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THE BREZIS–EKELAND PRINCIPLE FOR DOUBLY NONLINEAR EQUATIONS*

ULISSE STEFANELLI†

Abstract. The celebrated Brezis–Ekeland principle [*C. R. Acad. Sci. Paris Ser. A-B*, 282 (1976), pp. Ai, A1197–A1198, Aii, and A971–A974] characterizes trajectories of nonautonomous gradient flows of convex functionals as solutions to suitable minimization problems. This note extends this characterization to doubly nonlinear evolution equations driven by convex potentials. The characterization is exploited in order to establish approximation results for gradient flows, doubly nonlinear equations, and rate-independent evolutions.

Key words. Brezis–Ekeland principle, gradient flow, doubly nonlinear equation, rate-independent evolution, Mosco convergence, Young measures, approximation

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1. Introduction. Let H denote a Hilbert space and $T > 0$ be some final reference time. Moreover, let $\phi : H \to (-\infty, \infty]$ be convex, proper, and lower semicontinuous, $f \in L^2(0,T;H)$, and $u^0 \in D(\phi) := \{v \in H \ : \ \phi(v) \neq \infty\}$. The variational principle formulated by Brezis and Ekeland [17, 18] and Nayroles [77, 78] (see also [9, sect. 3.4]) characterizes solutions $u \in H^1(0,T;H)$ of the gradient flow

$$u' + \partial\phi(u) \ni f \quad \text{a.e. in } \ (0,T), \quad u(0) = u^0 \tag{1.1}$$

(where the prime stands for time differentiation and $\partial\phi$ is the subdifferential of ϕ in the sense of convex analysis; see below) as a global minimizer of the functional $J : H^1(0,T;H) \to [0,\infty]$ defined as

$$J(u) := \int_0^T \big(\phi(u) + \phi^*(f - u') - (f,u)\big) + \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2 + |u(0) - u^0|^2. \tag{1.2}$$

Here $(\cdot,\cdot)$ is the scalar product in H, $|\cdot|$ is the corresponding norm, and we have denoted by ϕ^* the conjugate of ϕ, i.e., $\phi^*(w) := \sup\{(w,u) - \phi(u), \ u \in H\}$ for all $w \in H$. Let us stress that $\phi(u) + \phi^*(w) \geq (w,u)$ for all $u, w \in H$ and that the equality holds iff $w \in \partial\phi(u)$. In particular, one readily checks that $J(v) \geq 0$ for all $v \in H^1(0,T;H)$ and that $J(u) = \min J = 0$ iff u solves the gradient flow (1.1). Namely, the unique solution to (1.1) and the unique minimizer of J coincide, and we have the following.

THEOREM 1.1 (Brezis and Ekeland [17, 18]). *u solves* (1.1) *iff* $J(u) = 0$.

The aim of this note is to extend the latter characterization result to the more general situation of doubly nonlinear equations. In particular, let a second convex, proper, and lower semicontinuous functional $\psi : H \to (-\infty, \infty]$ be given. We are interested in solving for $u \in W^{1,p}(0,T;H)$, $p \in [1,\infty]$, the equation

$$\partial\psi(u') + \partial\phi(u) \ni f \quad \text{a.e. in } \ (0,T), \quad u(0) = u^0, \tag{1.3}$$

where now $f \in L^q(0,T;H)$, with $1/p + 1/q = 1$ (usual convention: $1/\infty = 0$).

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http://www.siam.org/journals/sicon/47-3/68457.html
†IMATI - CNR, v. Ferrata 1, I-27100 Pavia, Italy (ulisse.stefanelli@imati.cnr.it).

The latter equation arises in a variety of different applicative contexts. In particular, inclusion (1.3) may represent a generalized balance relation in thermomechanics. The reader is referred to Moreau [74, 75] and Germain [30] for some justification and to Colli and Visintin [23] and Colli [22] for existence results for functionals ψ of p-growth for $1 < p < \infty$ (see also Barbu [12], Arai [3], and Senba [95] among others). The linear-growth case $p = 1$ is strictly related with the modeling of rate-independent evolution and has been considered in connection with elasto-plasticity [25, 26, 58, 59, 60, 61], damage [68], brittle fractures [27], delamination [56], ferroelectricity [73], shape-memory alloys [67, 71, 72], and vortex pinning in superconductors [94]; see Mielke [62] for a comprehensive survey of mathematical results.

We shall make precise problem (1.3) by introducing an auxiliary function. Namely, we consider $(u, v) \in W^{1,p}(0, T; H) \times L^q(0, T; H)$ such that

$$v \in \partial\psi(u') \quad \text{a.e. in } (0, T), \tag{1.4}$$

$$v + \partial\phi(u) \ni f \quad \text{a.e. in } (0, T), \tag{1.5}$$

$$u(0) = u^0. \tag{1.6}$$

The above relations have a clear mechanical interpretation. By letting u represent the displacement of a body from its reference configuration, ϕ can be interpreted as the corresponding energy, and ψ stands for the related dissipation potential. In particular, relation (1.5) expresses the balance among the system of conservative forces $\partial\phi(u)$, the dissipative (viscous) force v, and the external load f. On the other hand, relation (1.4) consists of a multivalued constitutive relation for the dissipative forces.

Let now the functional $I : W^{1,p}(0, T; H) \times L^q(0, T; H) \to [0, \infty]$ be defined as

$$\begin{aligned} I(u, v) := &\left(\int_0^T \big(\psi(u') + \psi^*(v) - (f, u') \big) + \phi(u(T)) - \phi(u^0) \right)^+ \\ &+ \int_0^T \big(\phi(u) + \phi^*(f - v) - (f - v, u) \big) + |u(0) - u^0|^2. \end{aligned} \tag{1.7}$$

Note that, exactly as for J, no derivatives of the potentials appear in the definition of I, making its formulation suited for nonsmooth situations.

The key point of this note is to check that solutions of (1.4)–(1.6) are precisely the (possibly nonunique) minimizers of the nonnegative functional I. In particular, we prove the following.

THEOREM 1.2. *(u, v) solves (1.4)–(1.6) iff $I(u, v) = 0$.*

Let us explicitly remark that Theorem 1.2 implies the Brezis–Ekeland characterization of Theorem 1.1; namely, the present analysis extends the former. In order to prove this fact, some care has to be used since in the quadratic case $\psi(\cdot) = |\cdot|^2/2$ the functionals $I(u, u')$ and $J(u)$ do not coincide. Precisely, we shall define $K : H^1(0, T; H) \to [0, \infty]$ by

$$K(u) := I(u, u') = \left(\int_0^T \big(|u'|^2 - (f, u') \big) + \phi(u(T)) - \phi(u^0) \right)^+ + J(u). \tag{1.8}$$

Theorem 1.1 will follow from Theorem 1.2 once we check that K and J have the same minimizer. This is exactly the point of the following.

LEMMA 1.3. *$K(u) = 0$ iff $J(u) = 0$.*

Indeed, by letting $J(u) = 0$ we shall prove that $K(u) = 0$ (the converse implication being obvious). Since u is a solution to (1.1) by Theorem 1.1, the chain rule [16, Lem. 3.3, p. 73] yields

$$\phi(u(T)) - \phi(u^0) = \int_0^T \frac{\mathrm{d}}{\mathrm{dt}}\phi(u) = \int_0^T (f - u', u').$$

Hence, the positive part in (1.8) is 0, and $K(u) = J(u) = 0$.

The functional J is convex and lower semicontinuous with respect to the weak topology of $H^1(0,T;H)$. Hence, one is tempted to exploit the Brezis–Ekeland characterization of Theorem 1.1 in order to obtain solutions to the gradient flow (1.1) by applying the direct method to J. This strategy is, however, much more involved than the classical maximal-monotone operator techniques (see Brezis [16]). The difficulty arises from the fact that one is not just asked to minimize J but also to prove that the minimum is 0 (the difficulty of proving the existence of a minimizer for J without using the equation was already pointed out in [18, Rem. 1]). Incidentally, note that J may fail to be coercive with respect to the weak topology of $H^1(0,T;H)$ unless ϕ^* has at least a quadratic growth and hence ϕ is quadratically bounded (take $\phi^*(\cdot) = |\cdot|$, $f = 0$, $u^0 = 0$, and u_n to be any $W^{1,1}(0,T;H)$-bounded sequence with $|u_n| \leq 1$ and u'_n unbounded in $L^2(0,T;H)$). On the other hand, K is clearly convex, lower semicontinuous, and coercive with respect to the weak topology of $H^1(0,T;H)$.

Conditional existence results for the gradient flow (1.1) by means of the direct method were first obtained by Rios [85, 88] (see also [86, 87]). Later on, Auchmuty [10] proved that in the controlled-growth case the minimum problem can be reformulated as a saddle point problem for which the minimax value 0 is achieved (see also [8]). Again in the controlled-growth case and by assuming ϕ to be continuously differentiable, Roubíček [90] directly checked that the optimality conditions imply (1.1) (see also the recent monograph [91, sect. 8.10]). Finally, the full extent of maximal-monotone methods has been recovered via the Brezis–Ekeland approach by Ghoussoub and Tzou [39]. In the latter paper, the authors eventually overcome the controlled-growth assumption and recast the problem within the far-reaching theory of (anti)self-dual Lagrangians by Ghoussoub [35, 34, 31, 32, 33, 37, 41, 40] (see also the monograph [36]). We mention some further results by Ghoussoub and McCann [38] for quadratic perturbations of convex functionals and the analysis of the long-time dynamics of autonomous gradient flows by Lemaire [51].

Our main focus here is, however, not on existence but rather on the application of the characterization result of Theorem 1.2 to the analysis of general approximation issues. Since solutions and minimizers of the respective functionals coincide, a quite natural idea in order to frame an abstract approach to limiting procedures is that of considering the corresponding approximating minimum problems via Γ-convergence [43, 24]. We shall apply this perspective and generalize some known approximation results for both gradient flows (section 6) and doubly nonlinear equations (section 7). Moreover, we obtain a new proof of a convergence result by Mielke, Roubíček, and Stefanelli [69] for the case of rate-independent problems in section 8.

The key step in the direction of approximations is a suitable Γ-lim inf tool settled in the frame of Young measures with values in separable and reflexive Banach spaces (see subsection 4.2). Let us mention that this perspective has already been considered by Pedregal [81, 82] and Michaille and Valadier [57] in the frame of Sobolev spaces. Here, moving from some recent version in weak topologies of the fundamental result by Balder [11, Thm. 1], we deduce a useful tool in order to pass to lower limits in sequences of integral functionals whose integrands fulfill a suitable Γ-lim inf inequality. As a by-product, we obtain some generalization of former results by Salvadori [92, Thm. 3.1].

By applying the above-mentioned approximation results, we show in subsection 7.1 the existence of solutions for a class of doubly nonlinear equations by passing to the limit via Theorem 1.2 within a suitable class of regularized problems. In particular, we recover by means of a variational technique a former existence result by Colli and Visintin [23, Thm. 2.1].

