Proseminar zu Algebraische Topologie

Sommersemester 2013, LVN: 250052

Di 12:05 – 12:50, 2A310 (UZA2)

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1. Let $I := [0,1] \subseteq \mathbb{R}$ denote the compact unit interval, consider the equivalence relation on $I \times I$ generated by $(x,0) \sim (x,1)$, $x \in I$, and $(0,y) \sim (1,y)$, $y \in I$, and let $p\colon I \times I \to X := (I \times I)/{\sim}$ denote the canonical projection onto the quotient space. Show that the map

$$f\colon I \times I \to S^1 \times S^1, \qquad f(x,y) := \bigl(e^{2\pi \mathbf{i} x}, e^{2\pi \mathbf{i} y}\bigr),$$

factorizes to a homeomorphism, $X \cong S^1 \times S^1$. More precisely, show that there exists a (unique) continuous map $\bar{f}\colon X \to S^1 \times S^1$ such that $\bar{f} \circ p = f$, and prove that $\bar{f}$ is a homeomorphism. Here $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle.

2. Let $R > r > 0$ and consider the following subspace (surface) in $\mathbb{R}^3$,

$$T := \Bigl\{(x,y,z) \in \mathbb{R}^3 : \bigl(\sqrt{x^2+y^2} - R\bigr)^2 + z^2 = r^2\Bigr\}.$$

Construct a homeomorphism $T \cong S^1 \times S^1$.

3. For $n \in \mathbb{N}_0$ let $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ denote the unit sphere. On $S^n \times [-1,1]$ consider the equivalence relation generated by $(x,1) \sim (y,1)$ und $(x,-1) \sim (y,-1)$, $x, y \in S^n$. Show that the quotient space $(S^n \times [-1,1])/{\sim}$ is homeomorphic to S^{n+1}. Provide drawings for $n = 0$ and $n = 1$.

4. Let (X, x_0) and (Y, y_0) be two pointed spaces such that $\pi_1(Y, y_0) = 0$. Show that the canonical projection, $p\colon X \times Y \to X$, $p(x,y) := x$, and the inclusion, $\iota\colon X \to X \times Y$, $\iota(x) := (x, y_0)$, induce mutually inverse group isomorphisms, $p_*\colon \pi_1\bigl(X \times Y, (x_0,y_0)\bigr) \to \pi_1(X, x_0)$ and $\iota_*\colon \pi_1(X,x_0) \to \pi_1\bigl(X \times Y, (x_0,y_0)\bigr)$, i.e. $p_* \circ \iota_* = \mathrm{id}_{\pi_1(X,x_0)}$ and $\iota_* \circ p_* = \mathrm{id}_{\pi_1(X \times Y,(x_0,y_0))}$.

5. Suppose $g, h\colon I \to X$ are two paths from $x_0 := g(0) = h(0)$ to $x_1 := g(1) = h(1)$. Show that the isomorphisms

$$\beta_g\colon \pi_1(X,x_1) \xrightarrow{\cong} \pi_1(X,x_0) \qquad \text{and} \qquad \beta_h\colon \pi_1(X,x_1) \xrightarrow{\cong} \pi_1(X,x_0)$$

coincide if and only if $[g\bar{h}]$ is contained in the center of $\pi_1(X,x_0)$.

6. A subset $X \subseteq \mathbb{R}^n$ is called *star shaped,* if there exists $z \in X$ with the following property: $x \in X,\ t \in [0,1] \Rightarrow (1-t)x + tz \in X$, i.e. the affine segment connecting x with z is entirely contained in X, for every $x \in X$. Any such z is called a *center* of X. Show that star shaped subsets are simply connected. Conclude that the $\mathbb{C} \setminus (-\infty, 0]$ is simply connected.

[1]Further problems will be posted at: `http://www.mat.univie.ac.at/~stefan/AT13.html`

7. Let $P \in S^n$, and show:

(i) If $Q \in S^n$ and $Q \neq P$, then $S^n \setminus \{P, Q\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

(ii) The closed upper hemisphere, $H := \{x \in S^n : \langle x, P\rangle \geq 0\}$, is homeomorphic to the closed unit disk, $D^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

8. Let $\mathrm{SO}_2 := \{U \in \mathcal{M}_{2\times 2}(\mathbb{R}) : U^tU = I, \det(U) = 1\}$ denote the group of orthogonal (2×2)-matrices with determinant 1, equipped with the topology induced from $\mathcal{M}_{2\times 2}(\mathbb{R}) = \mathbb{R}^4$. Show that

$$f\colon S^1 \to \mathrm{SO}_2, \qquad f(x,y) := \begin{pmatrix} y & x \\ -x & y \end{pmatrix},$$

is a homeomorphism, $S^1 \cong \mathrm{SO}_2$. Conclude that $\pi_1(\mathrm{SO}_2) \cong \mathbb{Z}$, and provide an explicit loop in SO_2, which represents a generator of $\pi_1(\mathrm{SO}_2)$.

