof. For every m, we consider a finite partition of measurable sets $(\Omega_i^m)_{i=1}^{I(m)}$ of Ω, and two finite partitions of intervals $(H_j^m)_{j=1}^{J(m)}$ of $[a-1, b+1]$ and $(K_k^m)_{k=1}^{K(m)}$ of $[c, d]$. We choose these three partitions in such a way that the diameter of each Ω_i^m, H_j^m, and K_k^m is less than $1/m$. Since the support of μ_1 is strictly contained in $[a-1, b+1]$, it is not difficult to see that we can always choose $(H_j^m)_j$ such that $\mu_1(\Omega_i^m \times \partial H_j^m) = 0$ for every $i = 1, \dots, I(m)$ and $j = 1, \dots J(m)$. Hence, $\mu_1(\partial(\Omega_i^m \times H_j^m)) = 0$ for every i and j, thanks to the projection property on Ω satisfied by the Young measure μ_1.

We now fix $n \in \mathbb{N}$ and, for every $i = 1, \dots, I(m)$, we define a family of subsets of Ω_i^m, which we term $(\Omega_{ij}^{m,n})_{j=1}^{J(m)}$, by setting

$$\Omega_{ij}^{m,n} := \{x \in \Omega_i^m : z_n^1(x) \in H_j^m\},$$

for every $j = 1, \dots, J(m)$. Since $(H_j^m)_j$ are pairwise disjoint, $(\Omega_{ij}^{m,n})_j$ are pairwise disjoint, too, and $\bigcup_{j=1}^{J(m)} \Omega_{ij}^{m,n} = \Omega_i^m$. We observe that $\sum_{k=1}^{K(m)} \mu(\Omega_i^m \times H_j^m \times K_k^m) = \mu(\bigcup_{k=1}^{K(m)} \Omega_i^m \times H_j^m \times K_k^m) = \mu(\Omega_i^m \times H_j^m \times [c, d])$, hence, if $\mu(\Omega_i^m \times H_j^m \times [c, d]) > 0$ we have

$$\frac{\mu(\Omega_i^m \times H_j^m \times K_k^m)}{\mu(\Omega_i^m \times H_j^m \times [c, d])} \leqq 1 \quad \text{for every } k = 1, \dots, K(m),$$

$$\sum_{k=1}^{K(m)} \frac{\mu(\Omega_i^m \times H_j^m \times K_k^m)}{\mu(\Omega_i^m \times H_j^m \times [c, d])} = 1.$$

Let us set $\mathcal{A} := \{(i, j) : \mu(\Omega_i^m \times H_j^m \times [c, d]) = 0\}$. Therefore, for every $(i, j) \notin \mathcal{A}$, it is possible to find a family of pairwise disjoint subsets of $\Omega_{ij}^{m,n}$, which we denote by $(\Omega_{ijk}^{m,n})_{k=1}^{K(m)}$, such that $\bigcup_{k=1}^{K(m)} \Omega_{ijk}^{m,n} = \Omega_{ij}^{m,n}$, and satisfying

$$|\Omega_{ijk}^{m,n}| = \frac{\mu(\Omega_i^m \times H_j^m \times K_k^m)}{\mu(\Omega_i^m \times H_j^m \times [c, d])} |\Omega_{ij}^{m,n}|.$$

Let us define $z_n^{m,2}(x) := \xi_k^m$, for some $\xi_k^m \in K_k^m$, whenever $x \in \Omega_{ijk}^{m,n}$ for $(i, j) \notin \mathcal{A}$, and $z_n^{m,2}(x) := c$ whenever $x \in \Omega_{ij}^{m,n}$ for $(i, j) \in \mathcal{A}$.

Since $\boldsymbol{\delta}_{z_n^1} \rightharpoonup \mu_1$, thanks to assumption (3.14), and $\mu_1(\partial(\Omega_i^m \times H_j^m)) = 0$ for every i, j, we have

$$|\Omega_{ij}^{m,n}| = \boldsymbol{\delta}_{z_n^1}(\Omega_i^m \times H_j^m) \longrightarrow \mu_1(\Omega_i^m \times H_j^m) = \mu(\Omega_i^m \times H_j^m \times [c, d]),$$

as $n \to \infty$. Therefore, for every m, there exists n_m such that

$$\left| \frac{|\Omega_{ij}^{m,n}|}{\mu(\Omega_i^m \times H_j^m \times [c, d])} - 1 \right| \leqq \frac{1}{m} \quad \text{for every } (i, j) \notin \mathcal{A}, \tag{3.15}$$

$$\sum_{(i,j) \in \mathcal{A}} |\Omega_{ij}^{m,n}| \leqq \frac{1}{m}. \tag{3.16}$$

whenever $n \geqq n_m$. Without loss of generality we can assume that $(n_m)_m$ is an increasing sequence of integers. We are now ready to define z_n^2 by setting $z_n^2(x) := z_n^{m,2}(x)$ whenever $n_m \leqq n < n_{m+1}$.

We now have to show that for every $f \in C_0(\Omega \times \mathbb{R}^2)$ and for every $\varepsilon > 0$ there exists N such that

$$\left| \int_{\Omega\times\mathbb{R}^2} f(x,\theta,\xi)\,\mathrm{d}\boldsymbol{\delta}_{(z_n^1(x),z_n^2(x))} - \int_{\Omega\times\mathbb{R}^2} f(x,\theta,\xi)\,\mathrm{d}\mu(x,\theta,\xi) \right| \leqq \varepsilon, \quad (3.17)$$

whenever $n \geqq N$.

Given n, let m be such that $n_m \leqq n < n_{m+1}$. Then we have

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^2} f(x,\theta,\xi)\ \mathrm{d}\boldsymbol{\delta}_{(z_n^1,z_n^2)}(x,\theta,\xi) = \int_{\Omega\times\mathbb{R}^2} f(x,\theta,\xi)\ \mathrm{d}\boldsymbol{\delta}_{(z_n^1,z_n^{m,2})}(x,\theta,\xi) \\
&\quad = \sum_{(i,j)\notin\mathcal{A}} \sum_k \int_{\Omega_{ijk}^{m,n}} f(x, z_n^1(x), \xi_k^m)\ \mathrm{d}x + \sum_{(i,j)\in\mathcal{A}} \int_{\Omega_{ij}^{m,n}} f(x, z_n^1(x), c)\,\mathrm{d}x.
\end{aligned}$$

In particular, for every $x_i^m \in \Omega_i^m$ and $\theta_j^m \in H_j^m$ we have

$$\begin{aligned}
&\Big| \int_{\Omega\times\mathbb{R}^2} f(x,\theta,\xi)\ \mathrm{d}\boldsymbol{\delta}_{(z_n^1,z_n^2)}(x,\theta,\xi) - \sum_{(i,j)\notin\mathcal{A}} \sum_k \int_{\Omega_{ijk}^{m,n}} f(x_i^m,\theta_j^m,\xi_k^m)\,\mathrm{d}x \\
&\qquad - \sum_{(i,j)\in\mathcal{A}} \int_{\Omega_{ij}^{m,n}} f(x_i^m,\theta_j^m,c)\,\mathrm{d}x \Big| \\
&\quad \leqq \sum_{(i,j)\notin\mathcal{A}} \sum_k \int_{\Omega_{ijk}^{m,n}} |f(x, z_n^1(x), \xi_k^m) - f(x_i^m,\theta_j^m,\xi_k^m)|\ \mathrm{d}x \\
&\qquad + \sum_{(i,j)\in\mathcal{A}} \int_{\Omega_{ij}^{m,n}} |f(x, z_n^1(x), c) - f(x_i^m,\theta_j^m,c)|\,\mathrm{d}x \\
&\quad \leqq \omega_f(2/m)\Big[\sum_{(i,j)\notin\mathcal{A}} \sum_k |\Omega_{ijk}^{m,n}| + \sum_{(i,j)\in\mathcal{A}} |\Omega_{ij}^{m,n}| \Big] = \omega_f(2/m) \sum_{ij} |\Omega_{ij}^{m,n}| \\
&\quad = \omega_f(2/m) \sum_i |\Omega_i^m| = \omega_f(2/m)|\Omega|, \qquad (3.18)
\end{aligned}$$

where ω_f is a modulus of continuity of f (f is uniformly continuous). Using the construction of $\Omega_{ijk}^{m,n}$, and the estimates (3.15) and (3.16), we get

$$\begin{aligned}
&\Big| \sum_{(i,j)\notin\mathcal{A}} \sum_k \int_{\Omega_{ijk}^{m,n}} f(x_i^m,\theta_j^m,\xi_k^m)\,\mathrm{d}x + \sum_{(i,j)\in\mathcal{A}} \int_{\Omega_{ij}^{m,n}} f(x_i^m,\theta_j^m,c)\,\mathrm{d}x \\
&\qquad - \sum_{(i,j)\notin\mathcal{A}} \sum_k f(x_i^m,\theta_j^m,\xi_k^m)\mu(\Omega_i^m \times H_j^m \times K_k^m) \\
&\qquad - \sum_{(i,j)\in\mathcal{A}} f(x_i^m,\theta_j^m,c)\mu(\Omega_i^m \times H_j^m \times [c,d]) \Big|
\end{aligned}$$

$$
\begin{aligned}
&\leqq \Big| \sum_{(i,j)\notin\mathcal{A}} \sum_k f(x_i^m,\theta_j^m,\xi_k^m)|\Omega_{ijk}^{m,n}| \\
&\quad - \sum_{(i,j)\notin\mathcal{A}} \sum_k f(x_i^m,\theta_j^m,\xi_k^m)\mu(\Omega_i^m \times H_j^m \times K_k^m)\Big| \\
&\quad + \Big| \sum_{(i,j)\in\mathcal{A}} f(x_i^m,\theta_j^m,c)|\Omega_{ij}^{m,n}| - 0\Big| \\
&\leqq \|f\|_\infty \sum_{(i,j)\notin\mathcal{A}} \sum_k \mu(\Omega_i^m \times H_j^m \times K_k^m)\Big[\frac{|\Omega_{ij}^{m,n}|}{\mu(\Omega_i^m \times H_j^m \times [c,d])} - 1\Big] \\
&\quad + \|f\|_\infty \sum_{(i,j)\in\mathcal{A}} |\Omega_{ij}^{m,n}| \\
&= \frac{\|f\|_\infty}{m}\Big[\sum_{ij} \mu(\Omega_i^m \times H_j^m \times [c,d]) + 1\Big] = \frac{\|f\|_\infty}{m}[|\Omega| + 1].
\end{aligned}
\tag{3.19}
$$

Finally, we have

$$
\begin{aligned}
&\Big| \sum_{(i,j)\notin\mathcal{A}} \sum_k f(x_i^m,\theta_j^m,\xi_k^m)\mu(\Omega_i^m \times H_j^m \times K_k^m) \\
&\quad + \sum_{(i,j)\in\mathcal{A}} f(x_i^m,\theta_j^m,c)\mu(\Omega_i^m \times H_j^m \times [c,d]) - \int_{\Omega\times\mathbb{R}^2} f(x,\theta,\xi)\, \mathrm{d}\mu(x,\theta,\xi)\Big| \\
&\leqq \sum_{(i,j)\notin\mathcal{A}} \sum_k \int_{\Omega_i^m \times H_j^m \times K_k^m} |f(x_i^m,\theta_j^m,\xi_k^m) - f(x,\theta,\xi)|\, \mathrm{d}\mu(x,\theta,\xi) \\
&\quad + \sum_{(i,j)\in\mathcal{A}} \int_{\Omega_i^m \times H_j^m \times [c,d]} |f(x,\theta,\xi)|\, \mathrm{d}\mu(x,\theta,\xi) \\
&\leqq \omega_f(3/m) \sum_{(i,j)\notin\mathcal{A}} \sum_k \mu(\Omega_i^m \times H_j^m \times K_k^m) \\
&\quad + \|f\|_\infty \sum_{(i,j)\in\mathcal{A}} \mu(\Omega_i^m \times H_j^m \times [c,d]) \\
&= \omega_f(3/m)|\Omega|.
\end{aligned}
\tag{3.20}
$$