Let us mention that some refined version of the functional I and Theorem 1.2 is discussed in [103] for the specific situation of linearized elastoplasticity with hardening. In particular, the (classical) well-posedness theory and the (more recent) convergence for time and space discretizations [47] is there recovered by means of a variational technique.

Let us close this introduction by observing that, besides the Brezis–Ekeland principle, a variety of global variational principles for dissipative evolutions have already been proposed. We mention Biot's work on irreversible thermodynamics [15] and Gurtin's principle for viscoelasticity and elastodynamics [44, 45, 46] among many others (see also the survey in Hlaváček [48]). We shall not attempt to give here a comprehensive report on the literature but rather concentrate on the specific case of doubly nonlinear evolutions. In this concern, the reader is referred to Visintin [105], where generalized solutions are obtained as minimal elements of a certain partial-order relation on the trajectories, and Mielke and Ortiz [63] (see also [70]), where solutions in the rate-independent case are recovered as suitable limits of relaxed global minimization problems.

Remark 1.4. The formulation in (1.2) is not the original one but is rather some modification due to Rockafellar (see again [18]) also considered in Ghoussoub and Tzou [39].

2. Characterization.

The Brezis–Ekeland characterization of Theorem 1.1 makes no essential use of the Hilbert-space structure. Hence, let us move from the very beginning to the reflexive Banach-space framework and start by enlisting our assumptions:

(A1) $p \in [1,\infty]$, $1/p+1/q=1$, and H is a real reflexive Banach space with norm $|\cdot|$. We shall use the symbol $(\cdot,\cdot)$ for the duality pairing between H^* (dual) and H.

(A2) $\phi, \psi : H \to (-\infty,\infty]$ are proper, convex, and lower semicontinuous.

(A3) $f \in L^q(0,T;H^*)$ and $u^0 \in D(\phi) := \{v \in H \ : \ \phi(v) \neq \infty\}$.

The subdifferentials occurring in the formulation of the Cauchy problem (1.4)–(1.6) are now acting from H to H^* being defined as

$$w \in \partial\phi(z) \quad \text{iff} \quad z \in D(\phi) \ \text{ and } \ (w, x-z) \leq \phi(x) - \phi(z) \ \ \forall x \in H$$

and analogously for $\partial\psi$. Notations have been chosen in such a way that the definition of the functional I in (1.7) still makes sense in the above Banach-space setting. For the sake of clarity, we shall restate here our main result.

THEOREM 2.1. *Under assumptions* (A1)–(A3)*, the pair* (u,v) *solves* (1.4)–(1.6) *iff* $I(u,v)=0$.

The proof of Theorem 2.1 relies on a suitable Banach version of the chain rule [16, Lem. 3.3, p. 73]. We state it here for the sake of completeness and provide a direct proof.

PROPOSITION 2.2 (chain rule). *Under assumption* (A1), *let* $\phi : H \to (-\infty, \infty]$ *be proper, convex, and lower semicontinuous,* $u \in W^{1,p}(0,T;H)$*, and* $w \in L^q(0,T;H^*)$ *be such that* $w \in \partial\phi(u)$ *almost everywhere in* $(0,T)$*. Then* $t \mapsto \phi(u(t))$ *is absolutely continuous and*

$$\phi(u(t)) - \phi(u(s)) = \int_s^t (z, u') \quad \forall 0 \le s \le t \le T,$$

for all $z \in L^q(0,T;H^*)$*, with* $z \in \partial\phi(u)$ *almost everywhere in* $(0,T)$*.*

Proof. Let us assume from the very beginning that both H and H^* are strictly convex (H can be equivalently renormed in such a way that this holds [5, 6]). We let $w_\varepsilon := \partial\phi_\varepsilon(u)$ for $\varepsilon > 0$, where ϕ_ε is the Yosida approximation of ϕ (see [13, Prop. 1.1, p. 42] for definition and properties). Since ϕ_ε is Gâteaux differentiable [13, Thm. 2.2, p. 57] we have

$$\phi_\varepsilon(u(t)) - \phi_\varepsilon(u(s)) = \int_s^t (w_\varepsilon, u') \quad \forall 0 \le s \le t \le T. \tag{2.1}$$

Moreover, one has $|w_\varepsilon|_* \le |w^\circ|_* \le |w|_*$ almost everywhere in $(0,T)$, where $|\cdot|_*$ is the norm of H^* and $w^\circ := (\partial\phi(u))^\circ$ is the element of minimal norm in $\partial\phi(u)$. Hence, for all $t \in (0,T)$ such that $|w_\varepsilon(t)|_* \le |w^\circ(t)|_* \le |w(t)|_*$, one can extract a not relabeled (and a priori depending on t) weakly convergent subsequence $w_\varepsilon(t)$. On the other hand, by using [13, Prop. 1.1, p. 42] we have $w_\varepsilon(t) \to w^\circ(t)$ weakly in H^*. In particular, the whole sequence $w_\varepsilon(t)$ converges, and we have $w_\varepsilon \to w^\circ$ weakly in H^* pointwise almost everywhere in $(0,T)$. By exploiting the convergence of Yosida approximations and the dominated convergence theorem and by passing to the limit as $\varepsilon \to 0$ in (2.1), we prove that $t \mapsto \phi(u(t))$ is absolutely continuous on $[0,T]$.

Let now $t \in (0,T)$ be a point, where $t \mapsto \phi(u(t))$ is differentiable and $u(t) \in D(\partial\phi) := \{v \in H \ : \ \partial\phi(v) \neq \emptyset\}$, and let $z \in \partial\phi(u(t))$. Then

$$(z, x - u(t)) \le \phi(x) - \phi(u(t)) \quad \forall x \in H.$$

By choosing $x = u(t \pm h)$ for $h > 0$ and passing to the limit as $h \to 0$ we readily check that

$$\frac{\mathrm{d}}{\mathrm{dt}}\phi(u(t)) = (z, u'(t)),$$

and the assertion follows. ∎

Proof of Theorem 2.1. Let $(u,v) \in W^{1,p}(0,T;H) \times L^q(0,T;H^*)$ solve (1.4)–(1.6). In particular, we have

$$\phi(u) + \phi^*(f - v) = (f - v, u), \quad \psi(u') + \psi^*(v) = (v, u') \quad \text{a.e. in } (0,T)$$

so that $I(u,v) < \infty$. Moreover, owing to Proposition 2.2, the map $t \mapsto \phi(u(t))$ is absolutely continuous and

$$\phi(u(T)) - \phi(u^0) = \int_0^T \frac{\mathrm{d}}{\mathrm{dt}}\phi(u) = \int_0^T (f - v, u'). \tag{2.2}$$

Hence, since $u(0) = u^0$,

$$I(u,v) = \left(\int_0^T \big(\psi(u') + \psi^*(v) - (v, u') \big) \right)^+ = 0.$$

On the other hand, let $(u,v) \in W^{1,p}(0,T;H) \times L^q(0,T;H^*)$ be such that $I(u,v) = 0$. The functional I results from the sum of three nonnegative contributions, namely, the positive-part term, the integral term, and the term taking into account the initial datum. As $I(u,v) = 0$, one has that all three terms must be 0. In particular, $u(0) = u^0$, and relation (1.5) holds. By using Proposition 2.2 and the already established (1.5) we have

$$\int_0^T \big(\psi(u') + \psi^*(v) - (v,u')\big)$$
$$= \int_0^T \big(\psi(u') + \psi^*(v) - (f,u')\big) + \phi(u(T)) - \phi(u^0) \leq 0,$$

where the last inequality follows from the fact that the positive-part term in $I(u,v)$ is 0. Finally, as $\psi(u') + \psi^*(v) \geq (v,u')$ almost everywhere in $(0,T)$, relation (1.4) holds as well. $\square$

Let us remark that the reflexive Banach frame of assumption (A1) includes the original Hilbert-space setting of Theorem 1.1. In particular, the Brezis–Ekeland characterization follows from Theorem 2.1 via Lemma 1.3.

Before going on, we shall comment that the alternative (and somehow more natural) choice $\tilde{I} : W^{1,p}(0,T;H) \times L^q(0,T;H) \to [0,\infty]$ given by

$$\tilde{I}(u,v) := \int_0^T \big(\psi(u') + \psi^*(v) - (v,u')\big)$$
$$+ \int_0^T \big(\phi(u) + \phi^*(f-v) - (f-v,u)\big) + |u(0) - u^0|^2$$

would lead to the same characterization of Theorem 2.1. On the other hand, the presence of the term (v,u') prevents the functional $\tilde{I}$ from being lower semicontinuous with respect to the natural topologies related to the doubly nonlinear problem (1.3) (see below), making the functional $\tilde{I}$ not interesting.

In the specific case of a Hilbert space H and a quadratic potential $\phi(\cdot) = |\cdot|^2/2$, relation (1.3) takes the form

$$\partial\psi(u') + u \ni f \quad \text{a.e. in } \ (0,T), \quad u(0) = u^0. \tag{2.3}$$

By letting the functional $Q : W^{1,p}(0,T;H) \to [0,\infty]$ be defined as

$$Q(u) := \int_0^T \big(\psi(u') + \psi^*(f-u) - (f,u')\big) + \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2 + |u(0) - u^0|^2,$$

one easily checks via Fenchel's duality that $u \in W^{1,p}(0,T;H)$ solves (2.3) iff $Q(u) = \min Q = 0$. On the other hand, in the same spirit of (1.8), one could consider $R : W^{1,p}(0,T;H) \to [0,\infty]$ as $R(u) := I(u, f-u)$, and the analogue of Lemma 1.3 holds.

The case of a quadratic potential ϕ bears some relevance with respect to applications (see section 8 and [103]). Hence, we shall explicitly consider the functional Q in the following.

3. Some extension. As already mentioned in the introduction, Theorem 2.1 is valid in some more general frames. In particular, we shall consider the doubly nonlinear relation

$$\partial\psi(u'(t)) + \partial\varphi(t,u(t)) \ni 0 \quad \text{for a.e. } \ t \in (0,T), \quad u(0) = u^0, \tag{3.1}$$

where the time dependence is now included in φ and the second subdifferential is referred to the variable u only.

Letting X be a separable metric space, we denote by $\mathcal{B}(X)$ its Borel σ-algebra, by $\mathcal{L}$ the σ-algebra of the Lebesgue measurable subsets of $(0,T)$, and by $\mathcal{L}\otimes\mathcal{B}(X)$ the respective product σ-algebra. A $\mathcal{L}\otimes\mathcal{B}(X)$-measurable function $g:(0,T)\times X\to(-\infty,\infty]$ is said to be a *normal integrand* if

$$u\mapsto g(t,u) \quad \text{is lower semicontinuous for a.e. } \ t\in(0,T).$$

The assumptions read as follows:

(A4) $\psi:H\to(-\infty,\infty]$ is convex, proper, and lower semicontinuous,
$\varphi:[0,T]\times H\to(-\infty,\infty]$ is such that:
$u\mapsto\varphi(t,u)$ is proper and convex for a.e. $t\in(0,T)$,
for all separable subspaces $X\subset H$, the restriction of φ to $[0,T]\times X$ is a normal integrand,
there exists $\pi:W^{1,p}(0,T;H)\to L^1(0,T)$ such that, for $u\in W^{1,p}(0,T;H)$ and $w\in L^q(0,T;H^*)$, with $w(t)\in\partial\varphi(t,u(t))$ for a.e. $t\in(0,T)$, the mapping $t\mapsto\varphi(t,u(t))$ is absolutely continuous and satisfies

$$\varphi(t,u(t))-\varphi(s,u(s))=\int_s^t(w,u')+\int_s^t\pi(u)\quad\forall 0\le s\le t\le T. \tag{3.2}$$

(A5) $u^0\in D(\varphi(0,\cdot)):=\{v\in H\ :\ \varphi(0,v)\neq\infty\}$.