9. Let $\mathrm{SU}_2 := \{U \in \mathcal{M}_{2\times 2}(\mathbb{C}) : U^*U = I, \det(U) = 1\}$ denote the group of unitary (2×2)-matrices with determinant 1, equipped with the topology induced from $\mathcal{M}_{2\times 2}(\mathbb{C}) = \mathbb{C}^4$. Consider the sphere $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ as a subspace of $\mathbb{C}^2$, and show that

$$f\colon S^3 \to \mathrm{SU}_2, \qquad f(z,w) := \begin{pmatrix} \bar{w} & z \\ -\bar{z} & w \end{pmatrix},$$

is a homeomorphism, $S^3 \cong \mathrm{SU}_2$. Conclude that SU_2 is simply connected.

10. Let $A \subseteq \mathbb{R}^n$ be an affine subspace of codimension $k := n - \dim(A)$. Show that $\mathbb{R}^n \setminus A$ is simply connected, provided $k \geq 3$. Furthermore, in the case $k = 2$, show that $\pi_1(\mathbb{R}^n \setminus A) \cong \mathbb{Z}$, and exhibit an explicit loop in $\mathbb{R}^n \setminus A$, which represents a generator of $\pi_1(\mathbb{R}^n \setminus A)$. Hint: Construct a homeomorphism $\mathbb{R}^n \setminus A \cong (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{\dim(A)}$.

11. Show that every continuous map $f\colon I^2 \to I^2$ has at least one fixed point, where $I^2 := I \times I$. Hint: Construct a homeomorphism $I^2 \cong D^2$, where $D^2 := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ denotes the closed unit disk.

12. Show: $CS^{n-1} \cong D^n$.

13. Show: $D^n/S^{n-1} \cong S^n$.

14. Let X, Y_1, Y_2 be topological spaces, and let $p_i\colon Y_1 \times Y_2 \to Y_i$ denote the canonical projections, $i = 1, 2$. Show that the two maps, $(p_i)_*\colon [X, Y_1 \times Y_2] \to [X, Y_i]$, $i = 1, 2$, determine a map,

$$[X, Y_1 \times Y_2] \xrightarrow{\cong} [X, Y_1] \times [X, Y_2],$$

which is a bijection.

15. Show that a continuous map, $f\colon X \to Y$, is a homotopy equivalence if and only if there are continuous maps $g\colon Y \to X$ and $h\colon Y \to X$, such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ h \simeq \mathrm{id}_Y$.

16. Put $Z := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$, and consider the subspace

$$X := \big(Z \times I\big) \cup \big(I \times \{0\}\big) \subseteq \mathbb{R}^2.$$

Moreover, let $P := (0,0) \in X$, $Q := (0,1) \in X$ and $A := I \times \{0\} \subseteq X$. Show:

(i) A is a deformation retract of X.
(ii) $\{P\}$ is a deformation retract of X.
(iii) X is contractible.
(iv) The inclusion $\{Q\} \to X$ is a homotopy equivalence.
(v) $\{Q\}$ is *not* a deformation retract of X.

Hint for (v): Suppose conversely, $H\colon X \times I \to X$ is a retracting deformation onto $\{Q\}$, i.e. $H_0 = \mathrm{id}_X$, $H_1(x) = Q$ for all $x \in X$, and $H_t(Q) = Q$ for all $t \in I$. Show that for every neighborhood U of Q there exists a neighborhood V von Q such that $H(V \times I) \subseteq U$. Conclude that each point in V can be connected with Q by a path in U. Choosing U sufficiently small, this leads to a contradiction.

17. Show that a continuous map, $f\colon S^1 \to S^1$, is a homotopy equivalence if and only if $\deg(f) = \pm 1$.

18. Prove the following generalization of Proposition I.4.4: Suppose $n \in \mathbb{N}$ and put $\zeta := e^{2\pi \mathbf{i}/n} \in S^1$, i.e. $\zeta^n = 1$. Moreover, let $f\colon S^1 \to S^1$ be a continuous map such that $f(\zeta z) = f(z)$, for all $z \in S^1$. Then $\deg(f) \equiv 0 \mod n$.