Therefore, putting together the estimates (3.18), (3.19), and (3.20) we obtain

$$
\begin{aligned}
&\Big| \int_{\Omega\times\mathbb{R}^2} f(x,\theta,\xi)\, \mathrm{d}\boldsymbol{\delta}_{(z_n^1, z_n^{m,2})}(x,\theta,\xi) - \int_{\Omega\times\mathbb{R}^2} f(x,\theta,\xi)\, \mathrm{d}\mu(x,\theta,\xi)\Big| \\
&\qquad \leqq |\Omega|\Big[\omega_f\Big(\frac{2}{m}\Big) + \frac{\|f\|_\infty}{m} + \omega_f\Big(\frac{3}{m}\Big)\Big] + \frac{\|f\|_\infty}{m}.
\end{aligned}
$$

In particular, for fixed $\varepsilon > 0$, condition (3.17) is satisfied for m sufficiently large, $m \geqq M$. Hence it is enough to choose N such that $n_M \leqq N \leqq n_{M+1}$. In this way, for every $n \geqq N$, we have $n_m \leqq n < n_{m+1}$ for some $m \geqq M$ and hence (3.17) holds true for every $n \geqq N$. □

Corollary 1. (Measure reconstruction with order) *In addition to the hypotheses of Theorem 3, if $a = c,\ b = d$, and μ satisfies the condition*

$$\mu^x(\{(\theta, \xi) \in \mathbb{R}^2 \,:\, \theta < \xi\}) = 0 \quad \text{for a.e. } x \in \Omega, \tag{3.21}$$

we can construct the sequence $(z_n^2)_n$ with the property

$$z_n^1(x) \geqq z_n^2(x) \quad \text{for a.e. } x \in \Omega.$$

Proof. For every m, we can assume that $(H_j^m)_j$ is ordered in the sense that $\theta_{j+1} > \theta_j$ whenever $\theta_j \in H_j^m,\ \theta_{j+1} \in H_{j+1}^m$. Since $[c, d] = [a, b]$, we can choose $K_k^m := H_k^m \cap [a, b]$ for every k. If $(i, j) \in \mathcal{A},\ z_n^{2,m}(x) = a \leqq z_n^1(x)$ for almost every $x \in \Omega_{ij}^{m,n}$. So let us consider from now on $(i, j) \notin \mathcal{A}$. Since $\mu^x(\{\theta < \xi\}) = 0$ for almost every $x \in \Omega$, due to assumption (3.21), we have that $\mu(\Omega_i^m \times H_j^m \times (H_k^m \cap [a, b])) = 0$ for every i and every $k > j$. In particular, $|\Omega_{ijk}^{m,n}| = 0$ whenever $k > j$. Nothing changes in the proof of Theorem 3 if we take ξ_k^m in the closure of $K_k^m \cap [a, b]$. In this way, for $k = j$ we are able to choose ξ_j^m with the property $\xi_j^m \leqq z_n^1(x)$ whenever $x \in \Omega_{ij}^{m,n}$ (notice that $z_n^1(x) \in [a, b]$), so that $z_n^{2,m}(x) \leqq z_n^1(x)$ whenever $x \in \Omega_{ijj}^{m,n}$. Finally, if $k < j$, then for every $\xi_k^m \in H_k^m \cap [a, b]$ we have $\xi_k^m \leqq z_n^1(x)$ whenever $x \in \Omega_{ij}^{m,n}$ and hence $z_n^{2,m}(x) \leqq z_n^1(x)$ whenever $x \in \Omega_{ijk}^{m,n}$. In conclusion, $z_n^{2,m}(x) \leqq z_n^1(x)$ for almost every $x \in \Omega$, and hence $z_n^2(x) \leqq z_n^1(x)$ for almost every $x \in \Omega$, too. □

3.3. Admissible Set in Terms of Young Measures

We now introduce the admissible set for the generalized notion of evolution we will consider. We recall that $\mu \in Y^2(\Omega; \mathbb{R}^{d\times d})$ is an H^1-gradient Young measure (H^1-GYM), if there exists a bounded sequence $(v_n)_n \in H^1(\Omega; \mathbb{R}^d)$ such that $\boldsymbol{\delta}_{\nabla v_n} \rightharpoonup \mu$ 2-weakly* as $n \to \infty$.

Definition 3. (*Admissible set*) Given a time interval $[0, T]$ and $\varphi\colon [0, T] \to W^{1,p}(\Omega; \mathbb{R}^d)$, for $p > 2$, we define $AY([0, T], \varphi)$ as the set of all $\nu \in Y^2(\Omega; \mathbb{R} \times \mathbb{R}^{d\times d}_{\text{sym}})^{[0,T]}$ such that for every $t \in [0, T]$ there exists a measure $\tilde{\nu}_t \in Y^2(\Omega; \mathbb{R} \times \mathbb{R}^{d\times d})$ with

$$\nu_t = S(\tilde{\nu}_t), \tag{3.22}$$
$$\operatorname{supp} \pi_1(\tilde{\nu}_t) = \operatorname{supp} \pi_1(\nu_t) \subseteq \Omega \times [0, 1], \tag{3.23}$$
$$\pi_2(\tilde{\nu}_t) \text{ is a } H^1\text{-GYM}, \tag{3.24}$$
$$\operatorname{bar}(\pi_2(\tilde{\nu}_t)) = \nabla v \quad \text{with } v \in \varphi(t) + H_0^1(\Omega; \mathbb{R}^d), \tag{3.25}$$

where $S(x, \theta, F) := (x, \theta, \frac{F+F^T}{2})$ for every $(x, \theta, F) \in \Omega \times \mathbb{R} \times \mathbb{R}^{d\times d}$, and π_1 and π_2 are projections, $\pi_1 : \Omega \times \mathbb{R} \times \mathbb{R}^{d\times d} \to \Omega \times \mathbb{R}$ and $\pi_2 : \Omega \to \mathbb{R} \times \mathbb{R}^{d\times d} \to \Omega \times \mathbb{R}^{d\times d}$, respectively.

From [13, Theorem 3.1], $\tilde{\nu}_t$ satisfies properties (3.23) and (3.24), (3.25) if and only if there exist a bounded sequence $(z_n)_n$ in $L^\infty(\Omega; [0, 1])$ and a bounded sequence $(v_n)_n$ in $H^1(\Omega; \mathbb{R}^{d\times d})$ such that $\boldsymbol{\delta}_{(z_n,\nabla v_n)} \rightharpoonup \tilde{\nu}_t$ 2-weakly* as $n \to \infty$. Moreover, by using, for instance, [1, Lemma 11.4.1], it is possible to choose $(v_n)_n$ in $\varphi(t) + H^1_0(\Omega; \mathbb{R}^{d\times d})$. Note that eventually $\boldsymbol{\delta}_{(z_n,\nabla v_n)} \rightharpoonup \tilde{\nu}_t$ 2-weakly* implies $\boldsymbol{\delta}_{(z_n,e(v_n))} \rightharpoonup S(\tilde{\nu}_t)$ 2-weakly*.

4. Main Result

We shall now aim at introducing the existence result for quasi-static damage evolution.

Before giving the definition of quasi-static damage evolution and stating the main result, we need to establish some extra notation.

Given $\nu \in Y^2(\Omega; \mathbb{R} \times \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\mu_{12} \in Y^1(\Omega; [0, 1]^2)$, we set

$$\langle W, \nu\rangle := \int_{\Omega\times\mathbb{R}\times\mathbb{R}^{d\times d}_{\mathrm{sym}}} W(\theta, \varepsilon)\, \mathrm{d}\nu(x, \theta, \varepsilon),$$

$$\langle d, \mu_{12}\rangle := \int_{\Omega\times\mathbb{R}^2} d(\theta_1, \theta_2)\, \mathrm{d}\mu_{12}(x, \theta_1, \theta_2).$$

Given $\mu_1, \mu_2 \in Y^1(\Omega; [0, 1])$, we define

$$\mathbb{D}(\mu_1, \mu_2) := \begin{cases} \rho\left[\int_{\Omega\times\mathbb{R}} \theta\, \mathrm{d}\mu_1(x, \theta) - \int_{\Omega\times\mathbb{R}} \theta\, \mathrm{d}\mu_2(x, \theta)\right] & \text{if } \mu_1 \succeq \mu_2 \\ \infty & \text{otherwise.} \end{cases}$$

The distance $\mathbb{D}(\mu_1, \mu_2)$ coincides with the infimum of $\langle d, \mu_{12}\rangle$ for μ_{12} varying in the set of measures in $Y^1(\Omega; [0, 1]^2)$ such that $\pi_1(\mu_{12}) = \mu_1$ and $\pi_2(\mu_{12}) = \mu_2$, where $\pi_1(x, \theta_1, \theta_2) := (x, \theta_1)$ and $\pi_2(x, \theta_1, \theta_2) := (x, \theta_2)$ for every $(x, \theta_1, \theta_2) \in \Omega\times\mathbb{R}^2$. Indeed, this is true if $\mu_1 \not\succeq \mu_2$, because, thanks to Theorem 2 and to the definition of d, in this case we have $\langle d, \mu_{12}\rangle = \infty$ for every μ_{12} satisfying the required projection properties. On the other hand, if $\mu_1 \succeq \mu_2$, by Theorem 2 there exists a measure μ_{12} satisfying the projection properties and with $\mu^x_{12}(\{\theta_1 < \theta_2\}) = 0$ for almost every $x \in \Omega$. Therefore, for every such measure μ_{12}, we have $\langle d, \mu_{12}\rangle < \infty$ and

$$\begin{aligned}
\langle d, \mu_{12}\rangle &= \int_{\Omega\times[0,1]^2} d(\theta_1, \theta_2)\, \mathrm{d}\mu_{12}(x, \theta_1, \theta_2)\\
&= \int_\Omega \Big(\int_{\{\theta_1\geqq\theta_2\}} d(\theta_1, \theta_2)\, \mathrm{d}\mu^x_{12}(\theta_1, \theta_2)\Big)\, \mathrm{d}x\\
&= \int_\Omega \Big(\int_{\{\theta_1\geqq\theta_2\}} \rho(\theta_1 - \theta_2)\, \mathrm{d}\mu^x_{12}(\theta_1, \theta_2)\Big)\, \mathrm{d}x\\
&= \rho\int_{\Omega\times[0,1]^2} (\theta_1 - \theta_2)\, \mathrm{d}\mu_{12}(x, \theta_1, \theta_2)\\
&= \rho\Big[\int_{\Omega\times\mathbb{R}} \theta_1\, \mathrm{d}\mu_1(x, \theta_1) - \int_{\Omega\times\mathbb{R}} \theta_2\, \mathrm{d}\mu_2(x, \theta_2)\Big].
\end{aligned}$$

Therefore, $\langle d, \mu_{12}\rangle$ is independent of the choice of μ_{12}, provided it has the required order property and coincides with $\mathbb{D}(\mu_1, \mu_2)$. In other words, $\mathbb{D}(\mu_1, \mu_2)$ corresponds to a Wasserstein-like distance associated with d between μ_1 and μ_2 (see for example [25]). Note that we may have $\mathbb{D}(\mu_1, \mu_2) = \infty$ even in cases where $\mathcal{D}(\mathrm{bar}(\mu_1), \mathrm{bar}(\mu_2)) < \infty$, because $\mu_1 \succeq \mu_2$ is a stronger condition than $\mathrm{bar}(\mu_1) \geqq \mathrm{bar}(\mu_2)$ almost everywhere in Ω, as explained in Remark 1.

Given a measure $\nu \in Y^2(\Omega; [0,1] \times \mathbb{R}^{d\times d})$, we will denote the projection of ν on $\Omega \times [0,1]$ by $\pi_1(\nu)$. We are now ready to define our solution notion for the quasi-static problem.