Assumption (A4) implies via Pettis' theorem that $t\mapsto\varphi(t,u(t))$ is measurable for all measurable $t\mapsto u(t)$.

As for the generalized chain rule stated in (A4), let us mention that π represents a power of external actions since, at least formally, $\pi=\partial_t\varphi$ (see also section 8 below). The chain rule (3.2) frequently holds in practice. In particular, it holds in the smooth case and if φ is a smooth perturbation of a convex function. The reader is referred to [64, Prop. 2.6] for a result in the nonperturbative case.

We will consider solutions $(u,v)\in W^{1,p}(0,T;H)\times L^q(0,T;H^*)$ of the Cauchy problem (see (1.4)–(1.6))

$$v\in\partial\psi(u')\quad\text{a.e. in }\ (0,T), \tag{3.3}$$

$$-v(t)\in\partial\varphi(t,u(t))\quad\text{for a.e. }\ t\in(0,T), \tag{3.4}$$

$$u(0)=u^0. \tag{3.5}$$

Hence, let us define the functional $\mathcal{I}$ acting on $W^{1,p}(0,T;H)\times L^q(0,T;H^*)$ as

$$\begin{aligned}\mathcal{I}(u,v):=&\left(\int_0^T\big(\psi(u')+\psi^*(v)-\pi(u)\big)+\varphi(T,u(T))-\varphi(0,u^0)\right)^+\\&+\int_0^T\big(\varphi(\cdot,u)+\varphi^*(\cdot,-v)+(v,u)\big)+|u(0)-u^0|^2,\end{aligned}$$

where the duality φ^* is taken with respect to the variable u only. The formulation of Theorem 1.2 in this setting reads as follows.

THEOREM 3.1. *Under assumptions* (A1) *and* (A4)–(A5), *the pair* (u,v) *solves* (3.3)–(3.5) *iff* $\mathcal{I}(u,v)=0$.

Sketch of the proof. This argument follows along the same lines as that of Theorem 1.2. All solutions to (3.3)–(3.5) are easily proved to be minimizers of $\mathcal{I}$ by means of the chain rule (3.2). On the other hand, let $(u,v) \in W^{1,p}(0,T;H) \times L^q(0,T;H^*)$ be such that $\mathcal{I}(u,v) = 0$. Then one has $u(0) = u^0$ and $-v(t) \in \partial\varphi(t,u(t))$ for almost every $t \in (0,T)$. Again by (3.2) one gets

$$\int_0^T \big(\psi(u') + \psi^*(v) - (v,u')\big)$$
$$= \int_0^T \big(\psi(u') + \psi^*(v) - \pi(u)\big) + \varphi(T,u(T)) - \varphi(0,u^0) \leq 0,$$

and the assertion follows. ∎

Remark 3.2. Let us stress that the present situation of (3.1) is not directly extending (1.3) as some extra time regularity is here exploited. As a matter of fact, the case $\varphi(t,u) = \phi(u) - (f(t),u)$ is included in the frame of (A4) for the smooth choice $f \in W^{1,q}(0,T;H^*)$ only.

Remark 3.3. We shall mention that the generalized situation of the equation

$$\partial\psi(u(t),u'(t)) + \partial\phi(t,u(t)) \ni 0 \quad \text{for a.e. } t \in (0,T), \quad u(0) = u^0, \tag{3.6}$$

where the functional $\psi : H \times H \to (-\infty,\infty]$ is convex in its second occurrence and subdifferentials are taken with respect to second variables, has recently attracted a good deal of attention. In particular (3.6) arises in connection with quasi-variational problems and has been considered in Aso, Frémond, and Kenmochi [4], Mielke [62], and Mielke, Rossi, and Savaré [66, 65]. The formulation of the above characterization result in Theorem 3.1 could be easily tailored to the case of (3.6) as well as to even more general situations (the nonlocal situations of [97, 98, 99, 100, 49], for instance) with no particular intricacy.

Before closing this section, let us explicitly mention that the choice H Hilbert space, $p = 2$, and $\psi(\cdot) = |\cdot|^2/2$ in relation (3.1) give rise to the generalized gradient flow

$$u'(t) + \partial\varphi(t,u(t)) \ni 0 \quad \text{a.e. in } (0,T), \quad u(0) = u^0, \tag{3.7}$$

whose consideration has to be traced back to Peralba [83, 84]. We shall consider (3.7) from the point of view of approximation in section 6 below. Let us explicitly observe there that the extension of the functional K to the latter time-dependent situation is $\mathcal{K} : H^1(0,T;H) \to [0,\infty]$ defined by

$$\mathcal{K}(u) := \mathcal{I}(u,u') = \left(\int_0^T \big(|u'|^2 - \pi(u)\big) + \varphi(T,u(T)) - \varphi(0,u^0)\right)^+$$
$$+ \int_0^T \big(\varphi(\cdot,u) + \phi^*(\cdot,-u')\big) + \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2 + |u(0) - u^0|^2.$$

4. Applications to approximation. We now apply the characterization results of Theorems 2.1 and 3.1 to the approximation of solutions of the gradient flows (1.1) and (3.7) and of the doubly nonlinear equations (1.3) and (3.1). As already mentioned in the introduction, we shall make here use of the notion of Γ-convergence. The reader is referred to the monographs by Attouch [7] and Dal Maso [24] for some comprehensive discussion of this topic as well as to subsection 4.1 for a (minimal) selection of results to be used later.

Our aim will be to present suitable assumptions for the corresponding approximating functionals to be Γ-converging (or rather Mosco-converging; see below). More precisely, since Theorems 2.1 and 3.1 directly quantify the value of the minima to be 0, what is actually needed for passing to limits are Γ-lim inf inequalities only. In subsection 4.2 we shall prepare a tool in order to deal with these kinds of problems in a quite general fashion. In particular, we will provide a Γ-lim inf result by exploiting the theory of Young measures for weak topologies in separable (and reflexive) Banach spaces. The application of the latter to the current convex situation is discussed in subsection 4.3 along with some relation with the former analysis by Salvadori [92]. Then in section 5 we systematically apply the Γ-lim inf tool to the functionals introduced in sections 1 and 3 and obtain the corresponding Γ-lim inf inequalities. Convergence results will then follow by simply checking the uniform coercivity of the approximating functionals with respect to suitable topologies. In the case of the gradient flows (1.1) and (3.7), we will show in section 6 that a natural choice is that of the weak topology in $H^1(0,T;H)$ (note that K is lower semicontinuous and coercive with respect to the latter). This leads us to generalize some known convergence results (see [7, sect. 3.9.2, p. 386]).

In the case of the doubly nonlinear equations (1.3) and (3.1) for $p \in (1,\infty)$, we will provide in section 7 some conditions implying uniform coercivity with respect to the topology

$$\mathcal{T}(p) = (\text{weak-}W^{1,p}(0,T;H) + \text{strong-}L^p(0,T;H)) \times \text{weak-}L^q(0,T;H^*) \tag{4.1}$$

(recall that I is lower semicontinuous with respect to $\mathcal{T}(p)$). Hence, under suitable nondegeneracy and growth-type assumptions (see (7.1)–(7.2)), section 7 leads to some generalization of former convergence results by Aizicovici and Yan [1].

The situation $p = 1$ is much more delicate, and we shall deal with it in the specific yet relevant case of rate-independent problems in section 8. By focusing on the functional frame of the recent well-posedness theory by Mielke and Rossi [64], we shall provide a new proof of a convergence result by Mielke, Roubíček, and Stefanelli [69], although in a simplified setting. Finally, the special case of (2.3) is discussed.

4.1. Preliminaries on Mosco convergence. Recall that H is a real reflexive Banach space. By letting $\phi_n, \phi : H \to (-\infty,\infty]$ be convex, proper, and lower semicontinuous, we say that the sequence ϕ_n is *uniformly proper* iff there exists a bounded sequence u_n such that $u_n \in D(\phi_n)$. Moreover, we say that $\phi_n \to \phi$ in the *Mosco sense* [7, 76] iff, for all $u \in H$,

$$\phi(u) \le \liminf_{n\to\infty} \phi_n(u_n) \quad \forall u_n \to u \ \text{ weakly in } \ H,$$
$$\exists u_n \to u \ \text{ strongly in } \ H \ \text{ such that } \ \phi(u) = \limsup_{n\to\infty} \phi_n(u_n).$$

In particular, $\phi_n \to \phi$ in the Mosco sense iff $\phi_n \to \phi$ in the sense of Γ-convergence with respect to both the weak and the strong topology in H. Let us stress that if $\phi_n \to \phi$ in the Mosco sense, then the sequence ϕ_n is uniformly proper. One has the following characterization.

LEMMA 4.1. *The following are equivalent:*

(i) $\phi_n \to \phi$ *in the Mosco sense in* H,

$$\text{(ii)} \begin{cases} \phi(u) \le \inf\big\{ \liminf_{n\to\infty} \phi_n(u_n) \ : \ u_n \to u \ \textit{ weakly in } \ H \big\}, \\ \phi^*(u) \le \inf\big\{ \liminf_{n\to\infty} \phi_n^*(u_n) \ : \ u_n \to u \ \textit{ weakly in } \ H^* \big\}, \\ \textit{and the sequence } \phi_n^* \textit{ is uniformly proper.} \end{cases}$$

Proof. Assumption (i) is equivalent to $\phi_n^* \to \phi^*$ in the Mosco sense [7, Thm. 3.18, p. 295]. Hence (ii) follows by the definition of Mosco convergence.

The converse implication (ii) $\Rightarrow$ (i) is more involved. Let us set for convenience, for all $u \in H$ and $v \in H^*$,

$$\phi_\sharp(u) := \min\big\{ \limsup_{n\to\infty} \phi_n(u_n) \ : \ u_n \to u \ \text{ strongly in } \ H\big\},$$
$$\phi_\flat(v) := \inf\big\{ \liminf_{n\to\infty} \phi_n^*(v_n) \ : \ v_n \to v \ \text{ weakly in } \ H^*\big\},$$

Owing to [7, Thm. 1.13, p. 29], the minimum in the definition of $\phi_\sharp$ is always attained in $(-\infty, \infty]$. In particular, for all $u \in H$,

$$\exists u_n \to u \ \text{ strongly in } \ H \ \text{ such that } \ \phi_\sharp(u) = \limsup_{n\to\infty} \phi_n(u_n). \tag{4.2}$$

The sequence ϕ_n^* is uniformly proper, and we can apply [7, Thm. 3.7, p. 271] in order to deduce that $\phi_\sharp = (\phi_\flat)^*$. Hence, since by the second part of (ii) we have $\phi^* \leq \phi_\flat$, we obtain by duality that $\phi_\sharp = (\phi_\flat)^* \leq \phi$. Finally, the first part of (ii) and (4.2) yield (i). ∎

Let us comment that, for the sake of establishing the following approximation results, only the implication (i) $\Rightarrow$ (ii) is exploited. We stated the above characterization in order to underline the fact that the Mosco convergence is the natural requirement for passing to the lim inf in the sum of functionals and their respective duals (which is precisely our situation).