19. Prove the following generalization of Theorem I.4.8: Suppose $n \in \mathbb{N}$ and put $\zeta := e^{2\pi \mathbf{i}/n} \in S^1$, i.e. $\zeta^n = 1$. Moroever, let $f\colon S^1 \to S^1$ be a continuous map such that $f(\zeta z) = \zeta f(z)$, for all $z \in S^1$. Then $\deg(f) \equiv 1 \mod n$. In particular, f is not nullhomotopic, provided $n \geq 2$. Hint: Replace the antipodal map A in the proof of Theorem I.4.8 by the rotation $R\colon S^1 \to S^1$, $R(z) := \zeta z$.

20. Let G_α be groups, $\alpha \in A$. Show that the canonical homomorphism,

$$\bigoplus_{\alpha \in A} G_\alpha^{\mathrm{ab}} \xrightarrow{\cong} \Big(\mathop{*}_{\alpha \in A} G_\alpha\Big)^{\mathrm{ab}},$$

is an isomorphism. Here $G^{\mathrm{ab}} := G/[G,G]$ denotes the Abelization of G.

21 (Hamilton's quaternions). Let $\mathbb{H}$ denote the set of all complex (2×2)-matrices of the form $\left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right)$, $z, w \in \mathbb{C}$. Show that, with respect to ordinary matrix addition and multiplication, $\mathbb{H}$ satisfies all axioms of a field, except commutativity of multiplication ($\mathbb{H}$ is a division ring/skew field). Put

$$1 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \mathbf{i} := \left(\begin{smallmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{smallmatrix}\right), \quad \mathbf{j} := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad \mathbf{k} := \left(\begin{smallmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{smallmatrix}\right).$$

Show that $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis of the real vector space underlying $\mathbb{H}$. Check that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, and

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}, \quad \mathbf{ji} = -\mathbf{k}, \quad \mathbf{kj} = -\mathbf{i}, \quad \mathbf{ik} = -\mathbf{j}.$$

We use the algebra homomorphisms $\mathbb{C} \to \mathbb{H}$, $z \mapsto \left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$, and $\mathbb{R} \to \mathbb{H}$, $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, to regard $\mathbb{R}$ and $\mathbb{C}$ as subalgebras of $\mathbb{H}$, respectively. For $x \in \mathbb{H}$, the conjugate quaternion is defined by $\bar{x} := x^*$ where x^* denotes the conjugate transposed of the matrix x. For instance, $\bar{1} = 1$, $\bar{\mathbf{i}} = -\mathbf{i}$, $\bar{\mathbf{j}} = -\mathbf{j}$ and $\bar{\mathbf{k}} = -\mathbf{k}$. Show $\bar{\bar{x}} = x$, $\overline{x+y} = \bar{x} + \bar{y}$ and $\overline{xy} = \bar{y}\bar{x}$ for alle $x, y \in \mathbb{H}$, and $\overline{ax} = a\bar{x}$ for all $a \in \mathbb{R}$ and $x \in \mathbb{H}$. Furthermore, show that $\bar{x} = x$ iff $x \in \mathbb{R} \subseteq \mathbb{H}$. The real part of $x \in \mathbb{H}$ is defined by $\mathrm{Re}(x) := (x + \bar{x})/2 = \mathrm{tr}(x)/2 \in \mathbb{R}$. In particular, $\mathrm{Re}(1) = 1$ and $\mathrm{Re}(\mathbf{i}) = \mathrm{Re}(\mathbf{j}) = \mathrm{Re}(\mathbf{k}) = 0$. Show $\mathrm{Re}(xy) = \mathrm{Re}(yx)$ for all $x, y \in \mathbb{H}$. Show that $\langle x, y\rangle := \mathrm{Re}(x\bar{y})$ defines in Euclidean inner product on $\mathbb{H}$ such that $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ becomes an orthonormal basis. Verify $\langle xy, z\rangle = \langle y, \bar{x}z\rangle$, $\langle yx, z\rangle = \langle y, z\bar{x}\rangle$ and $\langle \bar{x}, \bar{y}\rangle = \langle x, y\rangle$, for all $x, y, z, \in \mathbb{H}$. Show that the associated norm, $|x|^2 := \langle x, x\rangle = x\bar{x} = \bar{x}x$, is multiplicative, $|xy| = |x||y|$. Conclude that multiplication in $\mathbb{H}$ restricts to a group structure on $S^3 = \{x \in \mathbb{H} : |x| = 1\}$. Observe that this group coincides with SU_2.