Definition 4. (*Quasi-static evolution*) Given $\varphi\colon [0,T] \times \Omega \to \mathbb{R}^d$, $z_0\colon \Omega \to [0,1]$, $v_0\colon \Omega \to \mathbb{R}^d$, and $T > 0$, a *quasi-static damage evolution* with boundary datum φ and initial condition (z_0, v_0), in the time interval $[0,T]$, is $\nu \in AY([0,T], \varphi)$, satisfying the following conditions:

(E0) *initial condition*: $\nu_0 = \delta_{(z_0, e(v_0))}$,

(E1) *irreversibility*: $\pi_1(\nu_s) \succeq \pi_1(\nu_t)$, whenever $0 \leqq s < t \leqq T$,

(E2) *translational stability*: for every $t \in [0,T]$, we have

$$\langle W, \nu_t\rangle \leqq \langle W, \mathrm{Tr}_{(\tilde{z}, e(\tilde{u}))}(\nu_t)\rangle + \mathbb{D}(\pi_1(\nu_t), \mathrm{Tr}_{\tilde{z}}(\pi_1(\nu_t))),$$

for every $\tilde{z} \in L^1(\Omega)$ and every $\tilde{u} \in H^1_0(\Omega; \mathbb{R}^d)$,

(E3) *global-stability for the internal variable*: for every $t \in [0,T]$, we have

$$\langle W, \nu_t\rangle \leqq \langle W, (\tilde{\mu}^x \otimes \delta_{e(\tilde{v})(x)})_{x\in\Omega}\rangle + \mathbb{D}(\pi_1(\nu_t), \tilde{\mu}), \tag{4.1}$$

for every $\tilde{v} \in \varphi(t) + H^1_0(\Omega; \mathbb{R}^d)$, and every $\tilde{\mu} \in Y(\Omega; \mathbb{R})$,

(E4) *energy equality*: for every $t \in [0,T]$ the map

$$t \mapsto \langle \sigma(t), e(\dot{\varphi}(t))\rangle \tag{4.2}$$

is measurable on $[0,T]$, where $\sigma(t)$ is the function defined by

$$\sigma(t)(x) := \int_{\mathbb{R}\times\mathbb{R}^{d\times d}_{\mathrm{sym}}} \frac{\partial W}{\partial \varepsilon}(\theta, \varepsilon)\, \mathrm{d}\nu_t^x(\theta, \varepsilon) \quad \text{for a.e. } x \in \Omega.$$

Moreover, for every $t \in [0,T]$ we have

$$\langle W, \nu_t\rangle + \mathrm{Diss}(\nu; 0, t) = \mathcal{W}(z_0, v_0) + \int_0^t \langle \sigma(s), e(\dot{\varphi}(s))\rangle\, \mathrm{d}s,$$

with $\mathrm{Diss}(\nu; 0,t) := \sup \sum_{i=1}^k \mathbb{D}(\pi_1(\nu_{t_{i-1}}), \pi_1(\nu_{t_i}))$, where the supremum is taken among all finite partitions $0 = t_0 < \cdots < t_k = t$.

Remark 2. Due to irreversibility property (E1), the total dissipation $\mathrm{Diss}(\nu; 0, t)$ of the quasi-static evolution ν in a time interval $[0,t]$ reduces to $\mathbb{D}(\pi_1(\nu_0), \pi_1(\nu_t))$, therefore we have

$$\mathrm{Diss}(\nu; 0, t) = \rho\Big[\int_\Omega z_0\, \mathrm{d}x - \int_{\Omega\times\mathbb{R}} \theta\, \mathrm{d}\pi_1(\nu_t)(x,\theta)\Big].$$

The main result of this paper reads as follows.

Theorem 4. (Existence of a quasi-static evolution) *Let* $T > 0$, $p > 2$, $\varphi \in AC([0,T]; W^{1,p}(\Omega;\mathbb{R}^d))$, $z_0 \in L^1(\Omega;[0,1])$, *and* $v_0 \in \varphi(0) + H^1_0(\Omega;\mathbb{R}^d)$ *be such that*

$$\mathcal{W}(z_0, v_0) \leqq \mathcal{W}(\tilde{z}, \tilde{v}) + \mathcal{D}(z_0, \tilde{z}), \tag{4.3}$$

for every $\tilde{z} \in L^1(\Omega)$ *and every* $\tilde{v} \in \varphi(0) + H^1_0(\Omega;\mathbb{R}^d)$. *Then there exists a quasi-static evolution with boundary datum* φ *and initial condition* (z_0, v_0) *in the time interval* $[0,T]$.

The proof is obtained via time discretization, incremental minimization, and passage to the limit and is detailed in Section 5.

5. Proof of the Existence Theorem 4

5.1. The Incremental Minimum Problem

Let us fix a time step $\tau := T/n$, and let $t^i_\tau := i\tau$ and $\varphi^i_\tau := \varphi(t^i_\tau)$, for every $i = 0,\dots,n$. We will define (z^i_τ, v^i_τ) iteratively: set $(z^0_\tau, v^0_\tau) := (z_0, v_0)$, and, for $i > 0$, define (z^i_τ, v^i_τ) as a minimizer (see Lemma 3 below) of the functional

$$\mathcal{F}^i_\tau(z, v) := \mathcal{W}(z, e(v)) + \mathcal{D}(z^{i-1}_\tau, z), \tag{5.1}$$

among all $z \in L^1(\Omega)$ and $v \in \varphi^i_\tau + H^1_0(\Omega;\mathbb{R}^d)$.

Lemma 3. (Incremental minimization) *Let* (z_0, v_0) *be as in Theorem 4. Then, for every* i *the functional* $\mathcal{F}^i_\tau$ *has a minimizer* (z, v) *in* $L^1(\Omega) \times (\varphi^i_\tau + H^1_0(\Omega;\mathbb{R}^d))$. *Moreover,* (z, v) *satisfies the following properties:*

$$0 \leqq z \leqq z^{i-1}_\tau \textit{ a.e. in } \Omega, \tag{5.2}$$

$$v \textit{ is a } \frac{C_W}{c_W}\textit{-cubic quasi-minimum of the functional } v \mapsto \int_\Omega |e(v)|^2. \tag{5.3}$$

Remark 3. In particular, for every i and τ, we have that $0 \leqq z^i_\tau(x) \leqq z_0(x) \leqq 1$ for almost every $x \in \Omega$, since $z_0(x) \in [0,1]$ for almost every $x \in \Omega$.

Proof. Let us first observe that whenever $z^{i-1}_\tau \geqq 0$ almost everywhere in Ω, we have

$$\mathcal{F}^i_\tau(z, v) \geqq \mathcal{F}^i_\tau((z \wedge z^{i-1}_\tau)^+, v), \tag{5.4}$$

for every $(z, v) \in L^1(\Omega) \times H^1(\Omega;\mathbb{R}^d)$. Indeed, $\mathcal{F}^i_\tau(z, v) < \infty$ if and only if $z \leqq z^{i-1}_\tau$ almost everywhere in Ω, hence $\mathcal{F}^i_\tau(z, v) \geqq \mathcal{F}^i_\tau(z \wedge z^{i-1}_\tau, v)$. On the other hand, $W(\theta, \varepsilon) \equiv W(0, \varepsilon)$ if $\theta \leqq 0$ (see hypothesis $(W.5)$). Hence, $\mathcal{W}(z \wedge z^{i-1}_\tau, e(v)) = \mathcal{W}((z \wedge z^{i-1}_\tau)^+, e(v))$. Finally, since $z^{i-1}_\tau \geqq 0$ almost everywhere in Ω, $\mathcal{D}(z^{i-1}_\tau, (z \wedge z^{i-1}_\tau)^+) \leqq \mathcal{D}(z^{i-1}_\tau, z \wedge z^{i-1}_\tau)$ (with the strict inequality if

$z(x) \notin [0,1]$ for almost every $x \in \Omega$). In conclusion, $\mathcal{F}^i_\tau(z,v) \geqq \mathcal{F}^i_\tau(z \wedge z^{i-1}_\tau, v) \geqq \mathcal{F}^i_\tau((z \wedge z^{i-1}_\tau)^+, v)$. This implies that if z^i_τ exists, it satisfies

$$0 \leqq z^i_\tau(x) \leqq z^{i-1}_\tau(x) \quad \text{for a.e. } x \in \Omega, \tag{5.5}$$

whenever $z^{i-1}_\tau \geqq 0$ almost everywhere in Ω. Since $z_0(x) \in [0,1]$ for almost every $x \in \Omega$, by induction we get that, if z^i_τ exists, it fulfills (5.5).

Now fix $i = 1, \dots, n$, and let (z_k, v_k) be a minimizing sequence for $\mathcal{F}^i_\tau$. Then

$$\mathcal{F}^i_\tau(z_k, v_k) = \int_\Omega W(z_k, e(v_k))\,\mathrm{d}x + \mathcal{D}(z^{i-1}_\tau, z_k) < C,$$

for a suitable positive constant C. In particular, thanks to $(W.2)$ and the Korn–Poincaré inequality (3.1), the sequence $(v_k)_k$ is bounded in $\varphi^i_\tau + H^1_0(\Omega;\mathbb{R}^d)$. Since $z^{i-1}_\tau \in L^1(\Omega;[0,1])$, we can apply (5.4) in order to deduce that $((z_k \wedge z^{i-1}_\tau)^+, v_k)$ is still a minimizing sequence. Since $z^{i-1}_\tau \leqq 1$, $(z_k \wedge z^{i-1}_\tau)^+$ is bounded in $L^\infty(\Omega)$. Up to a subsequence, we can assume that v_k converges weakly in H^1 to a function $v \in \varphi^i_\tau + H^1_0(\Omega;\mathbb{R}^d)$, and $(z_k \wedge z^{i-1}_\tau)^+$ converges weakly* in L^∞ to a function z with values in $[0,1]$ almost everywhere in Ω. Since W is S-cross-quasiconvex, thanks to Lemma 1, the functional in (5.1) is sequentially lower semicontinuous with respect to the product of the weak* topology of L^∞ and the weak topology of H^1. This proves that (z,v) is a minimum of functional (5.1) and satisfies condition (5.2).

Hence, it remains to show that v is a $\frac{C_W}{c_W}$-cubic quasi-minimum of the functional $v \mapsto \int_\Omega |e(v)|^2$. Let w be a function such that $v - w \in H^1_0(\Omega \cap Q_R)$. We extend it to a function in $H^1(\Omega;\mathbb{R}^d)$ by setting $w := v$ on $\Omega \setminus Q_R$. Then (z,w) is a competitor for the minimum problem solved by (z,v). Hence,

$$\int_\Omega W(z(x), e(v)(x))\,\mathrm{d}x \leqq \int_\Omega W(z(x), e(w)(x))\,\mathrm{d}x.$$

By construction of w, this implies

$$\int_{\Omega\cap Q_R} W(z(x), e(v)(x))\,\mathrm{d}x \leqq \int_{\Omega\cap Q_R} W(z(x), e(w)(x))\,\mathrm{d}x.$$

Hence, by hypothesis $(W.2)$ on W, we get

$$c_W \int_{\Omega\cap Q_R} |e(v)(x)|^2\,\mathrm{d}x \leqq C_W \int_{\Omega\cap Q_R} |e(w)(x)|^2\,\mathrm{d}x,$$

which proves that v satisfies condition (5.3). □

Let (z_τ, v_τ) and φ_τ be the functions in $L^\infty([0,T]; L^1(\Omega) \times H^1(\Omega;\mathbb{R}^d))$ and $L^\infty([0,T]; H^1(\Omega;\mathbb{R}^d))$, respectively, defined by

$$(z_\tau(t), v_\tau(t)) := (z^i_\tau, v^i_\tau) \quad \text{if } t^i_\tau \leqq t < t^{i+1}_\tau,$$
$$\varphi_\tau(t) := \varphi^i_\tau \quad \text{if } t^i_\tau \leqq t < t^{i+1}_\tau, \quad i = 0, 1, \dots, n.$$

We define $\sigma_\tau \in L^\infty([0,T]; L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ (thanks to $(W.3)$) by

$$\sigma_\tau(t) := \frac{\partial W}{\partial \varepsilon}(z_\tau(t), e(v_\tau(t))), \tag{5.6}$$

for every $t \in [0,T]$.