4.2. A general Γ-lim inf result. We shall now discuss a technical tool which will be at the basis of the forthcoming analysis. In particular, we present here a Γ-lim inf result in the frame of Young measures for weak topologies. The reader is referred to Castaing, Raynaud de Fitte, and Valadier [20] for a comprehensive discussion of Young measures on separable Banach spaces.

Recall that, by letting X be a separable metric space, a $\mathcal{L} \otimes \mathcal{B}(X)$-measurable function $g : (0,T) \times X \to (-\infty, \infty]$ is said to be a *normal integrand* if

$$u \mapsto g(t,u) \quad \text{is lower semicontinuous for a.e. } \ t \in (0,T).$$

We denote by $\mathcal{M}(0,T;X)$ the set of all $\mathcal{L}$-measurable functions $w : (0,T) \to X$. Following [11], a sequence $w_n \in \mathcal{M}(0,T;X)$ is said to be *tight* if there exists a nonnegative normal integrand g such that

$$\{w \in X \ : \ g(t,w) \leq c\} \quad \text{is compact for a.e. } t \in (0,T) \text{ and all } c \geq 0, \tag{4.3}$$

$$\text{and} \quad \sup_n \int_0^T g(t, w_n(t))\,\mathrm{d}t < \infty. \tag{4.4}$$

In case X is a Banach space, a $\mathcal{L}\otimes\mathcal{B}(X)$-measurable function $h : (0,T)\times X \to (-\infty,\infty]$ is called a *weakly normal integrand* if

$$u \mapsto h(t,u) \quad \text{is weakly lower semicontinuous for a.e. } \ t \in (0,T), \tag{4.5}$$

and a sequence $u_n \in \mathcal{M}(0,T;X)$ is said to be *weakly tight* if there exists a nonnegative weakly normal integrand h such that

$$\lim_{|u|\to\infty} h(t,u) = \infty \quad \text{for a.e. } \ t \in (0,T), \tag{4.6}$$

$$\text{and} \quad \sup_n \int_0^T h(t, u_n(t))\,\mathrm{d}t < \infty. \tag{4.7}$$

We prepare a result which will turn out to be useful in the proof of Theorem 4.3.

LEMMA 4.2 (weak tightness implies tightness of the norms). *Let H be a separable and reflexive Banach space and u_n be weakly tight. Then $|u_n|$ is tight in $\mathbb{R}$.*

Proof. Let $h : (0,T) \times H \to [0,\infty]$ be a weakly normal integrand fulfilling (4.6)–(4.7), and define $\tilde{h} : (0,T) \times [0,\infty] \to [0,\infty]$ as

$$\tilde{h}(t,r) := \inf_{|w|\geq r} h(t,w). \tag{4.8}$$

We shall prove that $\tilde{h}$ is a normal integrand and that

$$\{r \geq 0 \ : \ \tilde{h}(t,r) \leq c\} \quad \text{is compact for a.e. } t \in (0,T) \text{ and all } c \geq 0, \tag{4.9}$$

$$\text{and} \quad \sup_n \int_0^T \tilde{h}(t,|u_n(t)|)\,\mathrm{d}t < \infty. \tag{4.10}$$

Ad measurability: One directly checks that, given $\alpha > 0$, the set $M = \{(t,u) \in (0,T)\times H \ : \ h(t,u) < \alpha\}$ belongs to $\mathcal{L}\otimes\mathcal{B}(H)$ and hence $N = \{(t,|u|) \ : \ (t,u) \in M\} \in \mathcal{L}\otimes\mathcal{B}([0,\infty))$. Note that, for all Borel sets $A \in \mathcal{L}\otimes\mathcal{B}([0,\infty))$, the set $\cup_{(t,\rho)\in A}\{t\}\times[0,\rho]$ belongs to $\mathcal{L}\otimes\mathcal{B}([0,\infty))$ as well. Now we have

$$\tilde{h}^{-1}\big((-\infty,\alpha)\big) = \{(t,r) \ : \ \exists (t,u) \in M \ \text{ such that } \ |u| \geq r\} = \bigcup_{(t,\rho)\in N} \{t\} \times [0,\rho].$$

In particular $\tilde{h}^{-1}((-\infty,\alpha)) \in \mathcal{L}\otimes\mathcal{B}([0,\infty))$, and $\tilde{h}$ is $\mathcal{L}\otimes\mathcal{B}([0,\infty))$-measurable.

Ad lower semicontinuity: We start by noting that $r \mapsto \tilde{h}(t,r)$ is nondecreasing for all $t \in (0,T)$. Assume by contradiction that there exists $t \in (0,T)$ such that $u \mapsto h(t,u)$ is lower semicontinuous while $r \to \tilde{h}(t,r)$ is not, namely, that there exist an increasing sequence $0 \leq r_n \to r$ and $\delta > 0$ such that, for all $n \in \mathbb{N}$,

$$\tilde{h}(t,r_n) + 2\delta \leq \tilde{h}(t,r). \tag{4.11}$$

Let now $w_n \in H$ be such that $|w_n| \geq r_n$ and $\tilde{h}(t,r_n) + \delta \geq h(t,w_n)$ (such w_n exist by (4.8)). Then surely $|w_n| \leq r$ as, if this was not the case, one would have

$$h(t,w_n) \overset{(4.8)}{\geq} \tilde{h}(t,r) \overset{(4.11)}{\geq} \tilde{h}(t,r_n) + 2\delta \geq h(t,w_n) + \delta.$$

Hence, we can extract a not relabeled subsequence w_n weakly converging to some w in H and compute

$$h(t,w) \overset{(4.5)}{\leq} \liminf_{n\to\infty} h(t,w_n) \leq \lim_{n\to\infty} \tilde{h}(t,r_n) + \delta \overset{(4.11)}{\leq} \tilde{h}(t,r) - 2\delta + \delta < \tilde{h}(t,r),$$

contradicting the very definition (4.8). Namely, $r \mapsto \tilde{h}(t,r)$ is lower semicontinuous for almost every $t \in (0,T)$.

As a consequence, the sets $\{(t,r) \ : \ \tilde{h}(t,r) \leq c\}$ are closed intervals for almost every $t \in (0,T)$. Moreover, they are also bounded for almost every $t \in (0,T)$ due to (4.7), and we have proved (4.9). Finally, one easily checks that

$$\sup_n \int_0^T \tilde{h}(t,|u_n(t)|)\,\mathrm{d}t \ \leq \ \sup_n \int_0^T h(t,u(t))\,\mathrm{d}t \overset{(4.7)}{<} \infty,$$

and (4.10) follows. ∎

A *parametrized measure* on X is a collection $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)}$ of Borel probability measures on X such that

$$t \mapsto \nu_t(B) \quad \text{is } \mathcal{L}\text{-measurable} \ \ \forall B \in \mathcal{B}(X),$$

and the set of all parametrized measures is denoted by $\mathcal{Y}(0,T;X)$.

THEOREM 4.3 (Γ-lim inf tool). *Let H be a separable and reflexive Banach space and $g_n, g_\infty : (0,T)\times H \to (-\infty,\infty]$ be weakly normal integrands such that*

$$g_\infty(t,u) \leq \inf\Big\{ \liminf_{n\to\infty} g_n(t,u_n) : \ u_n \to u \ \ \textit{weakly in} \ \ H\Big\} \qquad \forall u\in H \ \ \textit{and for a.e.} \ \ t\in(0,T). \tag{4.12}$$

Moreover let u_n be weakly tight. Then there exists a subsequence $k\mapsto n_k$ and a parametrized measure $\boldsymbol{\nu}\in\mathcal{Y}(0,T;H)$ such that, for a.e. $t\in(0,T)$,

$$\nu_t \ \ \textit{is concentrated on the set} \ \bigcap_{j=1}^{\infty} \mathrm{cl}_w\Big(\{u_{n_k}(t) \ : \ k\geq j\}\Big), \tag{4.13}$$

where cl_w denotes the weak closure in H, and, whenever $t\mapsto g^-_{n_k}(t,u_{n_k}(t)) = \max\{-g_{n_k}(t,u_{n_k}(t)),0\}$ are uniformly integrable, namely,

$$\lim_{|A|\to 0}\sup_{k\in\mathbb{N}}\int_A g^-_{n_k}(t,u_{n_k}(t))\,\mathrm{d}t = 0$$

(the limit being restricted to Lebesgue measurable sets $A\subset(0,T)$; $|A|$ denotes the Lebesgue measure), we have

$$\int_0^T\left(\int_H g_\infty(t,\xi)\,\mathrm{d}\nu_t(\xi)\right)\mathrm{d}t \leq \liminf_{k\to\infty}\int_0^T g_{n_k}(t,u_{n_k}(t))\,\mathrm{d}t.$$

Proof. The statement follows directly from the fundamental theorem by Balder [11, Thm. 1] adapted to weak topologies along the same lines of Rossi and Savaré [89, Thm. 3.2] (see also [64, Thm. B.1]). The idea is to rephrase the dependence of the functionals from $n\in N:=\mathbb{N}\cup\{\infty\}$ as an extra variable. As we shall see below, condition (4.12) ensures that the augmented integrand is still normal. The only difficulty arises from the fact that the weak topology of H is not globally metrizable (apart from finite-dimensional cases). By following closely the proof of [89, Thm. 3.2], we will circumvent this fact by considering the set

$$V := \big\{(u,r,n)\in H\times\mathbb{R}\times N \ : \ |u|\leq r\big\}\subset H\times\mathbb{R}\times N$$

and endowing it with the metric

$$d(v_1,v_2) := |||u_1-u_2|||^2 + |r_1-r_2| + |\arctan(n_1)-\arctan(n_2)| \qquad \forall v_i = (u_i,r_i,n_i)\in V, \quad i=1,2.$$

In the latter, we have used the convention $\arctan(\infty)=\pi/2$, and, given a countable dense subset w_n of the unit ball in H^*, we have (classically) defined

$$|||u|||^2 := \sum_{k=0}^{\infty} 2^{-k}|(w_k,u)|^2 \quad \forall u\in H.$$

Hence, (V, d) is a separable and complete metric space since

$$v_k = (u_k, r_k, n_k) \to v = (u, r, n) \quad \text{in} \quad V \quad \text{iff}$$
$$u_k \to u \ \text{ weakly in } \ H, \quad r_k \to r \ \text{ in } \ \mathbb{R}, \quad \text{and} \ \ n_k \to n \ \text{ in } \ N$$

(where N is endowed with the arctan metric). Let us remark that any bounded closed set in (V, d) is compact with respect to the latter topology. Hence, all intersections of closed balls of $H \times \mathbb{R} \times N$ with V are Borel subsets of V, namely,

$$B \in \mathcal{B}(H \times \mathbb{R} \times N) \quad \Rightarrow \quad B \cap V \in \mathcal{B}(V), \tag{4.14}$$

and any Borel measure on V can be trivially extended to a Borel measure on $H \times \mathbb{R} \times N$.