22. We consider $\mathbb{H}^n := \mathbb{H} \times \cdots \times \mathbb{H}$ as a left $\mathbb{H}$-modul, i.e. for $\lambda \in \mathbb{H}$ and $(x_1, \dots, x_n) \in \mathbb{H}^n$ we put $\lambda(x_1, \dots, x_n) := (\lambda x_1, \dots, \lambda x_n)$. Show that $x \sim y \Leftrightarrow \exists \lambda \in \mathbb{H} : \lambda x = y$ defines an equivalence relation on $\mathbb{H}^{n+1} \setminus \{0\}$. Show that the quotient space $\mathbb{H}\mathrm{P}^n := \big(\mathbb{H}^{n+1} \setminus \{0\}\big)/\!\sim$ is a compact Hausdorff space. Construct a continuous map $\varphi\colon S^{4n-1} \to \mathbb{H}\mathrm{P}^{n-1}$ such that

$$\mathbb{H}\mathrm{P}^n \cong \mathbb{H}\mathrm{P}^{n-1} \cup_\varphi D^{4n}.$$

Conclude that $\mathbb{H}\mathrm{P}^n$ is simply connected. Moreover, observe that $\mathbb{H}\mathrm{P}^1 \cong S^4$. Hint: Proceed as in the complex case.

23. Consider $S^3 := \{x \in \mathbb{H} : |x| = 1\}$ and

$$\mathbb{I} := 1^\perp = \{x \in \mathbb{H} : \bar{x} = -x\} = \{x \in \mathbb{H} : \mathrm{Re}(x) = 0\} \cong \mathbb{R}^3,$$

see Problem 21. Show that for $x \in S^3$ and $y \in \mathbb{I}$ the expression $\lambda_x(y) := xy\bar{x}$ defines an $\mathbb{R}$-linear map $\lambda_x\colon \mathbb{I} \to \mathbb{I}$. Show that λ_x is an isometry, i.e. $|\lambda_x(y)| = |y|$ for all $x \in S^3$ and $y \in \mathbb{I}$. Conclude that we obtain a continuous map $\lambda\colon S^3 \to \mathrm{SO}_3$. Show that λ is a surjective homomorphism of groups with kernel $\ker(\lambda) = \{\pm 1\}$. Show that λ factorizes to a homeomorphism, $\mathbb{R}\mathrm{P}^3 \cong \mathrm{SO}_3$. *Hint for the surjectivity of λ:* For $\pm 1 \neq x \in S^3$ the isometry λ_x is a rotation by the angle $2\arccos(\mathrm{Re}(x))$ around the axis spanned by $x - \bar{x}$. To see this verify:

(i) The points on the subspace spanned by $x - \bar{x}$ are fixed points of λ_x.
(ii) If $y \in \mathbb{I}$ and $\langle y, x - \bar{x}\rangle = 0$, then $\langle x, y\rangle = 0$, hence $y\bar{x} = xy$ and thus $2\langle \lambda_x(y), y\rangle = 2\big(2(\mathrm{Re}(x))^2 - 1\big)|y|^2$.
(iii) Use the relation $\arccos(2t^2 - 1) = 2\arccos(t)$, $0 \leq t \leq 1$, to show that the angle between $\lambda_x(y)$ and y is $2\arccos(\mathrm{Re}(x))$.

Finally, recall that every element of SO_3 can be written as a product of rotations.

Alternatively, one can observe that the differential of the map $\lambda\colon S^3 \to \mathrm{SO}_3$ at the identity is invertible and use the implicit function theorem to conclde that the image of λ contains a neighborhood of the identity in SO_3. Since λ is a homomorphism and since SO_3 is connected, this implies that λ is onto.

24. Determine the fundamental group of $X := (S^1 \times S^1)/(\{1\} \times S^1)$.

25. Let M_1 and M_2 be two connected topological n-manifolds.[2] Choose open subsets $U_i \subseteq M_i$, homeomorphisms $\varphi_i\colon U_i \to \mathbb{R}^n$ and put $\dot{M}_i := M_i \setminus \varphi_i^{-1}(B^n)$, $i = 1, 2$. Consider $A := \varphi_2^{-1}(S^{n-1}) \subseteq \dot{M}_2$ and the map $\varphi : A \to \dot{M}_1$, $\varphi := \varphi_1^{-1} \circ \varphi_2$. Show that the connected sum, $M_1 \sharp M_2 := \dot{M}_1 \cup_\varphi \dot{M}_2$, is a topological n-manifold. Moreover, show that $\pi_1(M_1 \sharp M_2) \cong \pi_1(M_1) * \pi_1(M_2)$, provided $n \geq 3$.

26. Let $L_1, \ldots, L_n$ be mutually different lines through the origin in $\mathbb{R}^3$. Determine $\pi_1\big(\mathbb{R}^3 \setminus (L_1 \cup \cdots \cup L_n)\big)$.