5.2. Improved Integrability

Since v_τ^i is a $\frac{C_W}{c_W}$-cubic quasi-minimum of the functional $v \mapsto \int_\Omega |e(v)|^2 \,\mathrm{d}x$, we use Theorem 1 (see also [12, Appendix]) to obtain the existence of two constants $\gamma > 0$ and $r > 1$, depending only on c_W, C_W, and φ, such that

$$\int_\Omega |e(v_\tau^i)|^{2r} \,\mathrm{d}x \leqq \int_\Omega |\nabla v_\tau^i|^{2r} \,\mathrm{d}x \leqq \gamma^{2r} \Big(\int_\Omega |\nabla v_\tau^i|^2 \,\mathrm{d}x + 1 \Big)^r$$
$$\leqq C(\Omega)^{2r} \gamma^{2r} \Big(\int_\Omega |e(v_\tau^i)|^2 \,\mathrm{d}x + 1 \Big)^r, \tag{5.7}$$

where $C(\Omega)$ is the Korn–Poincaré constant. In particular, all the above constants are independent of τ and i.

5.3. A Priori Estimates

Next, we obtain an a priori estimate for the piecewise constant interpolations (z_τ, v_τ).

Since $(z_\tau^{i-1}, v_\tau^{i-1} - \varphi_\tau^{i-1} + \varphi_\tau^i) \in L^1(\Omega) \times (\varphi_\tau^i + H_0^1(\Omega; \mathbb{R}^d))$, the minimality of (z_τ^i, v_τ^i) implies that

$$\mathcal{W}(z_\tau^i, e(v_\tau^i)) + \mathcal{D}(z_\tau^{i-1}, z_\tau^i) \leqq \mathcal{W}(z_\tau^{i-1}, e(v_\tau^{i-1} - \varphi_\tau^{i-1} + \varphi_\tau^i))$$
$$= \mathcal{W}(z_\tau^{i-1}, e(v_\tau^{i-1}))$$
$$+ \mathcal{W}(z_\tau^{i-1}, e(v_\tau^{i-1} - \varphi_\tau^{i-1} + \varphi_\tau^i)) - \mathcal{W}(z_\tau^{i-1}, e(v_\tau^{i-1})). \tag{5.8}$$

The last two terms of the right-hand side above may be controlled as follows

$$\mathcal{W}(z_\tau^{i-1}, e(v_\tau^{i-1} - \varphi_\tau^{i-1} + \varphi_\tau^i)) - \mathcal{W}(z_\tau^{i-1}, e(v_\tau^{i-1}))$$
$$= \int_{t_\tau^{i-1}}^{t_\tau^i} \Big[\int_\Omega \frac{\partial W}{\partial \varepsilon}(z_\tau^{i-1}, e(v_\tau^{i-1} - \varphi_\tau^{i-1} + \varphi(s))) : e(\dot\varphi(s)) \,\mathrm{d}x \Big] \mathrm{d}s$$
$$= \int_{t_\tau^{i-1}}^{t_\tau^i} \Big[\int_\Omega \sigma_\tau(s) : e(\dot\varphi(s)) \,\mathrm{d}x \Big] \mathrm{d}s$$
$$+ \int_{t_\tau^{i-1}}^{t_\tau^i} \Big[\int_\Omega \Big(\frac{\partial W}{\partial \varepsilon}(z_\tau^{i-1}, e(v_\tau^{i-1} - \varphi_\tau^{i-1} + \varphi(s)))$$
$$- \frac{\partial W}{\partial \varepsilon}(z_\tau^{i-1}, e(v_\tau^{i-1})) \Big) : e(\dot\varphi(s)) \,\mathrm{d}x \Big] \mathrm{d}s.$$

Taking the sum in (5.8) for $t \in [0, T]$, $\tau(t) := \max\{t_\tau^i \,:\, t_\tau^i \leqq t\}$, we have

$$\mathcal{W}(z_\tau(t), e(v_\tau(t))) + \mathrm{Diss}(z_\tau; 0, t)$$

$$
\begin{aligned}
&\leqq \mathcal{W}(z_0, v_0) + \int_0^{\tau(t)} \left[\int_\Omega \sigma_\tau(s){:}e(\dot\varphi(s))\,\mathrm{d}x \right] \mathrm{d}s \\
&\quad + \int_0^{\tau(t)} \Bigg[\int_\Omega \bigg(\frac{\partial W}{\partial \varepsilon}(z_\tau(s), e(v_\tau(s) - \varphi_\tau(s) + \varphi(s))) \\
&\qquad - \frac{\partial W}{\partial \varepsilon}(z_\tau(s), e(v_\tau(s))) \bigg){:}e(\dot\varphi(s))\,\mathrm{d}x \Bigg] \mathrm{d}s. \qquad (5.9)
\end{aligned}
$$

We observe that, thanks to $(W.3)$, we have

$$
\begin{aligned}
&\left| \int_0^{\tau(t)} \Bigg[\int_\Omega \bigg(\frac{\partial W}{\partial \varepsilon}(z_\tau(s), e(v_\tau(s) - \varphi_\tau(s) + \varphi(s))) \right. \\
&\qquad \left. - \frac{\partial W}{\partial \varepsilon}(z_\tau(s), e(v_\tau(s))) \bigg){:}e(\dot\varphi(s))\,\mathrm{d}x \Bigg] \mathrm{d}s \right| \\
&\leqq 2C \left(\sup_{t\in[0,T]} \|e(v_\tau(t))\|_2 + \sup_{t\in[0,T]} \|e(\varphi(t))\|_2 + 1 \right) \int_0^T \|e(\dot\varphi(s))\|_2 \,\mathrm{d}s.
\end{aligned}
$$

Now,

$$
\int_0^T \|e(\dot\varphi(s))\|_2 \,\mathrm{d}s + \sup_{t\in[0,T]} \|e(\varphi(t))\|_2 < \infty,
$$

since $\varphi \in AC([0,T], H^1(\Omega;\mathbb{R}^d))$. Hence, also owing to definition (5.6) and hypothesis $(W.2)$, we get

$$
\begin{aligned}
c_W \sup_{t\in[0,T]} \|e(v_\tau(t))\|_2^2 &\leqq \sup_{t\in[0,T]} \mathcal{W}(z_\tau(t), e(v_\tau(t))) + \mathrm{Diss}(z_\tau; 0, T) \\
&\leqq \mathcal{W}(z_0, v_0) + C \left(\sup_{t\in[0,T]} \|e(v_\tau(t))\|_2 + 1 \right),
\end{aligned}
$$

for every $t \in [0,T]$ and a positive constant C. Therefore, we deduce that there exists a positive constant K, independent of the choice of the time step τ, such that

$$
\sup_{t\in[0,T]} \|e(v_\tau(t))\|_2 \leqq K.
$$

In particular, we get

$$
\sup_{t\in[0,T]} \|\nabla v_\tau(t)\|_2 \leqq C(\Omega)K \qquad (5.10)
$$

$$
\sup_{t\in[0,T]} \|\sigma_\tau(t)\|_2 \leqq C_W(K+1). \qquad (5.11)
$$

Thanks to the improved regularity estimate (5.7), we get

$$
\sup_{t\in[0,T]} \|\nabla v_\tau(t)\|_{2r} \leqq C(\Omega)\gamma\sqrt{K^2+1}. \qquad (5.12)
$$

5.4. Passage to the Limit

Let us now consider a sequence of time steps $(\tau_n)_n$ converging to 0, and the associated interpolations $(z_{\tau_n}, v_{\tau_n})_n$. We want to define a family of measures $\nu \in AY([0,T];\varphi)$. We will do this by passing to the limit in the sequence of approximate solutions $(z_{\tau_n}(t), v_{\tau_n}(t))_n$. For technical reasons, which will appear patent in the proof, we need to proceed by defining ν_t on larger and larger time sets. In particular, we will first define ν_t for $t \in [0,T]\cap\mathbb{Q}$ and then in the rest of $[0,T]$.

Thanks to the uniform bound (5.10) and to the higher integrability estimate (5.12), and by using a diagonalization argument, we can find a not-relabeled subsequence (z_{τ_n}, v_{τ_n}) and $\tilde{\nu} \in Y^{2r}(\Omega;\mathbb{R}\times\mathbb{R}^{d\times d})^{[0,T]\cap\mathbb{Q}}$, such that

$$\boldsymbol{\delta}_{(z_{\tau_n}(t),\nabla v_{\tau_n}(t))} \rightharpoonup \tilde{\nu}_t \quad 2r\text{-weakly*},$$

for every $t \in [0,T]\cap\mathbb{Q}$.

For every $t \in [0,T]\setminus\mathbb{Q}$, let us choose an increasing sequence of integers n_k^t possibly depending on t, such that

$$\limsup_n \langle \sigma_{\tau_n}(t), e(\dot{\varphi}(t))\rangle = \lim_k \langle \sigma_{\tau_{n_k^t}}(t), e(\dot{\varphi}(t))\rangle. \tag{5.13}$$

Again, we are allowed to extract a further subsequence, still denoted by $(z_{\tau_{n_k^t}}, v_{\tau_{n_k^t}})_k$, satisfying (5.13) and such that there exists $\tilde{\nu}_t \in Y^{2r}(\Omega;\mathbb{R}\times\mathbb{R}^{d\times d})$ with

$$\boldsymbol{\delta}_{(z_{\tau_{n_k^t}}(t),\nabla v_{\tau_{n_k^t}}(t))} \rightharpoonup \tilde{\nu}_t \quad 2r\text{-weakly*, as } k\to\infty$$

for every $t \in [0,T]\setminus\mathbb{Q}$. Note that, for every $t \in [0,T]\setminus\mathbb{Q}$,

$$\begin{aligned}\limsup_n \langle \sigma_{\tau_n}(t), e(\dot{\varphi}(t))\rangle &= \lim_k \langle \sigma_{\tau_{n_k^t}}(t), e(\dot{\varphi}(t))\rangle\\ &= \lim_k \int_\Omega \frac{\partial W}{\partial \varepsilon}(z_{\tau_{n_k^t}}(t), e(v_{\tau_{n_k^t}}(t))):e(\dot{\varphi}(t))\,\mathrm{d}x = \langle \sigma(t), e(\dot{\varphi}(t))\rangle,\end{aligned}$$

where σ is defined by

$$\sigma(t,x) := \int_{\mathbb{R}\times\mathbb{R}^{d\times d}} \frac{\partial W}{\partial \varepsilon}(\theta,\varepsilon)\,\mathrm{d}S(\tilde{\nu})_t^x(\theta,\varepsilon),$$

for $S(\theta,F) := (\theta, \frac{F+F^T}{2})$, for every $(\theta,F)\in\mathbb{R}\times\mathbb{R}^{d\times d}$. Moreover, for every $t \in [0,T]\cap\mathbb{Q}$ we have

$$\limsup_n \langle \sigma_{\tau_n}(t), e(\dot{\varphi}(t))\rangle = \lim_n \langle \sigma_{\tau_n}(t), e(\dot{\varphi}(t))\rangle = \langle \sigma(t), e(\dot{\varphi}(t))\rangle.$$

This implies that the map in (4.2) is measurable on $[0,T]$.

In this way, we have defined $\tilde{\nu}$ in $Y^{2r}(\Omega;\mathbb{R}\times\mathbb{R}^{d\times d})^{[0,T]}$, satisfying by construction properties (3.23) and (3.24), (3.25) in Definition 3. Therefore, by letting $\nu_t := S(\tilde{\nu}_t)$ for every $t\in[0,T]$, we get that ν also satisfies condition (3.22), and hence $\nu \in AY([0,T],\varphi)$.