We apply Balder's theorem [11, Thm. 1] to the family $v_n = (u_n, |u_n|, n)$ which turns out to be tight by Lemma 4.2 as N is compact. Hence, we find a subsequence $k \mapsto n_k$ and a parametrized measure $\boldsymbol{\mu} = \{\mu_t\}_{t \in (0,T)} \in \mathcal{Y}(0, T; V)$ such that, for almost every $t \in (0, T)$,

$$\mu_t \ \text{ is concentrated on the set } \ \Lambda(t) := \bigcap_{j=1}^{\infty} \mathrm{cl}_V \Big(\{ v_{n_k}(t) \ : \ k \geq j \} \Big),$$

namely, the set of V-limit points of $v_{n_k}(t)$.

Let now $f : (0, T) \times V \to (-\infty, \infty]$ be defined by

$$f(t, v) := g_n(t, u) \quad \forall v = (u, r, n) \in V, \ t \in (0, T).$$

Given any $L \in \mathcal{L}, B \in \mathcal{B}(H)$, and $n \in N$, by using (4.14) one checks that

$$\big(L \times B \times \mathbb{R} \times \{n\} \big) \cap \big((0, T) \times V \big) \in \mathcal{L} \otimes \mathcal{B}(V).$$

Hence, we have

$$\big(\mathcal{L} \otimes \mathcal{B}(H) \times \mathbb{R} \times \{n\} \big) \cap \big((0, T) \times V \big) \subset \mathcal{L} \otimes \mathcal{B}(V).$$

On the other hand, for all $a \in \mathbb{R}$,

$$\begin{aligned} &\big\{ (t, v) \ : \ f(t, v) \leq a \big\} \\ &= \bigcup_{n=1}^{\infty} \Big(\big(\{ (t, u) \ : \ g_n(t, u) \leq a \} \times \mathbb{R} \times \{n\} \big) \cap \big((0, T) \times V \big) \Big) \in \mathcal{L} \otimes \mathcal{B}(V), \end{aligned}$$

since all elements under the union sign are in $\mathcal{L} \otimes \mathcal{B}(V)$. Hence, the function f is $\mathcal{L} \otimes \mathcal{B}(V)$-measurable. Moreover, f is a normal integrand. Indeed, let $v_k \to v$ in V. Then either $n_k = n$ definitely or $n_k \to \infty$. In the first case, lower semicontinuity follows from the weak sequential lower semicontinuity of g_n, whereas in the second case it can be deduced from (4.12).

Since $t \mapsto f^-(t, v_{n_k}(t)) = g^-_{n_k}(t, u_{n_k}(t))$ are uniformly integrable, again by [11, Thm. 1] we have

$$\begin{aligned} \int_0^T \left(\int_V f(t, \zeta) \, \mathrm{d}\mu_t(\zeta) \right) \mathrm{d}t &\leq \liminf_{k \to \infty} \int_0^T f(t, v_{n_k}(t)) \, \mathrm{d}t \\ &= \liminf_{k \to \infty} \int_0^T g_{n_k}(t, u_{n_k}(t)) \, \mathrm{d}t. \end{aligned} \tag{4.15}$$

By recalling that $\Lambda(t) \subset H \times \mathbb{R} \times \{\infty\}$ for all $t \in (0,T)$ and letting

$$\nu_t(B) := \mu_t\big((B \times \mathbb{R} \times N) \cap V\big) \quad \forall B \in \mathcal{B}(H),$$

we obtain a parametrized measure $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;H)$ which fulfills (4.13) and is such that

$$\int_0^T \left(\int_V f(t,\zeta)\, \mathrm{d}\mu_t(\zeta) \right) \mathrm{d}t = \int_0^T \left(\int_H g_\infty(t,\xi)\, \mathrm{d}\nu_t(\xi) \right) \mathrm{d}t,$$

which, together with (4.15), entails the result. $\square$

Let us remark that, under the frame of Theorem 4.3, whenever $u_n \to u$ weakly in $L^p(0,T;H)$ for some $p \in [1,\infty)$ (weakly star for $p = \infty$), then

$$u(t) = \int_H \xi \, \mathrm{d}\nu_t(\xi) \quad \text{for a.e. } t \in (0,T). \tag{4.16}$$

This fact was already remarked in [89, Thm. 3.2] for $p \in (1,\infty)$ (and hence, for $p = \infty$ as well). As for $p = 1$, one can readily choose the weakly normal integrands

$$g(t,u) := (w(t),u) \quad \forall u \in H, \text{ a.e. } t \in (0,T), \quad \text{with } w \in L^\infty(0,T;H^*),$$

and exploit Theorem 4.3 with the constant sequence $g_n = g$ (or [64, Thm. B.1]) in order to conclude.

4.3. The Γ-lim inf result for normal convex integrands. Let us now specify the result of Theorem 4.3 in the case of normal convex integrands $g_n, g_\infty : [0,T]\times H \to (-\infty,\infty]$.

COROLLARY 4.4. *Let $p \in [1,\infty]$, H be a separable and reflexive Banach space, and $g_n, g_\infty : (0,T) \times H \to (-\infty,\infty]$ be normal convex integrands such that* (4.12) *holds. Moreover, let $u_n \to u$ weakly in $L^p(0,T;H)$ (weakly star if $p = \infty$). Then, whenever $t \mapsto g_n^-(t,u_n(t))$ are uniformly integrable, we have*

$$\int_0^T g_\infty(t,u(t))\, \mathrm{d}t \le \liminf_{n\to\infty} \int_0^T g_n(t,u_n(t))\, \mathrm{d}t. \tag{4.17}$$

Proof. Let $j \mapsto n_j \in \mathbb{N}$ be an increasing sequence. As u_n are uniformly bounded in $L^p(0,T;H)$, the family u_{n_j} is weakly tight. Hence, by applying Theorem 4.3, we may extract a further subsequence $k \mapsto n_{j_k}$ such that

$$\begin{aligned}\liminf_{k\to\infty} \int_0^T g_{n_{j_k}}(t,u_{n_{j_k}}(t))\, \mathrm{d}t &\ge \int_0^T \left(\int_H g_\infty(t,\xi)\, \mathrm{d}\nu_t(\xi) \right) \mathrm{d}t \\ &\ge \int_0^T g_\infty(t,u(t))\, \mathrm{d}t,\end{aligned} \tag{4.18}$$

where the last inequality follows from (4.16) and Jensen's inequality. Namely, for all subsequences of u_n there exist further subsequences such that (4.18) holds. An easy argument ensures that indeed (4.18) holds for the whole sequence $g_n(\cdot,u_n(\cdot))$ as well. $\square$

Let us now comment on the uniform integrability condition of Corollary 4.4. First of all, it is clear that the latter holds if g_n are uniformly bounded below. More generally, one can consider the case

$$\begin{aligned}&g_n(t,u) \ge -c_0|u| - \gamma(t) \qquad \text{for some } c_0 > 0, \ \gamma \in L^1(0,T), \\ &\qquad \forall u \in H, \ n \in \mathbb{N}, \quad \text{for a.e. } t \in (0,T).\end{aligned} \tag{4.19}$$

In fact, whenever u_n converges weakly in $L^p(0,T;H)$ (weakly star in $p=\infty$), the functions $t\mapsto|u_n(t)|$ are uniformly integrable [28, Thm. 4, p. 104], and (4.19) entails the uniform integrability of $t\mapsto g_n^-(t,u_n(t))$.

We shall explicitly remark that, in case g_n are independent of time, the lower bound (4.19) follows directly from the condition (4.12). Indeed, owing to [7, Thm. 3.7, p. 271], by letting $v\in D(g^*_\infty)$ be fixed, there exist v_n such that $v_n\to v$ strongly in H^* and $\limsup_{n\to\infty} g_n^*(v_n)=g^*_\infty(v)$. Then (4.19) follows with the choice

$$c_0=\sup_n |v_n|_*,\ \ \gamma(t)=g^*_\infty(v)+1.$$

In particular, we have the following.

Corollary 4.5 (g_n independent of time). *Let $p\in[1,\infty]$, H be a separable and reflexive Banach space, and $g_n, g_\infty: H\to(-\infty,\infty]$ be convex, proper, and lower semicontinuous such that* (4.12) *holds. Moreover, let $u_n\to u$ weakly in $L^p(0,T;H)$ (weakly star if $p=\infty$). Then we have*

$$\int_0^T g_\infty(u(t))\,\mathrm{d}t\le\liminf_{n\to\infty}\int_0^T g_n(u_n(t))\,\mathrm{d}t. \tag{4.20}$$

Before moving on, we remark that some result in the direction of Corollary 4.4 was already contained in the convergence analysis by Salvadori [92, Thm. 3.1]. The latter was focused on establishing conditions under which the integral functionals

$$G_n(u)=\begin{cases}\displaystyle\int_0^T g_n(t,u(t))dt & \text{if } \ t\mapsto g_n^+(t,u(t))\in L^1(0,T),\\ \infty & \text{otherwise}\end{cases}$$

would Mosco-converge to the limit functional

$$G_\infty(u)=\begin{cases}\displaystyle\int_0^T g_\infty(t,u(t))dt & \text{if } \ t\mapsto g_\infty^+(t,u(t))\in L^1(0,T),\\ \infty & \text{otherwise}\end{cases}$$

(and analogously for G_n^* and G^*_∞, which are defined from g_n^* and g^*_∞, respectively). The Mosco-convergence result in [92, Thm. 3.2] was obtained under some uniform quantitative properness of the functionals. In particular, both sequences G_n and G_n^* are asked to be uniformly proper on $L^p(0,T;H)$ and $L^q(0,T;H^*)$, respectively. Moreover, by letting u_n and v_n be the corresponding bounded sequences, the existence of two functions $f,f_*\in L^1(0,T)$ such that

$$|g_n(t,u_n(t))|\le f(t)\quad\text{and}\quad |g_n^*(t,v_n(t))|\le f_*(t)\ \ \text{for a.e. } \ t\in(0,T)$$

is required.

The frame of Corollary 4.4 is quite weaker, since we are not assuming any control on the domains of the functionals but rather some standard uniform integrability of negative parts of the integrands. Hence, by exploiting the characterization of Lemma 4.1 and restricting to the case where $p\in(1,\infty)$, we can obtain a refined version of [92, Thm. 3.1] as follows.