27 (Geometric realization of simplicial complexes). An (abstract) *simplicial complex*, Δ, is a set of finite subsets of some set S, i.e. $\Delta \subseteq 2^S$, with the following property: if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. If $\sigma \in \Delta$ has precisely $k+1$ elements, then σ is said to be of *dimension* k and is called k*-simplex* of Δ. 0-simplices are also called *vertices* of Δ, the set of all vertices will be denoted by $V(\Delta)$. A simplicial complex is called *finite* if Δ is finite, equivalently, $V(\Delta)$ is finite.

A *simplicial map*, $f\colon \Delta' \to \Delta$, between two simplicial complexes, Δ' and Δ, is a map $f\colon V(\Delta') \to V(\Delta)$ such that $f(\sigma) \in \Delta$ for all $\sigma \in \Delta'$. Convince yourself, that the composition of simplicial maps is again a simplicial map, and so is the identical map, id_Δ. A simplicial map $f\colon \Delta' \to \Delta$ is called *isomorphism*, if there exists a simplicial map $g\colon \Delta \to \Delta'$ such that $f \circ g = \mathrm{id}_\Delta$ and $g \circ f = \mathrm{id}_{\Delta'}$.

A subset $\Delta' \subseteq \Delta$ of a simplicial complex Δ is called a *subcomplex* if Δ' is a simplicial complex itself. In this case the natural inclusion $V(\Delta') \to V(\Delta)$ is a simplicial map $\Delta' \to \Delta$. Let $\mathrm{Sk}_k(\Delta) \subseteq \Delta$ denote the k*-skeleton* of Δ, i.e. set of all simplices of dimension at most k. Observe that $\mathrm{Sk}_k(\Delta)$ is subcomplex of Δ.

The geometric realization, $|\Delta|$, of a finite simplicial complex Δ with vertices $V := V(\Delta)$, is the compact topological space

$$|\Delta| := \big\{\lambda\colon V \to \mathbb{R} \;\big|\; \lambda \geq 0,\ \textstyle\sum_{v \in V} \lambda(v) = 1,\ \mathrm{supp}(\lambda) \in \Delta\big\} \subseteq \mathbb{R}^V \cong \mathbb{R}^n,$$

where n denotes the cardinality of V. Show that

$$|\Delta| = \bigcup_{\sigma \in \Delta} \langle\sigma\rangle$$

where $\langle\sigma\rangle$ denotes the convex hull of the unit vectors in $\mathbb{R}^V$ corresponding to $\sigma \subseteq V$. The geometric realization of a simplicial map, $f\colon \Delta' \to \Delta$, is the continuous map $|f|\colon |\Delta'| \to |\Delta|$ where $|f|(\lambda')(v) := \sum_{f(v')=v} \lambda'(v')$. Show that this is indeed well defined and continuous. Moreover, verify

$$|f \circ g| = |f| \circ |g| \qquad \text{and} \qquad |\mathrm{id}_\Delta| = \mathrm{id}_{|\Delta|}$$

for any other simplicial map $g\colon \Delta'' \to \Delta'$. Conclude that the geometric realization of a simplicial isomorphism is a homeomorphism.

[2]Recall that a topological n-manifold is a paracompact Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

28.

(i) Let Δ^n denote the abstract simplicial complex consisting of all subsets of $\{0,\dots,n\}$ and show $|\Delta^n| \cong D^n$.

(ii) Let $\dot\Delta^n = \mathrm{Sk}_{n-1}(\Delta^n)$ denote the abstract simplicial complex consisting of all subsets of $\{0,\dots,n\}$ which have at most n elements, and show $|\dot\Delta^n| \cong S^{n-1}$.

(iii) Describe an abstract simplicial complex Δ with geometric realization $|\Delta| \cong S^1 \times S^1$.

29. Let Δ be a finite abstract simplicial complex. Show that the geometric realization of its k-skeleton can be obtained from the geometric realization of the $(k-1)$-skeleton by attaching simplicies $|\Delta^k|$ along maps defined on $|\dot\Delta^k|$. More precisely, show that

$$|\mathrm{Sk}_k(\Delta)| \cong |\mathrm{Sk}_{k-1}(\Delta)| \cup_\varphi \bigsqcup |\Delta^k|$$

for an appropriate continuous map $\varphi\colon \bigsqcup |\dot\Delta^k| \to |\mathrm{Sk}_{k-1}(\Delta)|$. Conclude that the inclusion $\mathrm{Sk}_2(\Delta) \to \Delta$ induces an isomorphism

$$\pi_1\big(|\mathrm{Sk}_2(\Delta)|, x_0\big) \cong \pi_1\big(|\Delta|, x_0\big).$$

30. A simplicial complex Δ is called connected if its geometric realization, $|\Delta|$, is (path)connected. Show that Δ is connected if and only if the following holds true: for any two vertices x, y of Δ there exist vertices $x = x_0, x_1, \dots, x_n = y$ such that $\{x_{i-1}, x_i\}$ is a simplex of Δ, for all $i = 1, \dots, n$. Hint: Show that Δ is connected iff its 1-skeleton $\mathrm{Sk}_1(\Delta)$ is connected.