In particular we have:

$$\begin{aligned} \boldsymbol{\delta}_{(z_{\tau_n}(t), e(v_{\tau_n}(t)))} &\rightharpoonup \nu_t \quad 2r\text{-weakly*, for } t \in [0,T] \cap \mathbb{Q}, \\ \boldsymbol{\delta}_{(z_{\tau_{n_k^t}}(t), e(v_{\tau_{n_k^t}}(t)))} &\rightharpoonup \nu_t \quad 2r\text{-weakly*, for } t \in [0,T] \setminus \mathbb{Q}. \end{aligned} \tag{5.14}$$

Since $(z_{\tau_n}(0), v_{\tau_n}(0)) = (z_0, v_0)$ for every n, the initial condition $(E0)$ is automatically satisfied.

5.5. Irreversibility

Let us consider $0 \leqq s < t \leqq T$ and fix $q \in [s,t] \cap \mathbb{Q}$.

Up to not-relabeled subsequences, we have that there exist $\mu_{sq}, \mu_{qt} \in Y(\Omega; [0,1]^2)$ with

$$\begin{aligned} \boldsymbol{\delta}_{(z_{\tau_{n_k^s}}(s), z_{\tau_{n_k^s}}(q))} &\rightharpoonup \mu_{sq} \quad \text{weakly*}, \\ \boldsymbol{\delta}_{(z_{\tau_{n_k^t}}(q), z_{\tau_{n_k^t}}(t))} &\rightharpoonup \mu_{qt} \quad \text{weakly*}. \end{aligned}$$

Thanks to the construction of ν, we have that μ_{sq} has projections $\pi_1(\nu_s)$ and $\pi_1(\nu_q)$, respectively, and μ_{qt} has projections $\pi_1(\nu_q)$ and $\pi_1(\nu_t)$, respectively.

Now, we have $z_{\tau_n}(t) \leqq z_{\tau_n}(q) \leqq z_{\tau_n}(s)$ almost everywhere in Ω, for every n. This implies that $\boldsymbol{\delta}_{(z_{\tau_n}(s), z_{\tau_n}(q))}(E \times \{\theta_1 < \theta_2\}) = 0$ and $\boldsymbol{\delta}_{(z_{\tau_n}(q), z_{\tau_n}(t))}(E \times \{\theta_1 < \theta_2\}) = 0$ for $E \subseteq \Omega$ open, for every n.

Since $\boldsymbol{\delta}_{(z_{\tau_{n_k^s}}(s), z_{\tau_{n_k^s}}(q))} \rightharpoonup \mu_{sq}$ weakly* as $n \to \infty$, and $E \times \{\theta_1 < \theta_2\}$ is open, we have

$$\mu_{sq}(E \times \{\theta_1 < \theta_2\}) \leqq \liminf_k \boldsymbol{\delta}_{(z_{\tau_{n_k^s}}(s), z_{\tau_{n_k^s}}(q))}(E \times \{\theta_1 < \theta_2\}) = 0;$$

therefore $\mu_{sq}^x(\{\theta_1 < \theta_2\}) = 0$ for almost every $x \in \Omega$. The same holds for μ_{qt}: $\mu_{qt}^x(\{\theta_1 < \theta_2\}) = 0$ for almost every $x \in \Omega$.

This implies, by Theorem 2, that $\pi_1(\nu_s) \succeq \pi_1(\nu_q)$ and $\pi_1(\nu_q) \succeq \pi_1(\nu_t)$. By transitivity, this implies $\pi_1(\nu_s) \succeq \pi_1(\nu_t)$, namely the irreversibility condition $(E1)$.

5.6. Stability

Let $\tilde{z} \in L^1(\Omega)$ and $\tilde{u} \in H_0^1(\Omega; \mathbb{R}^d)$. Let us observe that if $\tilde{z} > 0$ on $\Omega' \subseteq \Omega$ with $|\Omega'| > 0$, then $\mathbb{D}(\mu, \mathrm{Tr}_{\tilde{z}}(\mu)) = \infty$, for every $\mu \in Y(\Omega; [0,1])$. Indeed, if $\mu \succeq \mathrm{Tr}_{\tilde{z}}(\mu)$, then for every $\alpha \in \mathbb{R}$ we would have $\mu^x(\alpha, \infty) \geqq \mathrm{Tr}_{\tilde{z}}(\mu)^x(\alpha, \infty) = \mu^x(\alpha - \tilde{z}(x), \infty)$. Therefore, for $x \in \Omega'$, we would have $\mu^x(\alpha - \tilde{z}(x), \alpha] = 0$, for every $\alpha \in \mathbb{R}$. This would imply $\mu^x([0,1]) = 0$, for $x \in \Omega'$, which is a contradiction with the fact that μ^x is a probability measure on $[0,1]$, for almost every x. In conclusion, if $\tilde{z} > 0$ on a subset of Ω with positive measure, $(E2)$ is automatically satisfied.

Hence, we reduce to the case $\tilde{z} \leqq 0$ almost everywhere in Ω. For every n and every $i = 1, \ldots, n$, the function $(z_{\tau_n}^i + \tilde{z}, v_{\tau_n}^i + \tilde{u})$ is an admissible competitor for the minimum problem defining $(z_{\tau_n}^i, v_{\tau_n}^i)$. Therefore, we have

$$\mathcal{W}(z_{\tau_n}^i, v_{\tau_n}^i) + \mathcal{D}(z_{\tau_n}^{i-1}, z_{\tau_n}^i) \leqq \mathcal{W}(z_{\tau_n}^i + \tilde{z}, v_{\tau_n}^i + \tilde{u}) + \mathcal{D}(z_{\tau_n}^{i-1}, z_{\tau_n}^i + \tilde{z}).$$

Since $\tilde{z} \leqq 0$ almost everywhere in Ω, we have that $z^i_{\tau_n} + \tilde{z} \leqq z^i_{\tau_n} \leqq z^{i-1}_{\tau_n}$. This implies that $\mathcal{D}(z^{i-1}_{\tau_n}, z^i_{\tau_n} - \tilde{z}) = \rho \int_\Omega (z^{i-1}_{\tau_n} - z^i_{\tau_n} + \tilde{z})\,\mathrm{d}x$ and hence

$$\mathcal{D}(z^{i-1}_{\tau_n}, z^i_{\tau_n} + \tilde{z}) - \mathcal{D}(z^{i-1}_{\tau_n}, z^i_{\tau_n}) = \mathcal{D}(z^i_{\tau_n}, z^i_{\tau_n} + \tilde{z}) = \mathcal{D}(0, \tilde{z}).$$

Hence, we get

$$\mathcal{W}(z^i_{\tau_n}, v^i_{\tau_n}) \leqq \mathcal{W}(z^i_{\tau_n} + \tilde{z}, v^i_{\tau_n} + \tilde{u}) + \mathcal{D}(0, \tilde{z}).$$

This means that, for every $t \in [0, T]$, we have

$$\mathcal{W}(z_{\tau_n}(t), v_{\tau_n}(t)) \leqq \mathcal{W}(z_{\tau_n}(t) + \tilde{z}, v_{\tau_n}(t) + \tilde{u}) + \mathcal{D}(0, \tilde{z}). \tag{5.15}$$

We observe that $\theta + \tilde{z}(x) \leqq 1$ for almost every $x \in \Omega$, for every $\theta \in [0, 1]$ ($\tilde{z} \leqq 0$ almost everywhere in Ω). Hence, thanks to $(W.2)$, we have

$$|W(\theta + \tilde{z}(x), \varepsilon + e(\tilde{u}))| \leqq C(|e(\tilde{u})(x)|^2 + |\varepsilon|^2).$$

Therefore, by using the convergence (5.14) and Lemma 2 we get

$$\begin{aligned}
&\int_\Omega W(z_{\tau_{n^t_k}}(t), e(v_{\tau_{n^t_k}}(t)))\,\mathrm{d}x \longrightarrow \int_{\Omega\times\mathbb{R}\times\mathbb{R}^{d\times d}_{\mathrm{sym}}} W(\theta, \varepsilon)\,\mathrm{d}\nu_t(x, \theta, \varepsilon),\\
&\int_\Omega W(z_{\tau_{n^t_k}}(t) + \tilde{z}(x), e(v_{\tau_{n^t_k}}(t)) + e(\tilde{u})(x))\,\mathrm{d}x\\
&\longrightarrow \int_{\Omega\times\mathbb{R}\times\mathbb{R}^{d\times d}_{\mathrm{sym}}} W(\theta + \tilde{z}(x), \varepsilon + e(\tilde{u})(x))\,\mathrm{d}\nu_t(x, \theta, \varepsilon)\\
&= \langle W, \mathrm{Tr}_{(\tilde{z}, e(\tilde{u}))}(\nu_t)\rangle,
\end{aligned}$$

for every $t \in (0, T]$, as $n \to \infty$. Therefore, we can deduce the translational stability $(E2)$ passing to the limit in inequality (5.15). For $t = 0$, relation $(E2)$ comes immediately from the hypothesis on the initial datum (4.3).

Now we want to prove global stability for the internal variable $(E3)$. Let us denote $\pi_1(\nu_t)$ by μ_t, for every $t \in [0, T]$.

Let us start by proving $(E3)$ for $\tilde{\mu} \in Y(\Omega; [0, 1])$. From the minimality of $(z^i_{\tau_n}, v^i_{\tau_n})$, we get that for every $(\tilde{z}, \tilde{v}) \in L^1(\Omega) \times (\varphi^i_{\tau_n} + H^1_0(\Omega; \mathbb{R}^d))$,

$$\mathcal{W}(z^i_{\tau_n}, v^i_{\tau_n}) + \mathcal{D}(z^{i-1}_{\tau_n}, z^i_{\tau_n}) \leqq \mathcal{W}(\tilde{z}, \tilde{v}) + \mathcal{D}(z^{i-1}_{\tau_n}, \tilde{z}).$$

Hence, using the triangle inequality for $\mathcal{D}$, we get

$$\mathcal{W}(z^i_{\tau_n}, v^i_{\tau_n}) \leqq \mathcal{W}(\tilde{z}, \tilde{v}) + \mathcal{D}(z^i_{\tau_n}, \tilde{z}).$$

Therefore, we deduce that for every n, $t \in [0, T]$, and $(\tilde{z}, \tilde{v}) \in L^1(\Omega) \times (\varphi(t) + H^1_0(\Omega; \mathbb{R}^d))$, we have

$$\begin{aligned}
\mathcal{W}(z_{\tau_n}(t), e(v_{\tau_n}(t))) &\leqq \mathcal{W}(\tilde{z}, e(\tilde{v} - \varphi(t) + \varphi_{\tau_n}(t))) + \mathcal{D}(z_{\tau_n}(t), \tilde{z})\\
&= \mathcal{W}(\tilde{z}, e(\tilde{v})) + \mathcal{D}(z_{\tau_n}(t), \tilde{z}) + R_n(t),
\end{aligned} \tag{5.16}$$

where

$$R_n(t) := \mathcal{W}(\tilde{z}, e(\tilde{v} - \varphi(t) + \varphi_{\tau_n}(t))) - \mathcal{W}(\tilde{z}, e(\tilde{v})).$$

Arguing as in Section 5.3, it is not difficult to show that

$$|R_n(t)| \leqq 2C\Big(\sup_{t\in[0,T]} \|e(\varphi(t))\|_2 + \|e(\tilde{v})\|_2 + 1\Big) \int_{t-\tau_n}^{t} \|e(\dot{\varphi}(s))\|_2 \,\mathrm{d}s.$$

Since $\dot{\varphi} \in L^1([0,T]; H^1(\Omega;\mathbb{R}^d))$, we have that, for every $t \in [0,T]$

$$R_n(t) \to 0, \quad \text{as } n \to \infty. \tag{5.17}$$

Let us now fix $t \in [0,T]$ and a competitor $\tilde{\mu} \in Y(\Omega;[0,1])$. If $\mu_t \not\succeq \tilde{\mu}$, we have $\mathbb{D}(\mu_t, \tilde{\mu}) = \infty$ and hence (4.1) holds true, so we can assume that $\mu_t \succeq \tilde{\mu}$. Thanks to Theorem 2, there exists a measure $\mu_{12,t}$ such that

$$\pi_1(\mu_{12,t}) = \mu_t, \quad \pi_2(\mu_{12,t}) = \tilde{\mu},$$
$$\mu^x_{12,t}(\{\theta_1 < \theta_2\}) = 0 \quad \text{for a.e. } x \in \Omega.$$