Corollary 4.6. *Let $p\in(1,\infty)$, H be a separable and reflexive Banach space, and $g_n, g_\infty:(0,T)\times H\to(-\infty,\infty]$ be normal convex integrands such that $g_n\to g_\infty$*

in the Mosco sense. Moreover, assume that

$$t \to g_n^-(t, u_n(t)) \;\; \text{and} \;\; t \to (g_n^*)^-(t, v_n(t)) \;\; \text{are uniformly integrable} \quad \forall (u_n, v_n) \to (u, v) \text{ weakly in } L^p(0,T;H) \times L^q(0,T;H^*), \tag{4.21}$$

$$G_n \;\; \text{and} \;\; G_n^* \;\; \text{are proper on } L^p(0,T;H) \text{ and } L^q(0,T;H^*), \text{ respectively, and either } G_n \text{ or } G_n^* \text{ is uniformly proper.} \tag{4.22}$$

Then $G_n \to G_\infty$ in the Mosco sense in $L^p(0,T;H)$ and $G_n^* \to G_\infty^*$ in the Mosco sense in $L^q(0,T;H^*)$. In particular, both sequences G_n and G_n^* are uniformly proper.

Proof. Owing to (4.21), Corollary 4.4 gives the Γ-lim inf inequalities for G_n and G_n^*. Owing to the properness of G_n and G_n^*, we have that $(G_n)^* = G_n^*$ and $(G_n^*)^* = G_n$ [21, Thm. VII.7, p. 200]. Since $L^p(0,T;H)$ is reflexive, the assertion follows from the uniform properness in (4.22) and Lemma 4.1. ∎

In the same spirit of Corollary 4.5, whenever g_n are independent of time, both the uniform integrability condition (4.21) and the uniform properness condition (4.22) are straightforward.

5. Lim inf inequalities. We shall now apply this functional convergence machinery to our problems. Throughout the remainder of the paper, we will assume that

(A6) $p \in [1, \infty]$, $1/p + 1/q = 1$, and
H is a real, separable, and reflexive Banach space.

As for the sequences of approximating functionals we will systematically ask that

$$t \to g_n^-(t, u_n(t)) \;\; \text{and} \;\; t \to (g_n^*)^-(t, v_n(t)) \;\; \text{are uniformly integrable} \quad \forall (u_n, v_n) \to (u, v) \text{ weakly (star) in } L^p(0,T;H) \times L^q(0,T;H^*) \tag{5.1}$$

for the choices $g_n = \phi_n, \psi_n, \varphi_n$ and admit the limiting cases $p = 1, \infty$ as well.

Let us start from the case of the gradient flow (1.1). Assume that we are given data ϕ_n, f_n, and u_0^n as in section 1, and define the corresponding approximating functionals $J_n, K_n : H^1(0,T;H) \to [0, \infty]$ for all $u \in H^1(0,T;H)$ as

$$J_n(u) := \int_0^T \big(\phi_n(u) + \phi_n^*(f_n - u') - (f_n, u)\big) + \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2 + |u(0) - u_n^0|^2,$$

$$K_n(u) := \left(\int_0^T \big(|u'|^2 - (f_n, u')\big) + \phi_n(u(T)) - \phi_n(u_n^0)\right)^+ + J_n(u).$$

We have the following.

LEMMA 5.1 (lim inf inequality for K_n). *Assume* (A6)*, and let H be a Hilbert space, $\phi_n \to \phi$ in the Mosco sense, $f_n \to f$ strongly in $L^2(0,T;H)$, $u_n^0 \to u^0$ in H, and $\phi_n(u_n^0) \to \phi(u^0)$. Then, for all $u \in H^1(0,T;H)$,*

$$K(u) \le \inf\big\{ \liminf_{n\to\infty} K_n(u_n) \; : \; u_n \to u \;\; \text{weakly in} \;\; H^1(0,T;H)\big\}.$$

Proof. By applying Corollary 4.5 we readily check that

$$\int_0^T \big(\phi(u) + \phi^*(f - u')\big) \le \liminf_{n\to\infty} \int_0^T \big(\phi_n(u_n) + \phi_n^*(f_n - u_n')\big).$$

On the other hand, owing to the strong convergence of f_n we have

$$\int_0^T (f_n, u_n) \to \int_0^T (f, u) \quad \text{and} \quad \int_0^T (f_n, u_n') \to \int_0^T (f, u'),$$

and, by the pointwise weak convergences $u_n(T) \to u(T)$ and $u_n(0) \to u(0)$ in H and the Mosco convergence of ϕ_n, one obtains

$$\begin{aligned}
&\phi(u(T)) \leq \liminf_{n\to\infty} \phi_n(u_n(T)) \quad \text{and} \\
&\frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2 + |u(0) - u^0|^2 = \frac{1}{2}|u(T)|^2 + \frac{1}{2}|u(0)|^2 + |u^0|^2 - 2(u(0), u^0) \\
&\leq \liminf_{n\to\infty} \left(\frac{1}{2}|u_n(T)|^2 + \frac{1}{2}|u_n(0)|^2 + |u_n^0|^2 - 2(u_n(0), u_n^0) \right) \\
&= \liminf_{n\to\infty} \left(\frac{1}{2}|u_n(T)|^2 - \frac{1}{2}|u_n(0)|^2 + |u_n(0) - u_n^0|^2 \right).
\end{aligned}$$

Finally, the assertion follows by lower semicontinuity. □

In fact, an analogous result holds for J_n as well, the convergence of the initial energies $\phi_n(u_n^0)$ not being needed. We prefer to state the lim inf inequality for K_n since, as already remarked, the sublevels of K_n are uniformly bounded in $H^1(0, T; H)$ whereas those of J_n are not, in general.

Let us now move to the doubly nonlinear situation by fixing $p \in [1, \infty]$ and recalling that $1/p + 1/q = 1$. Assume that we are given ϕ_n, ψ_n, f_n, and u_0^n as in section 2, and define the corresponding approximating functionals $I_n : W^{1,p}(0, T; H) \times L^q(0, T; H^*) \to [0, \infty]$ for all $(u, v) \in W^{1,p}(0, T; H) \times L^q(0, T; H^*)$ as

$$\begin{aligned}
I_n(u, v) := &\left(\int_0^T \big(\psi_n(u') + \psi_n^*(v) - (f_n, u')\big) + \phi_n(u(T)) - \phi_n(u_n^0) \right)^+ \\
&+ \int_0^T \big(\phi_n(u) + \phi_n^*(f_n - v) - (f_n - v, u)\big) + |u(0) - u_n^0|^2.
\end{aligned}$$

Hence, by recalling (4.1), we have the following lemma.

LEMMA 5.2 (lim inf inequality for I_n). *Assume* (A6)*, and let $\phi_n \to \phi$ and $\psi_n \to \psi$ in the Mosco sense and fulfill* (5.1)*, $f_n \to f$ strongly in $L^q(0, T; H^*)$, $u_n^0 \to u^0$ in H, and $\phi_n(u_n^0) \to \phi(u^0)$. Then, for all $(u, v) \in W^{1,p}(0, T; H) \times L^q(0, T; H^*)$,*

$$I(u, v) \leq \inf \big\{ \liminf_{n\to\infty} I_n(u_n, v_n) \ : \ (u_n, v_n) \to (u, v) \ \text{ in } \ \mathcal{T}(p) \big\}.$$

Proof. The proof of the latter lemma follows along the very same lines as that of Lemma 5.1 by systematically considering the $p - q$ frame and additionally applying once again Corollary 4.5 in order to deduce that

$$\int_0^T \big(\psi(u) + \psi^*(v)\big) \leq \liminf_{n\to\infty} \int_0^T \big(\psi_n(u_n) + \psi_n^*(v_n)\big). \qquad \square$$

We shall explicitly mention that, in the separable Hilbert-space setting and in the (quadratic) case of (2.3), namely, for $\phi(\cdot) = \phi_n(\cdot) = |\cdot|^2/2$, the above lim inf inequality holds also in some stronger form. By introducing the approximating functionals Q_n :

$W^{1,p}(0,T;H) \to [0,\infty]$ as

$$Q_n(u) := \int_0^T \big(\psi_n(u') + \psi_n^*(f_n - u) - (f_n, u')\big) + \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2 + |u(0) - u_n^0|^2,$$

we have the following.

LEMMA 5.3 (lim inf inequality for Q_n). *Assume that H is a separable Hilbert space, $\psi_n \to \psi$ in the Mosco sense and fulfill* (5.1)*, $f_n \to f$ strongly in $L^q(0,T;H)$, and $u_n^0 \to u^0$ in H. Then, for all $u \in W^{1,p}(0,T;H)$,*

$$Q(u) \le \inf\big\{ \liminf_{n\to\infty} Q_n(u_n) \ : \ u_n \to u \ \ \textit{weakly (star) in} \ \ W^{1,p}(0,T;H)\big\}. \tag{5.2}$$

The assertion follows via the same arguments of the proofs above, this case being indeed simplified since ϕ is quadratic.

Finally, we consider the generalized situation of (3.1) by letting ψ_n, φ_n, π_n, and u_0^n be as in section 3 and defining the functionals $\mathcal{I}_n : W^{1,p}(0,T;H) \times L^q(0,T;H^*) \to [0,\infty]$ for all $(u,v) \in W^{1,p}(0,T;H) \times L^q(0,T;H^*)$ as

$$\mathcal{I}_n(u,v) := \left(\int_0^T \big(\psi_n(u') + \psi_n^*(v) - \pi_n(u)\big) + \varphi_n(T,u(T)) - \varphi_n(0,u_n^0)\right)^+ + \int_0^T \big(\varphi_n(\cdot,u) + \varphi_n^*(\cdot,-v) + (u,v)\big) + |u(0) - u_n^0|^2.$$

In order to establish a lim inf inequality result for $\mathcal{I}_n$, the limiting behavior of π_n has to be prescribed. We shall directly ask that, for all $(u,v) \in W^{1,p}(0,T;H) \times L^q(0,T;H^*)$,

$$\int_0^T \pi(u) \ge \sup\left\{ \limsup_{n\to\infty} \int_0^T \pi_n(u_n) \ : \ (u_n,v) \to (u,v) \ \text{ in } \ \mathcal{T}(p)\right\}. \tag{5.3}$$

The latter is readily fulfilled if $\varphi_n(t,u) = \phi_n(u) - (f_n(t),u)$, $\varphi(t,u) = \phi(u) - (f(t),u)$, and $f_n \to f$ strongly in $W^{1,p}(0,T;H^*)$; see Remark 3.2. We have the following.