31 (Trees). A *tree* is a simply connected simplicial complex of dimension at most one.[3] Suppose T is a tree in a 1-dimensional simplicial complex Δ, i.e. T is a subcomplex of Δ. Show that the natural quotient map $|\Delta| \to |\Delta|/|T|$ is a homotopy equivalence. Conclude that a 1-dimensional simplicial complex is a tree iff its is contractible.

32 (Maximal trees). Observe that every simplicial complex Δ contains a maximal (with respect to inclusion of subcomplexes) tree. Assuming Δ to be connected, show that a tree in Δ is maximal iff it contains all vertices of Δ.

33 (Fundamental group of 1-dimensional simplicial complexes). Let Δ be a connected 1-dimensional simplicial complex, suppose T is a maximal tree in Δ, and let $e_1, \dots, e_k$ denote the 1-simplices of Δ which are not contained in T. Show that the fundamental group of Δ is free of rank k with generators corresponding to $e_1, \dots, e_k$. Moreover, show that $\chi(\Delta) = 1 - k$ where the Euler characteristics $\chi(\Delta)$ is defined as the number of 0-simplices minus the number of 1-simplices of Δ. Conclude that a connected 1-dimensional simplicial complex is a tree if and only if $\chi(\Delta) = 1$.

[3] A simplicial complex is called simply connected if its geometric realization is simply connected.

34 (Fundamental group of general simplicial complexes). Let Δ be a connected simplicial complex, and suppose T is a maximal tree in Δ. Let $e_1, \dots, e_k$ denote the 1-simplices of Δ which are not contained in T, and fix an orientation for each e_i. Moreover, let $\sigma_1, \dots, \sigma_l$ denote all 2-simplices of Δ. For every $j \in \{1, \dots, l\}$ define an element r_j in the free group $F(\{e_1, \dots, e_k\})$ by following the three edges (i.e. 1-simplices) of σ_j consecutively (starting at any vertex, proceeding according to either orientation), writing down e_i or e_i^{-1} for each edge which happens to be among the $e_1, \dots, e_k$, and disregarding the others (i.e. those edges which are in T). More precisely, we write e_i if the orientations match and e_i^{-1} if they don't. Show that:

$$\pi_1(|\Delta|) \cong \langle e_1, \dots, e_k \mid r_1, \dots, r_l \rangle.$$

Hint: Use the van Kampen theorem, Problem 29 and Problem 33.

35. Use the previous problem to compute the fundamental groups of the 2-dimensional torus and the Kleinian bottle.

36. Suppose $n, p \in \mathbb{N}$, $n \geq 2$, $q_1, \dots, q_n \in \mathbb{Z}$ are such that p and q_i are coprime, $i = 1, \dots, n$, and let $L := L(p; q_1, \dots, q_n)$ denote the associated lense space. Show that $[L, K] = 0$, where K denotes the Kleinian bottle. In other words, show that any two continuous maps $L \to K$ are homotopic. *Hint:* Show that every homomorphism $\pi_1(L) \to \pi_1(K)$ is trivial, and use the covering $\mathbb{R}^2 \to K$.

37 (Nielsen–Schreier theorem). Show that every subgroup of a free group is free. Proceed as follows:

A 1-dimensional CW complex is topological space X which is homeomorphic to space obtained by attaching any number of 1-cells to a discrete space, i.e.

$$X \cong X_0 \cup_\varphi \bigsqcup_{\lambda \in \Lambda} D^1$$

where X_0 is a discrete space, Λ is an index set, $D^1 = [-1, 1]$ denotes the 1-dimensional disk, $\varphi\colon \bigsqcup_{\lambda \in \Lambda} S^0 \to X_0$ is a (continuous) map, and $S^0 = \partial D^1 = \{-1, 1\}$ denotes the 0-dimensional sphere. Both, X_0 and Λ, may be infinite. Thus, a 1-dimensional CW complex is the same thing as a graph.