Let us consider the sequence $(z_{\tau_{n_k^t}}(t), v_{\tau_{n_k^t}}(t))_k$ such that $\boldsymbol{\delta}_{(z_{\tau_{n_k^t}}(t), e(v_{\tau_{n_k^t}}(t)))} \rightharpoonup \nu_t$ $2r$-weakly*. This implies, by Lemma 2, that

$$\mathcal{W}(z_{\tau_{n_k^t}}(t), e(v_{\tau_{n_k^t}}(t))) = \langle W, \boldsymbol{\delta}_{(z_{\tau_{n_k^t}}(t), e(v_{\tau_{n_k^t}}(t)))}\rangle \to \langle W, \nu_t\rangle, \tag{5.18}$$

as $k \to \infty$. Moreover $\boldsymbol{\delta}_{z_{\tau_{n_k^t}}(t)} \rightharpoonup \pi_1(\mu_{12,t})$ weakly*, so we can apply Theorem 3 and Corollary 1 to construct a sequence $(\tilde{z}_k)_k$ in $L^1(\Omega;[0,1])$ such that, as $k \to \infty$,

$$\tilde{z}_k \leqq z_{\tau_{n_k^t}}(t) \quad \text{a.e. in } \Omega,$$
$$\boldsymbol{\delta}_{(z_{\tau_{n_k^t}}(t), \tilde{z}_k)} \rightharpoonup \mu_{12,t} \quad \text{weakly*},$$
$$\boldsymbol{\delta}_{\tilde{z}_k} \rightharpoonup \tilde{\mu} \quad \text{weakly*}.$$

We can apply Lemma 2 to obtain

$$\begin{aligned} \mathcal{W}(\tilde{z}_k, e(\tilde{v})) &= \int_{\Omega\times\mathbb{R}} W(\theta, e(\tilde{v}(x))\,\mathrm{d}\boldsymbol{\delta}_{\tilde{z}_k}(x,\theta) \\ &\to \int_{\Omega\times\mathbb{R}} W(\theta, e(\tilde{v}(x))\,\mathrm{d}\tilde{\mu}(x,\theta) = \langle W, (\tilde{\mu}^x \otimes \boldsymbol{\delta}_{e(\tilde{v})(x)})_{x\in\Omega}\rangle, \end{aligned} \tag{5.19}$$

as $k \to \infty$. As $\tilde{z}_k \leqq z_{\tau_{n_k^t}}(t)$ almost everywhere in Ω, we have that

$$\begin{aligned} \mathcal{D}(z_{\tau_{n_k^t}}(t), \tilde{z}_k) &= \int_\Omega d\Big(z_{\tau_{n_k^t}}(x,t), \tilde{z}_k(x)\Big)\,\mathrm{d}x = \int_\Omega \rho\Big(z_{\tau_{n_k^t}}(x,t) - \tilde{z}_k(x)\Big)\,\mathrm{d}x \\ &= \int_{\Omega\times\mathbb{R}^2} \rho(\theta_1 - \theta_2)\,\mathrm{d}\boldsymbol{\delta}_{(z_{\tau_{n_k^t}}(t), \tilde{z}_k)} \\ &\to \int_{\Omega\times\mathbb{R}^2} \rho(\theta_1 - \theta_2)\,\mathrm{d}\mu_{12,t}(x,\theta_1,\theta_2) \\ &= \rho\Big[\int_{\Omega\times\mathbb{R}} \theta_1\,\mathrm{d}\mu_t(x,\theta_1) - \int_{\Omega\times\mathbb{R}} \theta_2\,\mathrm{d}\tilde{\mu}(x,\theta_2)\Big] \\ &= \mathbb{D}(\mu_t, \tilde{\mu}) = \mathbb{D}(\pi_1(\nu_t), \tilde{\mu}), \end{aligned} \tag{5.20}$$

as $k \to \infty$.

Therefore, putting together inequality (5.16) for $\tilde{z} = \tilde{z}_k$, and the convergence properties (5.17), (5.18), (5.19), and (5.20), we get (4.1).

Let us now consider a general $\hat{\mu} \in Y(\Omega; \mathbb{R})$. If $\operatorname{supp}(\hat{\mu}) \nsubseteq \Omega \times (-\infty, 1]$, then $\mu_t \not\succeq \hat{\mu}$. Therefore, $\mathbb{D}(\mu_t, \hat{\mu}) = \infty$ and $(E3)$ is proved, so let us assume that $\operatorname{supp}(\hat{\mu}) \subseteq \Omega \times (-\infty, 1]$. We define $\tilde{\mu} \in Y(\Omega; [0, 1])$, by setting:

$$\int_{\Omega\times\mathbb{R}} f(x,\theta)\ \mathrm{d}\tilde{\mu}(x,\theta) := \int_{\Omega\times(0,1]} f(x,\theta)\ \mathrm{d}\hat{\mu}(x,\theta) + \int_{\Omega\times(-\infty,0]} f(x,0)\ \mathrm{d}\hat{\mu}(x,\theta),$$

for every bounded Borel function $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$. It can be seen immediately that if $\mu_t \succeq \hat{\mu}$, then $\mu_t \succeq \tilde{\mu}$. Indeed, let $\alpha \in [0, 1]$, then $\tilde{\mu}^x(\alpha, 1] = \hat{\mu}^x(\alpha, 1] \leqq \mu_t^x(\alpha, 1]$, and, if $\alpha < 0$, $\tilde{\mu}^x(\alpha, 1] = \tilde{\mu}^x[0, 1] = 1 = \mu_t^x[0, 1] = \mu_t^x(\alpha, 1]$, for almost every $x \in \Omega$.

We claim that

$$\begin{aligned} &\langle W, (\tilde{\mu}^x \otimes \delta_{e(\tilde{v})(x)})_{x\in\Omega}\rangle - \int_{\Omega\times\mathbb{R}} \theta\ \mathrm{d}\tilde{\mu}(x,\theta) \\ &\leqq \langle W, (\hat{\mu}^x \otimes \delta_{e(\tilde{v})(x)})_{x\in\Omega}\rangle - \int_{\Omega\times\mathbb{R}} \theta\ \mathrm{d}\hat{\mu}(x,\theta). \end{aligned}$$

Indeed, we have

$$\begin{aligned} \int_{\Omega\times\mathbb{R}} \theta\ \mathrm{d}\tilde{\mu}(x,\theta) &= \int_{\Omega\times(0,1]} \theta\ \mathrm{d}\hat{\mu}(x,\theta) + \int_{\Omega\times(-\infty,0]} 0\ \mathrm{d}\hat{\mu}(x,\theta) \\ &\geqq \int_{\Omega\times(0,1]} \theta\ \mathrm{d}\hat{\mu}(x,\theta) + \int_{\Omega\times(-\infty,0]} \theta\ \mathrm{d}\hat{\mu}(x,\theta) = \int_{\Omega\times\mathbb{R}} \theta\ \mathrm{d}\hat{\mu}(x,\theta). \end{aligned}$$

On the other hand, thanks to $(W.5)$ we have

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}} W(\theta, e(\tilde{v})(x))\ \mathrm{d}\tilde{\mu}(x,\theta) \\ &= \int_{\Omega\times(0,1]} W(\theta, e(\tilde{v})(x))\ \mathrm{d}\hat{\mu}(x,\theta) + \int_{\Omega\times(-\infty,0]} W(0, e(\tilde{v})(x))\ \mathrm{d}\hat{\mu}(x,\theta) \\ &= \int_{\Omega\times(0,1]} W(\theta, e(\tilde{v})(x))\ \mathrm{d}\hat{\mu}(x,\theta) + \int_{\Omega\times(-\infty,0]} W(\theta, e(\tilde{v})(x))\ \mathrm{d}\hat{\mu}(x,\theta) \\ &= \int_{\Omega\times\mathbb{R}} W(\theta, e(\tilde{v})(x))\ \mathrm{d}\hat{\mu}(x,\theta). \end{aligned}$$

The claim is hence proved, and we have that

$$\langle W, (\tilde{\mu}^x \otimes \delta_{e(\tilde{v})(x)})_{x\in\Omega}\rangle + \mathbb{D}(\mu_t, \tilde{\mu}) \leqq \langle W, (\hat{\mu}^x \otimes \delta_{e(\tilde{v})(x)})_{x\in\Omega} + \mathbb{D}(\mu_t, \hat{\mu})\rangle.$$

We have checked that the global stability for the internal variable $(E3)$ holds for $\tilde{\mu} \in Y(\Omega; \mathbb{R})$ as well.

5.7. Upper Energy Estimate

First of all we observe that, thanks to the irreversibility property $(E1)$ and Theorem 2, we have, for every $t \in [0, T]$,

$$\mathrm{Diss}(\nu; 0, t) = \int_{\Omega\times\mathbb{R}} \rho(z_0(x) - \theta)\,\mathrm{d}\pi_1(\nu_t)(x, \theta).$$

Since $z_{\tau_n}(s) \geqq z_{\tau_n}(t)$ almost everywhere in Ω, whenever $s \leqq t$, we have

$$\mathrm{Diss}(z_{\tau_n}; 0, t) = \int_{\Omega\times\mathbb{R}} \rho(z_0(x) - \theta)\, d\boldsymbol{\delta}_{z_{\tau_n}(t)}(x, \theta).$$

We have that $\boldsymbol{\delta}_{z_{\tau_{n_k^t}}(t)} \rightharpoonup \pi_1(\nu_t)$ weakly*, and hence we get $\mathrm{Diss}(z_{\tau_{n_k^t}}; 0, t) \to \mathrm{Diss}(\nu; 0, t)$ as $k \to \infty$. Let us fix $t \in [0, T]$. We have

$$\langle W, \nu_t\rangle + \mathrm{Diss}(\nu; 0, t) \leqq \liminf_k \left[\mathcal{W}(z_{\tau_{n_k^t}}(t), e(v_{\tau_{n_k^t}}(t))) + \mathrm{Diss}(z_{\tau_{n_k^t}}; 0, t)\right].$$

By using estimate (5.9), we deduce that

$$\begin{aligned}
&\langle W, \nu_t\rangle + \mathrm{Diss}(\nu; 0, t)\\
&\quad\leqq \liminf_k \left[\mathcal{W}(z_0, e(v_0)) + \int_0^{\tau_{n_k^t}(t)} \langle \sigma_{\tau_{n_k^t}}(s), e(\dot\varphi(s))\rangle\,\mathrm{d}s + \rho_{n_k^t}\right]\\
&\quad\leqq \mathcal{W}(z_0, e(v_0)) + \limsup_n \int_0^{\tau_n(t)} \langle \sigma_n(s), e(\dot\varphi(s))\rangle\,\mathrm{d}s + \limsup_n \rho_n,
\end{aligned}$$

where

$$\begin{aligned}
\rho_n := \int_0^{\tau_n(t)} \Bigg[\int_\Omega \Bigg(&\frac{\partial W}{\partial \varepsilon}(z_{\tau_n}(s), e(v_{\tau_n}(s) - \varphi_{\tau_n}(s) + \varphi(s)))\\
&- \frac{\partial W}{\partial \varepsilon}(z_{\tau_n}(s), e(v_{\tau_n}(s)))\Bigg) : e(\dot\varphi(s))\,\mathrm{d}x\Bigg]\,\mathrm{d}s.
\end{aligned}$$

Since $\sup_{t,n} \|\sigma_{\tau_n}(t)\|_2$ is finite thanks to estimate (5.11), by Fatou's Lemma we get

$$\begin{aligned}
\limsup_n \int_0^{\tau_n(t)} \langle \sigma_n(s), e(\dot\varphi(s))\rangle\,\mathrm{d}s &\leqq \int_0^T \limsup_n 1_{[0,\tau_n(t)]}\langle \sigma_n(s), e(\dot\varphi(s))\rangle\,\mathrm{d}s\\
&= \int_0^t \langle \sigma(s), e(\dot\varphi(s))\rangle\,\mathrm{d}s.
\end{aligned}$$

Finally we apply the following lemma with $X = \Omega$, $H = \frac{\partial W}{\partial \varepsilon}$, $q = 2$, $\Phi_n = (z_{\tau_n}(s), e(v_{\tau_n}(s)))$, $\Psi_n := (0, e(\varphi_{\tau_n}(s) - \varphi(s)))$, and $\Phi = e(\dot\varphi(s))$.