LEMMA 5.4 (lim inf inequality for $\mathcal{I}_n$). *Assume* (A6)*, and let $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in the Mosco sense and fulfill* (5.1) *and* (5.3)*, $u_n^0 \to u^0$ in H, and $\varphi_n(0,u_n^0) \to \varphi(0,u^0)$. Then, for all $(u,v) \in W^{1,p}(0,T;H) \times L^q(0,T;H^*)$,*

$$\mathcal{I}(u,v) \le \inf\big\{ \liminf_{n\to\infty} \mathcal{I}_n(u_n,v_n) \ : \ (u_n,v_n) \to (u,v) \ \ \textit{in} \ \ \mathcal{T}(p)\big\}.$$

Sketch of the proof. This proof differs from that of Lemma 5.2 for the sole use of (5.3) instead of the strong convergence of f_n. ∎

Before closing this section, let us explicitly consider the case of the nonautonomous gradient flow (3.7). To this aim, let the approximating functionals $\mathcal{K}_n : H^1(0,T;H) \to [0,\infty]$ be defined as

$$\mathcal{K}_n(u) := \left(\int_0^T \big(|u'|^2 - \pi_n(u)\big) + \varphi_n(T,u(T)) - \varphi_n(0,u_n^0)\right)^+ + \int_0^T \big(\varphi_n(\cdot,u) + \varphi_n^*(\cdot,-u')\big) + \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u(0)|^2 + |u(0) - u_n^0|^2.$$

In order to possibly obtain some stronger convergence result in this case, we shall need some additional convergence property for π_n. Namely, we ask, for all $u \in H^1(0,T;H)$, that

$$(5.4)\qquad \int_0^T \pi(u) \geq \sup\left\{ \limsup_{n\to\infty} \int_0^T \pi_n(u_n) \;:\; u_n \to u \ \text{ weakly in } \ H^1(0,T;H) \right\}.$$

Again, in the case where $\varphi_n(t,u) = \phi_n(u) - (f_n(t),u)$, $\varphi(t,u) = \phi(u) - (f(t),u)$, the latter is fulfilled if $f_n \to f$ strongly in $H^1(0,T;H^*)$. Hence, we have the following.

Lemma 5.5 (lim inf inequality for $\mathcal{K}_n$). *Assume* (A6)*, and let $\varphi_n \to \varphi$ in the Mosco sense and fulfill* (5.1) *and* (5.4)*, $u_n^0 \to u^0$ in H, and $\varphi_n(0,u_n^0) \to \varphi(0,u^0)$. Then, for all $u \in H^1(0,T;H)$,*

$$\mathcal{K}(u) \leq \inf\left\{ \liminf_{n\to\infty} \mathcal{K}_n(u_n) \;:\; u_n \to u \ \text{ weakly in } \ H^1(0,T;H) \right\}.$$

The proof of the latter is obtained by easily adapting the arguments of Lemmas 5.1 and 5.4. Let us stress that, in case the weaker convergence (5.3) holds, a lim inf inequality for $\mathcal{K}_n$ is still available as a corollary to Lemma 5.4.

6. Approximation of gradient flows. We shall exploit Theorems 2.1 and 3.1 and Lemmas 5.1 and 5.5 in order to recover and further generalize some approximation results for gradient flows under the separability assumption in (A6). By assuming the above notations and recalling the uniform coercivity of K_n with respect to the weak topology in $H^1(0,T;H)$, we obtain a first convergence result which we state below by omitting the easy proof.

Lemma 6.1 (convergence for gradient flows). *Under the assumptions of Lemma* 5.1*, let $u_n \in H^1(0,T;H)$ be such that $K_n(u_n) \to 0$. Then $u_n \to u$ weakly in $H^1(0,T;H)$ and $K(u) = 0$.*

Note in particular that the whole approximating sequence u_n converges since K admits a unique minimizer u.

By reducing ourselves to the case $K_n(u_n) = 0$ (i.e., letting u_n be solutions to the respective differential problems), the above lemma recovers the result by Attouch on the approximation of gradient flows under the Mosco convergence of the functionals [7, Thm. 3.74(2), p. 388] under the separability assumption in (A6) (here no strong $H^1(0,T;H)$ nor energy convergence is proved, though). Let us, however, remark that our result turns out to be slightly more general than the former since u_n are a priori not required to be solutions at level n. In particular, the functions u_n could be approximated solutions of the corresponding gradient flows as well. Moreover, the functional frame is here extended from (separable) Hilbert to separable reflexive Banach spaces. One has, however, to mention that the specific case of regularization by means of the Yosida approximation (in Hilbert spaces) was already discussed within the existence proof by Ghoussoub and Tzou [39]. In case the convergence of the initial energies $\phi_n(u_n^0)$ does not hold, one is still in the position of proving a convergence result by arguing on J_n if $J_n(u_n) = 0$ (or, more generally, in case of a weakly $H^1(0,T;H)$-precompact sequence u_n such that $J_n(u_n) \to 0$).

Our second convergence result concerns the generalized situation of (3.7). Again, the following lemma is implied by the fact that, by assuming π_n to be uniformly linearly bounded, $\mathcal{K}_n$ are uniformly coercive with respect to the weak topology of $H^1(0,T;H)$.

Lemma 6.2 (convergence for generalized gradient flows). *Under the assumptions of Lemma* 5.5*, let π_n to be uniformly linearly bounded and $u_n \in H^1(0,T;H)$ be such that $\mathcal{K}_n(u_n) \to 0$. Then $u_n \to u$ weakly in $H^1(0,T;H)$ and $\mathcal{K}(u) = 0$.*

The latter convergence result for the nonautonomous gradient flow (3.7) is to be compared with the former results by Ortner [79, sect. 3.2] which hold in the general metric and λ-geodesically convex setting but under more restrictive functional convergence assumptions (see also [80]). We shall, however, stress that here the approximating u_n need not be solutions to the corresponding differential problems at level n.

Before closing this section, we shall mention the work by Mabrouk [52, 53] where the Brezis–Ekeland principle is exploited within an approximation procedure in order to establish the existence of generalized solutions to some semilinear parabolic equation with measure data (see also the results by the same author [54, 55] for some second order in time equations). Moreover, we mention that some identification result for nonlinear parabolic problems based on (a variational technique related to) the Brezis–Ekeland principle has been obtained by Barbu and Kunisch [14]. Finally, the issue of approximating nonconvex gradient flows has recently attracted some attention (see Ambrosio, Gigli, and Savaré [2] and Sandier and Serfaty [93]).

7. Approximation of doubly nonlinear equations. Let us now move to the situation of the doubly nonlinear relation (1.3) and fix from the very beginning and throughout this section

$$p \in (1, \infty).$$

For Lemma 5.2 to serve as the basis for a convergence result, one just needs to provide coercivity for I_n with respect to the topology $\mathcal{T}(p)$ (see (4.1)). The latter holds, for instance, in the situation of potentials ψ_n of p-growth and functionals ϕ_n with compact sublevels. In particular, we let

$$c_1|w|^p - c_2 \le \psi(w) \le c_3(|w|^p + 1) \quad \forall w \in H, \tag{7.1}$$

$$\phi(w) \ge c_4\|w\|^p - c_5 \quad \forall w \in V \subset H, \tag{7.2}$$

where the injection of the reflexive Banach space V into H is compact, $\|\cdot\|$ is the norm in V, and $c_1, c_3, c_4 > 0$, $c_2, c_5 \ge 0$ are given. In this case, it may be checked that $c_6|w|_*^q - c_7 \le \psi^*(w)$ for all $w \in H^*$ and with some constants $c_6, c_7 > 0$ depending on c_3 and p ($|\cdot|_*$ is the norm in H^*). Hence, all sublevels of I are relatively compact with respect the topology $\mathcal{T}(p)$ by means of well-known compactness results (see, e.g., Simon [96]). Namely, we have the following.

Lemma 7.1 (convergence for doubly nonlinear equations). *Under the assumptions of Lemma* 5.2, *let ϕ_n and ψ_n fulfill* (7.1)–(7.2) *uniformly with respect to n. Moreover, let $(u_n, v_n) \in W^{1,p}(0,T;H) \times L^q(0,T;H^*)$ be such that $I_n(u_n, v_n) \to 0$. Then there exists a (not relabeled) subsequence such that $(u_n, v_n) \to (u, v)$ in $\mathcal{T}(p)$ and $I(u, v) = 0$.*

Here the convergence of the whole sequence (u_n, v_n) cannot be expected since the limiting minimum problem for I need not admit a unique minimizer.

By restricting to the case of a sequence (u_n, v_n) of solutions to the approximating doubly nonlinear problem (i.e., imposing $I_n(u_n, v_n) = 0$), we recover the convergence result of Aizicovici and Yan [1, Thm. 3.1] under the extra separability assumption in (A6). The referred result is in fact slightly stronger, since the subdifferentials $\partial\psi_n$ are replaced by general maximal monotone operators A_n and the growth and compactness requirements are weaker (although quite similar). On the other hand, under assumptions (7.1)–(7.2), our result turns out to be more general than the former, since the approximating (u_n, v_n) need not be solutions to the corresponding equations. This fact allows some possible extra freedom in the choice of the approximating sequence.

Finally, we are in the position of providing a convergence lemma which applies to the generalized situation of (3.1). The following is to our knowledge the first result in this direction.

THEOREM 7.2 (convergence for generalized doubly nonlinear equations). *Under the assumptions of Lemma* 5.4*, let* $\varphi_n(t,\cdot)$ *and* ψ_n *fulfill* (7.1)–(7.2) *for almost every* $t \in (0,T)$ *and uniformly with respect to* n*. Moreover, let* $(u_n, v_n) \in W^{1,p}(0,T;H) \times L^q(0,T;H^*)$ *be such that* $\mathcal{I}_n(u_n, v_n) \to 0$*. Then there exists a (not relabeled) subsequence such that* $(u_n, v_n) \to (u, v)$ *in* $\mathcal{T}(p)$ *and* $\mathcal{I}(u, v) = 0$*.*

LEMMA 7.3 (convergence for case $\phi(\cdot) = \phi_n(\cdot) = |\cdot|^2/2$). *Under the assumptions of Lemma* 5.3*, for* $p \in (1,\infty]$ *let* ψ_n *fulfill* (7.1) *uniformly with respect to* n*. Moreover, let* $u_n \in W^{1,p}(0,T;H)$ *be such that* $Q_n(u_n) \to 0$*. Then there exists a (not relabeled) subsequence such that* $u_n \to u$ *weakly in* $W^{1,p}(0,T;H)$ *and* $Q(u) = 0$*.*

Note that, in the frame of Lemma 7.3, if $f \in W^{1,1}(0,T;H)$, then the solution of (2.3) is unique and the whole sequence u_n of the statement converges.

7.1. Existence for some doubly nonlinear equation via the Brezis–Ekeland approach. We shall now apply the above-developed convergence theory in order to possibly (re)obtain the existence of solutions of some specific doubly nonlinear equation (1.3) via the variational characterization of Theorem 3.1. For the sake of simplicity, we reduce our attention to the Hilbert-space framework of Colli and Visintin [23]. More specifically, we shall ask that

$$H \text{ is a real Hilbert space and } p = 2. \tag{7.3}$$

LEMMA 7.4. *Under assumptions* (A2) *and* (A3) *and* (7.1)–(7.3)*, there exists a solution* (u, v) *of* (1.4)–(1.6)*.*

Note that the latter stands as a weaker version of [23, Thm. 2.1] where indeed $\partial\psi$ is replaced by a general maximal monotone operator and some weaker coercivity assumption on ϕ is considered.