(i) Show that 1-dimensional CW complexes are locally contractible, whence locally path connected and semi locally simply connected.
(ii) Show that every covering of a 1-dimensional CW complex is a 1-dimensional CW complex.
(iii) Show that any compact subset of X intersects only finitely many of the attached disks.
(iv) Show that the fundamental group of a connected 1-dimensional CW complex is free. Hint: Use the lemma of Zorn and (iii) to show that there exists a subset $\Lambda' \subseteq \Lambda$ such that $T := X_0 \cup_\varphi \bigsqcup_{\lambda \in \Lambda'} D^1$ is simply connected (maximal tree); observe that $X/T \cong \bigvee_{\lambda \in \Lambda \setminus \Lambda'} S^1$; and show that the canonical projection, $X \to X/T$, is a homotopy equivalence.

Given a subgroup G of a free group F, construct a connected 1-dimensional CW complex X such that $\pi_1(X) \cong F$, consider a covering $\tilde{X} \to X$ with characteristic subgroup G, and recall that $\pi_1(\tilde{X}) \cong G$.

38. Determine all 2 and 3-sheated connected coverings of $S^1 \vee S^1$. Which of these coverings are normal? Hint: Determine all conjugacy classes of subgroups in $\mathbb{Z} * \mathbb{Z}$ of index 2 and 3.

39 (Orientation bundle). Let $p\colon E \to X$ be a real vector bundle of rank k.[4] The orientation bundle of E is a 2-fold covering, $\mathcal{O}_E \to X$, which can be described in either of the following ways:

- (i) $\mathcal{O}_E = \bigsqcup_{x\in X} \mathcal{O}_{E_x}$, where $\mathcal{O}_{E_x}$ denotes the set (with two elements) of orientations of the vector space E_x. The topology is defined using vector bundle charts $E|_U \cong U \times \mathbb{R}^k$ and declaring the induced bijections, $\mathcal{O}_E|_U = \bigsqcup_{x\in U} \mathcal{O}_{E_x} \cong U \times \mathcal{O}_{\mathbb{R}^k}$, to be homeomorphisms.
- (ii) $\mathcal{O}_E = P/\!\sim$, where P denotes the frame bundle[5] of E, and two frames over $x \in X$ are considered equivalent iff they define the same orientation of the vector space E_x.
- (iii) $\mathcal{O}_E = (\Lambda^k E \setminus 0)/\mathbb{R}^+$, where $\Lambda^k E \setminus 0$ denotes the k-fold exterior produkt[6] with the zero section removed, and the group $\mathbb{R}^+$ acts by scalar multiplication on (the fibers of) $\Lambda^k E$.

Show that these definition actually describe the same, i.e. canonically isomorphic, 2-fold coverings of X. Moreover, show that the following are equivalent:

- (iv) One can choose orientations of each fiber E_x which depend continuously on $x \in X$, in the sense of (i).
- (v) The covering $\mathcal{O}_E \to X$ is trivializable.
- (vi) The line bundle[7] $\Lambda^k E$ is trivializable.
- (vii) E admits a vector bundle atlas whose transition functions take values in $\mathrm{SL}(\mathbb{R}^k)$.
- (viii) E admits a vector bundle atlas whose transition functions take values in $\mathrm{GL}_+(\mathbb{R}^k)$.

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[4] Recall that a real vector bundle of rank k is a continuous map $p\colon E \to X$ together with the structure of a k-dimensional real vector space on each fiber $E_x := p^{-1}(x)$, $x \in X$, which is locally trivial in the following sense: Every point in X admits a neighborhood U such that there exists a fiberwise linear homeomorphism $\varphi\colon E|_U := p^{-1}(U) \to U \times \mathbb{R}^n$, i.e. $\mathrm{pr}_1 \circ \varphi = p|_U$ and $\varphi_x\colon E_x \to \{x\} \times \mathbb{R}^n = \mathbb{R}^n$ is a linear isomorphism, for all $x \in U$.

[5] $P := \big\{(e_1, \dots, e_k) \in E \times \cdots \times E \;\big|\; \exists x \in X : e_1, \dots, e_k \text{ is a basis of } E_x\big\}$ with the topology induced from $E \times \cdots \times E$.

[6] $\Lambda^q E := \bigsqcup_{x\in X} \Lambda^q E_x$ is a vector bundle over X obtained by replacing each fiber E_x with its q-fold exterior product. The topology on $\Lambda^q E$ can be described by using vector bundle charts $E|_U \cong U \times \mathbb{R}^k$ and declaring the induced fiber wise linear bijections, $\Lambda^q E|_U = \bigsqcup_{x\in U} \Lambda^q E_x \cong U \times \Lambda^q \mathbb{R}^k = U \times \mathbb{R}^{\binom{k}{q}}$ to be vector bundle charts for $\Lambda^q E$.