Lemma 4. [10, Lemma 4.9] *Let* $(X, \mathcal{A}, \mu)$ *be a finite measure space, let* $q > 1$*, let* $m, n \geqq 1$*, and let* $H\colon X \times \mathbb{R}^N \to \mathbb{R}^m$ *be a Carathéodory function. Assume that there exist a constant* $a \geqq 0$ *and a nonnegative function* $b \in L^{q'}(X)$*, with* $q' = q/(q-1)$*, such that*

$$|H(x,\xi)| \leqq a|\xi|^{q-1} + b(x)$$

for every $(x,\xi) \in X \times \mathbb{R}^N$*. Let* Φ_n *and* Ψ_n *be two sequences in* $L^q(X;\mathbb{R}^N)$*. Assume that* Φ_n *is bounded in* $L^q(X;\mathbb{R}^N)$ *and* Ψ_n *converges to* 0 *strongly in* $L^q(X;\mathbb{R}^N)$*. Then*

$$\int_X [H(x, \Phi_n(x) + \Psi_n(x)) - H(x, \Phi_n(x))]\Phi(x)\,\mathrm{d}\mu(x) \to 0$$

for every $\Phi \in L^q(X;\mathbb{R}^m)$*.*

We obtain

$$\int_\Omega \left(\frac{\partial W}{\partial \varepsilon}(z_{\tau_n}(s), e(v_{\tau_n}(s) - \varphi_{\tau_n}(s) + \varphi(s))) - \frac{\partial W}{\partial \varepsilon}(z_{\tau_n}(s), e(v_{\tau_n}(s)))\right):e(\dot\varphi(s))\,\mathrm{d}x \to 0,$$

as $n \to \infty$, for almost every $s \in [0,T]$. Moreover, we have

$$\begin{aligned}&\left|\int_\Omega \left(\frac{\partial W}{\partial \varepsilon}(z_{\tau_n}(s), e(v_{\tau_n}(s) - \varphi_{\tau_n}(s) + \varphi(s))) - \frac{\partial W}{\partial \varepsilon}(z_{\tau_n}(s), e(v_{\tau_n}(s)))\right):e(\dot\varphi(s))\,\mathrm{d}x\right|\\ &\leqq \tilde c\left(\sup_{n,t}\|e(v_{\tau_n}(t))\|_2 + \sup_t \int_{t-\tau_n}^t \|e(\dot\varphi(s))\|_2\,\mathrm{d}s + 1\right)\|e(\dot\varphi(s))\|_2\\ &\leqq C\|e(\dot\varphi(s))\|_2 \in L^1([0,T]),\end{aligned}$$

for almost every $s \in [0,T]$. Therefore, by Dominated Convergence we get $\lim_n \rho_n = 0$, and we can deduce that

$$\langle W, \nu_t\rangle + \mathrm{Diss}(\nu; 0, t) \leqq \mathcal{W}(z_0, e(v_0)) + \int_0^t \langle \sigma(s), e(\dot\varphi(s))\rangle\,\mathrm{d}s. \quad (5.21)$$

5.8. Lower Energy Estimate

To prove the lower energy estimate, we proceed in the same way as in [12, Subsection 7.6]. We recall the main passages for the reader's convenience. Let us denote $\pi_1(\nu_t)$ by μ_t for every $t \in [0,T]$. Let $s < t$, with $s \in [0,T] \cap \mathbb{Q}$ and $t \in [0,T]$. Thanks to the minimality property satisfied by (z_{τ_n}, v_{τ_n}), the fact that $z_{\tau_n}(s) \geqq z_{\tau_n}(t)$ almost everywhere in Ω, and the triangle inequality for $\mathcal{D}$, we get

$$\begin{aligned}&\mathcal{W}(z_{\tau_n}(s), e(v_{\tau_n}(s)))\\ &\quad\leqq \mathcal{W}(z_{\tau_n}(t), e(v_{\tau_n}(t) - \varphi(t) + \varphi(s))) + \mathcal{D}(z_{\tau_n}(s), z_{\tau_n}(t))\\ &\qquad + R_n(s,t),\end{aligned} \quad (5.22)$$

where now

$$R_n(s,t) := \mathcal{W}(z_{\tau_n}(t), e(v_{\tau_n}(t) + \varphi_{\tau_n}(s) - \varphi_{\tau_n}(t))) \\ -\mathcal{W}(z_{\tau_n}(t), e(v_{\tau_n}(t) - \varphi(t) + \varphi(s))).$$

As in Section 5.3, it is easy to see that $R_n(s,t) \to 0$ as $n \to \infty$.

Since $s \in [0,T] \cap \mathbb{Q}$, we have

$$\begin{aligned} \boldsymbol{\delta}_{(z_{\tau_n}(s), e(v_{\tau_n}(s)))} &\rightharpoonup \nu_s \quad 2r\text{-weakly* as } n \to \infty, \\ \boldsymbol{\delta}_{\left(z_{\tau_{n_k^t}}(t), e(v_{\tau_{n_k^t}}(t))\right)} &\rightharpoonup \nu_t \quad 2r\text{-weakly* as } k \to \infty, \end{aligned} \tag{5.23}$$

where n_k^t is the subsequence chosen in Section 5.4, if $t \notin [0,T] \cap \mathbb{Q}$.

Hence, passing to the limit in inequality (5.22) we get

$$\langle W, \nu_s \rangle \leqq \langle W, \nu_t \rangle + \mathbb{D}(\mu_s, \mu_t) - \int_s^t \langle \sigma(\tau), e(\dot{\varphi}(\tau)) \rangle \, d\tau + R(s,t), \tag{5.24}$$

where

$$R(s,t) := \int_s^t \left\{ \int_{\Omega \times \mathbb{R} \times \mathbb{R}^{d\times d}} \left[-\frac{\partial W}{\partial \varepsilon}(\theta, \varepsilon + e(\varphi(\tau) - \varphi(t))) + \frac{\partial W}{\partial \varepsilon}(\theta, \varepsilon) \right] \right. \\ \left. : e(\dot{\varphi}(\tau)) \, d\nu_t(x, \theta, \varepsilon) \right\} d\tau.$$

By changing the choice of the subsequence in (5.23), we obtain inequality (5.24) for $s \in [0,T]$ and $t \in [0,T] \cap \mathbb{Q}$.

Now we use a measure theoretic result (see [10], or [9, Lemma 4.12] for a detailed proof), which allows us to approximate a Lebesgue integral by Riemann sums. For the reader's convenience we recall the statement of this result in the formulation of [12].

Lemma 5. *Let X be a Banach space, and let $F\colon [0,t] \to X$ be a Bochner integrable function. Then, there exists a sequence of partitions $\mathcal{S}_j := \{s_j^i,\ 0 \leqq i \leqq n_j\},\ j \in \mathbb{N}$ of the interval $[0,t]$, with*

$$0 = s_j^0 < \cdots < s_j^{n_j - 1} < s_j^{n_j} = t, \\ s_j^1 \leqq 1/j, \quad t - s_j^{n_j - 1} \leqq 1/j, \tag{5.25}$$

$$s_j^i - s_j^{i-1} = 1/j \quad \textit{for} \quad i = 2, \ldots, n_j - 1, \tag{5.26}$$

such that

$$\lim_j \sum_{i=1}^{n_j} \int_{s_j^{i-1}}^{s_j^i} \|F(s_j^i) - F(\tau)\| \, d\tau = 0.$$

We apply this Lemma to the functional defined by

$$F\ :\ [0,t]\ni\tau\mapsto(e(\dot\varphi(\tau)),\langle\sigma(\tau),e(\dot\varphi(\tau))\rangle)\in L^2(\Omega;\mathbb{R}^d)\times\mathbb{R}$$

in order to find a sequence of partitions $\mathcal{S}_j$ of $[0,t]$ satisfying requirements (5.25) and (5.26), and such that

$$\lim_j\sum_{i=1}^{n_j}\int_{s_j^{i-1}}^{s_j^i}\|e(\dot\varphi(s_j^i)-\dot\varphi(\tau))\|_2\,\mathrm{d}\tau=0, \tag{5.27}$$

$$\lim_j\sum_{i=1}^{n_j}\int_{s_j^{i-1}}^{s_j^i}|\langle\sigma(s_j^i),e(\dot\varphi(s_j^i))\rangle-\langle\sigma(\tau),e(\dot\varphi(\tau))\rangle|\,\mathrm{d}\tau=0. \tag{5.28}$$

Whenever both s_j^{i-1} and s_j^i belong to $[0,T]\setminus\mathbb{Q}$, we consider $t_j^{i-1}\in(s_j^{i-1},s_j^{i-1}+1/j^2)\cap\mathbb{Q}$, so that the estimate (5.24) holds true for s_j^{i-1},t_j^{i-1} and t_j^{i-1},s_j^i. Hence we get

$$\begin{aligned}\langle W,\nu_{s_j^{i-1}}\rangle&\leqq\langle W,\nu_{s_j^i}\rangle+\mathbb{D}(\mu_{s_j^{i-1}},\mu_{t_j^{i-1}})+\mathbb{D}(\mu_{t_j^{i-1}},\mu_{s_j^i})\\&-\int_{s_j^{i-1}}^{s_j^i}\langle\sigma(s_j^i),e\,(\dot\varphi(\tau))\rangle\,\mathrm{d}\tau-\int_{s_j^{i-1}}^{t_j^{i-1}}\langle(\sigma(t_j^{i-1})-\sigma(s_j^i)),e\,(\dot\varphi(\tau))\rangle\,\mathrm{d}\tau\\&+R(s_j^{i-1},t_j^{i-1})+R(t_j^{i-1},s_j^i).\end{aligned}$$

Summing up with respect to i and using $(E0)$, we get

$$\begin{aligned}&\mathcal{W}(z_0,e(v_0))-\langle W,\nu_t\rangle-\mathrm{Diss}(\nu;0,t)\\&\leqq-\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{s_j^i}\langle\sigma(s_j^i),e(\dot\varphi(\tau))\rangle\,\mathrm{d}\tau-\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{t_j^{i-1}}\langle(\sigma(t_j^{i-1})-\sigma(s_j^i)),e(\dot\varphi(\tau))\rangle\,\mathrm{d}\tau\\&\quad+\sum_{i=1}^{i_j}[R(s_j^{i-1},t_j^{i-1})+R(t_j^{i-1},s_j^i)].\end{aligned}$$

By arguing as in [11, Lemma 7.5], we deduce that

$$\sum_{i=1}^{i_j}[R(s_j^{i-1},t_j^{i-1})+R(t_j^{i-1},s_j^i)]\to 0\quad\text{as } j\to\infty.$$

We now use Hölder's inequality and the fact that $\sup_t\|\sigma(t)\|_2$ is bounded by estimate (5.11) in order to deduce that

$$\left|\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{t_j^{i-1}}\langle(\sigma(t_j^{i-1})-\sigma(s_j^i)),e(\dot\varphi(\tau))\rangle\,\mathrm{d}\tau\right|\to 0\quad\text{as } j\to\infty.$$

We have

$$
\begin{aligned}
&\left| \sum_{i=1}^{i_j} \int_{s_j^{i-1}}^{s_j^i} \langle \sigma(s_j^i), e(\dot\varphi(\tau))\rangle \, \mathrm{d}\tau - \int_0^t \langle \sigma(\tau), e(\dot\varphi(\tau))\rangle \, \mathrm{d}\tau \right| \\
&\quad \leqq \left| \sum_{i=1}^{i_j} \int_{s_j^{i-1}}^{s_j^i} \langle \sigma(s_j^i), e(\dot\varphi(\tau))\rangle \, \mathrm{d}\tau - \sum_{i=1}^{i_j} \int_{s_j^{i-1}}^{s_j^i} \langle \sigma(s_j^i), e(\dot\varphi(s_j^i))\rangle \, \mathrm{d}\tau \right| \\
&\qquad + \left| \sum_{i=1}^{i_j} \int_{s_j^{i-1}}^{s_j^i} \langle \sigma(s_j^i), e(\dot\varphi(s_j^i))\rangle \, \mathrm{d}\tau - \int_0^t \langle \sigma(\tau), e(\dot\varphi(\tau))\rangle \, \mathrm{d}\tau \right|.
\end{aligned} \tag{5.29}
$$

Using properties (5.27) and (5.28) it is now possible to show that the two last lines of (5.29) converge to 0 as $j \to \infty$, and hence we get

$$
\mathcal{W}(z_0, e(v_0)) + \int_0^t \langle \sigma(\tau), e(\dot\varphi(\tau))\rangle \, \mathrm{d}\tau \leqq \langle W, \nu_t \rangle + \mathrm{Diss}(\nu; 0, t),
$$

which, together with inequality (5.21), gives $(E4)$.