Proof. For all $n \in \mathbb{N}$ let

$$\psi_n(u) = \frac{1}{2n}|u|^2 + \psi(u) \quad \text{and} \quad \phi_n(u) = \inf_{v \in H}\left(\frac{n}{2}|v-u|^2 + \phi(v)\right).$$

Namely, ϕ_n is the Yosida approximation of ϕ at level $1/n$. The corresponding regularized problem reads now as follows:

$$v \in \frac{1}{n}u' + \partial\psi(u'), \quad v + \partial\phi_n(u) = f \quad \text{a.e. in } (0,T), \quad u(0) = u^0,$$

and clearly admits a unique solution $(u_n, v_n) \in H^1(0,T;H) \times L^2(0,T;H)$ since $(1/n + \partial\psi)^{-1} \circ \partial\phi_n$ is Lipschitz continuous. Hence, by Theorem 2.1 we have that $I_n(u_n, v_n) = 0$.

We now aim at applying the lim inf result of Lemma 5.2. First of all, we have $\psi_n \to \psi$ and $\phi_n \to \phi$ in the Mosco sense [7, Thm. 3.26, p. 305]. Second, condition (5.1) is trivially satisfied by ψ_n since we are requiring (7.1). In particular, $\psi \le \psi_n$ and $\psi_n^* \ge -\psi(0)$. Moreover, owing to the fact that $(\phi_n)^*(u) = \phi^*(u) + |u|^2/(2n)$ (see, for instance, [42, subsect. 4.1]), we have

$$\phi_n(u) = \sup_{v \in H}\left((v,u) - \phi^*(v) - \frac{1}{2n}|v|^2\right) \ge (w,u) - \phi^*(w) - \frac{1}{2}|w|^2$$

for any fixed $w \in D(\phi^*)$. Finally, the fact that

$$(\phi_n)^*(u) \ge \phi^*(u) \ge (u,z) - \phi(z) \quad \forall u \in H,$$

for some $z \in D(\phi)$ fixed, entails that condition (5.1) holds for ϕ_n as well.

It is a standard matter to determine new constants c_1', c_2', and c_3' in such a way that ψ_n fulfills the corresponding nondegeneracy and growth assumption (7.1) uniformly with respect to n. In particular, we have the fact that

$$u_n' \text{ and } v_n \text{ are bounded in } L^2(0,T;H) \text{ independently of } n. \tag{7.4}$$

Since $D(\phi_n) = H$, the coercivity (7.2) cannot hold at level n. In fact one can check that

$$\phi_n(u) = \frac{1}{2n}|\partial\phi_n(u)|^2 + \phi(j_n u) \geq c_4\|j_n u\|^2 - c_5 \quad \forall u \in H,$$

where $j_n := (1 + (1/n)\partial\phi)^{-1}$ is the resolvent. By recalling that j_n is Lipschitz continuous, uniformly with respect to n, we have the fact that

$$j_n u_n \text{ is bounded in } H^1(0,T;H) \cap L^2(0,T;V) \text{ independently of } n. \tag{7.5}$$

The bounds (7.4)–(7.5) imply that (u_n, v_n) is precompact in $\mathcal{T}(2)$. By extracting a not relabeled subsequence $(u_n, v_n) \to (u, v)$ in $\mathcal{T}(2)$ and applying Lemma 5.2, we get $I(u,v) = 0$, and the assertion follows from Theorem 2.1. □

Let us comment that the Hilbert-space frame of (7.3) is chosen as a possible first illustration of this technique and that the Brezis–Ekeland approach applies to the more general Banach case as well. However, in the latter case, one should consider a time-discretized problem rather than a regularized one (this was exactly the strategy in [22]). The development of a discrete version of the characterization of Theorem 3.1 is presented and applied to the convergence of time discretizations for (1.3) in [101]. As a by-product, the existence of solutions to the doubly nonlinear equation (1.3) in the Banach framework is there recovered by a purely variational technique.

8. Approximation of rate-independent evolutions. Let us now focus on a specific class of potentials ψ of growth $p = 1$ (see (7.1)). In particular, in addition to (A4) we shall ask that

$$\psi \text{ is positively homogeneous of degree } 1, \tag{8.1}$$

letting the evolution problem be rate-independent (see Mielke [62]). Equivalently, ψ is required to be the support function of a convex and closed set $C \subset H^*$ containing 0, namely,

$$\psi(w) = \sup\{(v,w) \ : \ v \in C\} \quad \forall w \in H. \tag{8.2}$$

Note in particular that $D(\psi) = H$. In the present rate-independent situation we are allowed to weaken the assumptions on ψ in (7.1) and ask for the upper bound only. Owing to positive homogeneity, we take with no loss of generality

$$\psi(w) \leq c_3|w| \quad \forall w \in H, \tag{8.3}$$

which in particular says that C is contained in a ball of center 0 and radius c_3. As for φ, besides (A4) we require

$$t \mapsto \varphi(t,u) \text{ differentiable and } t \mapsto \partial_t\varphi(t,u) \text{ measurable } \forall u \in H. \tag{8.4}$$

Moreover, we ask for a nonnegative function $\lambda \in L^1(0,T)$ and a constant $c_8 > 0$ such that

$$|\partial_t\varphi(t,u)| \leq \lambda(t)(\varphi(t,u) + 1) \quad \forall u \in H, \tag{8.5}$$

$$|\partial_t\varphi(t,u) - \partial_t\varphi(t,w)| \leq c_8|u - w| \quad \forall u, w \in H, \text{ for a.e. } t \in (0,T). \tag{8.6}$$

The latter and [64, Prop. 2.6] entail that the choice $\pi(u(t)) := \partial_t\varphi(t,u(t))$ is admissible and fulfills the chain rule (3.2). Finally, we ask for the uniform convexity of φ, namely,

$$\begin{aligned}&\exists\kappa>0 \ \text{ such that, } \forall u_0,u_1\in H,\ \forall t\in[0,T],\ \forall\theta\in[0,1],\\ &\varphi(t,(1-\theta)u_0+\theta u_1)\\ &\qquad\le(1-\theta)\varphi(t,u_0)+\theta\varphi(t,u_1)-\frac{\kappa}{2}\theta(1-\theta)|u_0-u_1|^2.\end{aligned}\tag{8.7}$$

In [64, subsect. 4.2] the authors discuss a nontrivial situation inspired by continuum mechanics where the latter conditions (8.2)–(8.7) are met. The crucial point now is that uniform convexity yields the Lipschitz time regularity of the solutions. In particular, under assumptions (8.2)–(8.7), all solutions (u,v) to the doubly nonlinear equation (3.1) fulfill [64, Thm. 3.2]

$$\|u'\|_{L^\infty(0,T;H)}\le c_8/\kappa.\tag{8.8}$$

LEMMA 8.1 (convergence for rate-independent problems). *Under the assumptions of Lemma* 5.4*, let* ψ_n *fulfill* (8.1) *and* (8.3) *and* φ_n *fulfill* (7.2) *and* (8.4)–(8.7) *uniformly with respect to* n*. Moreover, let* $(u_n,v_n)\in W^{1,1}(0,T;H)\times L^\infty(0,T;H^*)$ *be such that* $\mathcal{I}_n(u_n,v_n)=0$*. Then there exists a (not relabeled) subsequence such that* $(u_n,v_n)\to(u,v)$ *weakly star in* $W^{1,\infty}(0,T;H)\times L^\infty(0,T;H^*)$ *and* $\mathcal{I}(u,v)=0$*.*

Proof. First of all, one has that $v_n\in C$ almost everywhere in $(0,T)$ and hence are uniformly bounded in $L^\infty(0,T;H^*)$. Moreover, since (u_n,v_n) are solutions to (3.1), the Lipschitz continuity estimate (8.8) holds uniformly with respect to n. Additionally, u_n are uniformly bounded in $L^1(0,T;V)$ due to (7.2). Hence, Lemma 5.4 yields the result. ∎

We shall mention that, differently from Lemmas 6.1 and 7.2, the latter result holds for sequences of solutions only since the estimate (8.8) is crucially used in order to obtain strong compactness in $L^1(0,T;H)$ for u_n. In particular, we are not entitled to approximate a rate-independent situation by means of rate-dependent approximations. The reader is referred instead to Efendiev and Mielke [29] and Mielke, Rossi, and Savaré [66, 65] for some results in this direction.

The above convergence result could be alternatively obtained by applying the abstract analysis by Mielke, Roubíček, and Stefanelli [69, Thm. 3.1]. In the latter, besides the convergences of the functionals $\psi_n\to\psi$ and $\varphi_n\to\varphi$, some extra closure condition, indeed fulfilled in the present situation, is crucially required [69, equation (2.11)]. Let us mention that [69] is devoted to a quite more general situation where the state space is just a Hausdorff topological space (in particular, no convexity is assumed on φ_n).

We specialize further the results on rate-independent evolutions by explicitly discussing the fundamental case of (2.3), i.e., the so-called *play operator* in a Hilbert space H. The latter stands as the basic element for the construction of a relevant class of hysteresis operators, namely, the so-called Prandtl–Ishlinskiĭ operators. The reader is referred to the classic monographs by Brokate and Sprekels [19], Krejčí [50], and Visintin [104] for a comprehensive collection of results on these operators. In particular, let us mention the convergence result [50, Thm. 3.12, p. 34] where the approximation of play operators under the Hausdorff convergence of the related characteristic convex sets C_n (see (8.2)) is discussed. Here we exploit instead the quite weaker situation of $C_n\to C$ in the Mosco sense [7] (namely, the corresponding indicator functions Mosco-converge).

Comparing the case of the play operator with the above general result for rate-independent problems, we stress that, owing to the lim inf inequality (5.2), no strong compactness is here needed, and (7.2) can be omitted. We have the following convergence result.

LEMMA 8.2 (convergence for the play operator). *Let H be a separable Hilbert space and $\psi_n \to \psi$ in the Mosco sense and fulfill* (5.1)*,* (8.1)*, and* (8.3) *uniformly with respect to n. Moreover, let $f_n \to f$ strongly in $C([0,T];H^*)$, f_n be uniformly Lipschitz continuous, and $u_n^0 \to u^0$ in H. Finally, let $u_n \in W^{1,1}(0,T;H)$ be such that $Q_n(u_n) = 0$. Then $u_n \to u$ weakly star in $W^{1,\infty}(0,T;H)$ and $Q(u) = 0$.*

Proof. The Lipschitz regularity of f_n entails (8.5)–(8.6). Hence, the uniform control of (8.3) yields via (8.8) the uniform bound of u_n in $W^{1,\infty}(0,T;H)$. The assertion follows by extracting weakly star convergent subsequences and exploiting Lemma 5.3. In particular, the convergence of the whole sequence is ensured by the uniqueness of the solution of the limit problem [50, Thm. 3.1, p. 27 and Prop. 3.9, p. 33]. ∎

The latter convergence result extends the former analysis by this author [102, Lemma 4.4] to the more natural setting $W^{1,1}$. One has, however, to mention that the former result was including the possibility of approximating the play operator with non-rate-independent evolutions (such as those stemming from penalizations or singular perturbations, for instance) while Lemma 8.2 is restricted to approximating plays only. On the other hand, the present convergence result is slightly more precise than the former since no strong convergence on the derivatives f_n' is required. The reader is referred to [103] for further results in this direction.

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