Remark 4. (Properties of the barycentre of the evolution) Let W be a convex function, $\varphi \in AC([0,T]; W^{1,p}(\Omega;\mathbb{R}^d))$, $p > 2$, $z_0 \in L^1(\Omega;[0,1])$, $v_0 \in \varphi(0) + H_0^1(\Omega;\mathbb{R}^d)$, and $(\nu_t)_{t\in[0,T]}$ be a damage quasi-static evolution. Let $(z_b(t), e(v_b(t))$ be the barycentre of ν_t, for every t. A natural question is whether $(z_b(t), e(v_b(t))$ can be seen as a quasi-static evolution, too. Let us focus on the stability condition. Thanks to Jensen's inequality, the global stability for the internal variable $(E3)$, satisfied by ν_t, gives

$$
\begin{aligned}
\mathcal{W}(z_b(t), e(v_b(t))) &\leqq \langle W, \nu_t \rangle \\
&\leqq \langle W, (\tilde\mu^x \otimes \delta_{e(\tilde v(x))}) \rangle + \rho \left[\int_{\Omega\times\mathbb{R}} \theta \, \mathrm{d}\mu_t(x,\theta) - \int_{\Omega\times\mathbb{R}} \theta \, \mathrm{d}\tilde\mu(x,\theta) \right],
\end{aligned}
$$

for every $\tilde\mu$ with $\mu_t \succeq \tilde\mu$ and every $\tilde v \in \varphi(t) + H_0^1(\Omega;\mathbb{R}^d)$. In particular, let us consider $\tilde\mu := \boldsymbol{\delta}_{\tilde z}$, for $\tilde z \in L^1(\Omega)$; we get

$$
\mathcal{W}(z_b(t), e(v_b(t))) \leqq \mathcal{W}(\tilde z, e(\tilde v)) + \int_\Omega \rho(z_b(t) - \tilde z) \, \mathrm{d}x,
$$

whenever $\mu_t \succeq \boldsymbol{\delta}_{\tilde z}$. Since $\mu_t \succeq \boldsymbol{\delta}_{\tilde z}$ implies $z_b(t) \geqq \tilde z$ almost everywhere in Ω (see Remark 1), we get

$$
\mathcal{W}(z_b(t), e(v_b(t))) \leqq \mathcal{W}(\tilde z, e(\tilde v)) + \mathcal{D}(z_b(t), \tilde z). \tag{5.30}
$$

Unfortunately, as observed in Remark 1, it may happen that $\tilde z \leqq z_b(t)$ almost everywhere in Ω, but $\mu_t \not\succeq \boldsymbol{\delta}_{\tilde z}$. Therefore, the minimality condition (5.30) is true, again, only for a restricted class of competitors $\tilde z$ (specifically, for those with $\mu_t \succeq \boldsymbol{\delta}_{\tilde z}$), and is not the desired complete stability property.

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Appendix

In this appendix, we prove a result which has been used in the proof of Theorem 2 in order to construct a discrete version of a Young measure coupling two other given measures.

Theorem 5. (Matrix reconstruction) *Given fixed $n \in \mathbb{N}$, let $(A_i)_{i=1}^n$, $(B_j)_{j=1}^n$ be two vectors in $[0,1]^n$ satisfying the following conditions:*

$$\sum_{i=1}^{k} A_i \leqq \sum_{j=1}^{k} B_j \quad \textit{for every } k \leqq n, \tag{6.1}$$

$$\sum_{i=1}^{n} A_i = \sum_{j=1}^{n} B_j. \tag{6.2}$$

Then there exist a matrix $(C_{ij})_{i,j=1}^n$ with entries in $[0,1]$ such that

$$\sum_{i=1}^{n} C_{ij} = B_j, \tag{6.3}$$

$$\sum_{j=1}^{n} C_{ij} = A_i, \tag{6.4}$$

$$C_{ij} = 0 \quad \textit{if } i < j. \tag{6.5}$$

The following lemma will be used to prove Theorem 5, by induction.

Lemma 6. (Iteration) *Given two vectors $(A_i)_{i=1}^n$ and $(B_j)_{j=1}^n$ in $[0,1]^n$ satisfying assumptions (6.1) and (6.2), there exists a vector $(C_{i1})_{i=1}^n$ in $[0,1]^n$ such that*

$$C_{11} = A_1, \tag{6.6}$$

$$C_{i1} \leqq A_i \quad \textit{for every } i, \tag{6.7}$$

$$\sum_{i=1}^{n} C_{i1} = B_1, \tag{6.8}$$

$$\sum_{i=2}^{k} (A_i - C_{i1}) \leqq \sum_{j=2}^{k} B_j, \quad \textit{for every } 2 \leqq k \leqq n, \tag{6.9}$$

$$\sum_{i=2}^{n} (A_i - C_{i1}) = \sum_{j=2}^{n} B_j. \tag{6.10}$$

Proof. According to (6.6), let us recursively define

$$C_{11} := A_1, \quad C_{i1} := A_i - \Big[A_i - \Big(B_1 - \sum_{k=1}^{i-1} C_{k1}\Big)\Big]^+, \text{ for } i > 1.$$

We observe that $B_1 - C_{11} = B_1 - A_1 \geqq 0$ by assumption (6.1), and that for $i > 2$

$$\begin{aligned} B_1 - \sum_{k=1}^{i-1} C_{k1} &= B_1 - \sum_{k=1}^{i-2} C_{k1} - C_{i-1,1} \\ &= B_1 - \sum_{k=1}^{i-2} C_{k1} - \left\{ A_{i-1} - \left[A_{i-1} - \left(B_1 - \sum_{k=1}^{i-2} C_{k1} \right) \right]^+ \right\} \\ &= -\left[A_{i-1} - \left(B_1 - \sum_{k=1}^{i-2} C_{k1} \right) \right] + \left[A_{i-1} - \left(B_1 - \sum_{k=1}^{i-2} C_{k1} \right) \right]^+ \\ &= 0 \vee \left\{ -\left[A_{i-1} - \left(B_1 - \sum_{k=1}^{i-2} C_{k1} \right) \right] \right\} \geqq 0. \end{aligned}$$

In particular, we have $A_i - (B_1 - \sum_{k=1}^{i-2} C_{k1}) \leqq A_i$ and $0 \leqq C_{i1} = A_i - [A_i - (B_1 - \sum_{k=1}^{i-2} C_{k1})]^+ \leqq A_i \leqq 1$. Hence, $C_{i1} \in [0, 1]$ for every i, and condition (6.7) holds true.

Now, we show that there exists i such that $C_{i1} = B_1 - \sum_{k=1}^{i-1} C_{k1}$. By contradiction, let us suppose that for every $i = 1, \dots, n$ we have $C_{i1} = A_i$ and hence $A_i < B_1 - \sum_{k=1}^{i-1} C_{k1}$. In particular, thanks to assumption (6.2), we have

$$A_n < B_1 - \sum_{k=1}^{n-1} C_{k1} = B_1 - \sum_{k=1}^{n-1} A_k = B_1 - \sum_{k=1}^{n} B_k + A_n = -\sum_{k=2}^{n} B_k + A_n,$$

which is a contradiction since $B_j \geqq 0$ for every j. Hence, there exists $\bar{i}$ such that $A_{\bar{i}1} = B_1 - \sum_{k=1}^{\bar{i}-1} C_{k1}$. This implies that $C_{i1} = 0$ for every $i > \bar{i}$ and that $\sum_{i=1}^{n} C_{i1} = \sum_{i=1}^{\bar{i}} C_{i1} = \sum_{i=1}^{\bar{i}-1} C_{i1} + B_1 - \sum_{i=1}^{\bar{i}-1} C_{i1} = B_1$, so condition (6.8) is satisfied.

Using $C_{11} = A_1$ and (6.8), we obtain condition (6.10). Indeed, we have

$$\sum_{i=2}^{n} (A_i - C_{i1}) = \sum_{i=1}^{n} A_i - \sum_{i=1}^{n} C_{i1} = \sum_{i=1}^{n} A_i - B_1 = \sum_{j=1}^{n} B_j - B_1 = \sum_{j=2}^{n} B_j.$$

It remains only to show inequality (6.9). We prove it by induction on k. For $k = 2$, we have $A_2 - C_{21} = [A_2 - (B_1 - A_1)]^+ = [A_1 + A_2 - B_1]^+ = 0 \vee [A_1 + A_2 - B_1] \leqq 0 \vee B_2 = B_2$, thanks to assumption (6.1). Let us now assume that inequality (6.9) holds for $k - 1$. Thanks to condition (6.7) and assumption (6.1), we have

$$\sum_{i=2}^{k} (A_i - C_{i1}) = \sum_{i=2}^{k-1} (A_i - C_{i1}) + A_k - C_{k1}$$

$$= \sum_{i=2}^{k-1} (A_i - C_{i1}) + \Big[A_k - \Big(B_1 - \sum_{i=1}^{k-1} C_{i1}\Big)\Big]^+$$
$$= \sum_{i=2}^{k-1} (A_i - C_{i1}) + \Big[0 \vee \Big(A_k - B_1 + \sum_{i=1}^{k-1} C_{i1}\Big)\Big]$$
$$\leqq \sum_{i=2}^{k-1} (A_i - C_{i1}) \vee \Big(\sum_{i=1}^{k} A_i - B_1\Big) \leqq \sum_{i=2}^{k-1} (A_i - C_{i1}) \vee \sum_{j=2}^{k} B_j;$$

the inductive hypothesis implies that $\sum_{i=2}^{k-1}(A_i - C_{i1}) \leqq \sum_{j=2}^{k-1} B_j \leqq \sum_{j=2}^{k} B_j$, and hence we can conclude that (6.9) holds true for every $k \geqq 2$. □

We are now able to prove Theorem 5.

Proof of Theorem 5. For $j = 1$ we define C_{i1} as in Lemma 6. For $2 \leqq j \leqq n$, we repeat the construction of Lemma 6, with $(A_i)_{i=1}^n$, $(B_j)_{j=1}^n$ substituted by the vectors $(A_i - \sum_{k=1}^{j-1} C_{ik})_{i=j}^n$ and $(B_k)_{k=j}^n$. Thanks to properties (6.9) and (6.10) we can prove by induction that the vectors $(A_i - \sum_{k=1}^{j-1} C_{ik})_{i=j}^n$ and $(B_k)_{k=j}^n$ satisfy the assumption of the lemma. For $i < j$, we define $C_{ij} := 0$, so that condition (6.5) is satisfied. Due to identity (6.8), condition (6.3) holds true for every j. Thanks to this construction, we have $C_{ii} = A_i - \sum_{k=1}^{i-1} C_{ik}$ for every i. In particular, $\sum_{j=1}^{n} C_{ij} = \sum_{j=1}^{i} C_{ij} = C_{ii} + \sum_{j=1}^{i-1} C_{ij} = A_i$, for every i, and therefore property (6.4) is fulfilled. □

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