[7] i.e. vector bundle of rank 1.

If these equivalent conditions are satisfied, then the vector bundle E is called orientable. Any trivialization $\mathcal{O}_E \cong X \times \{\pm 1\}$ of the orientation bundle (covering) is called an orientation of E. How many orientations, does an orientable vector bundle possess? Provide conditions on X which ensure that every vector bundle over X is orientable.

40. Let M be a smooth manifold of dimension n. The covering $\tilde{M} := \mathcal{O}_{TM}$ of M is called the orientation covering of M, cf. the last problem. Show that the following are equivalent:

(i) M admits an oriented atlas, i.e. an atlas with orientation preserving transition functions.
(ii) TM admits a vector bundle atlas whose transition functions take values in $\mathrm{GL}_+(\mathbb{R}^n)$.
(iii) TM admits a vector bundle atlas whose transition functions take values in $\mathrm{SL}(\mathbb{R}^n)$.
(iv) The line bundle $\Lambda^n TM$ is trvializeable.
(v) The orientation covering $\tilde{M} \to M$ is trivializeable.

Conclude that every simply connected manifold is orientable. More generally, if M is connected and its fundamental group does not admit a subgroup of index two, then M is orientable.

41. Let $\tilde{M} \to M$ be the orientation covering of a smooth n-manifold M. Show that $\tilde{M}$ is orientable. Is the non-trivial deck transformation, $\tilde{M} \to \tilde{M}$, orientation preserving or reversing?

42 (Algebraic Morse inequalities). Let C_* be a finitely generated chain complex. Show that

$$(-1)^k \sum_{q \le k} (-1)^q b_q(C) \le (-1)^k \sum_{q \le k} (-1)^q \operatorname{rank}(C_q)$$

for all k. Here $b_q(C) := \operatorname{rank}(H_q(C))$ denotes the q-th Betti number of C_*. What do we get for $k \to \infty$?

43. For an endomorphism of a finitely generated abelian group, $\varphi\colon A \to A$, define its trace by

$$\operatorname{tr}(\varphi) := \operatorname{tr}_{\mathbb{Q}}\big(A \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\varphi \otimes \mathrm{id}_{\mathbb{Q}}} A \otimes_{\mathbb{Z}} \mathbb{Q}\big).$$

Show that this trace has the following properties:

(i) $\operatorname{tr}(\varphi) \in \mathbb{Z}$.
(ii) $\operatorname{tr}(\mathrm{id}_A) = \operatorname{rank}(A)$.
(iii) $\operatorname{tr}(\varphi + \psi) = \operatorname{tr}(\varphi) + \operatorname{tr}(\psi)$, for any two endomorphisms $\varphi, \psi\colon A \to A$.
(iv) $\operatorname{tr}(\varphi \circ \psi) = \operatorname{tr}(\psi \circ \varphi)$, for any two endomorphisms $\varphi, \psi\colon A \to A$.

(v) $\operatorname{tr}(\varphi) + \operatorname{tr}(\rho) = \operatorname{tr}(\psi)$, for every commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\rho} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

of finitely generated abelian groups with exact rows.

For an endomorphism of a finitely generated graded abelian group, $\varphi\colon A_* \to A_*$, define its graded (super) trace as $\operatorname{str}(\varphi) := \sum_q (-1)^q \operatorname{tr}(\varphi_q\colon A_q \to A_q)$. Spell out and prove analoguous properties of this graded trace.

44. Suppose $\varphi\colon C_* \to C_*$ is a chain map on a finitely generated chain complex, and let $\varphi_*\colon H_*(C) \to H_*(C)$ denote the induced homomorphism in homology. Show that

$$\operatorname{str}\bigl(\varphi\colon C_* \to C_*\bigr) = \operatorname{str}\bigl(\varphi_*\colon H_*(C) \to H_*(C)\bigr).$$

Hint: Apply the graded version of Problem 43(v) to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_* & \longrightarrow & Z_* & \longrightarrow & H_*(C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi|_{B_*}} & & \downarrow{\scriptstyle\varphi|_{Z_*}} & & \downarrow{\scriptstyle\varphi_*} & & \\
0 & \longrightarrow & B_* & \longrightarrow & Z_* & \longrightarrow & H_*(C) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_* & \longrightarrow & C_* & \xrightarrow{\partial} & (\Sigma B)_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi|_{Z_*}} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\Sigma\varphi} & & \\
0 & \longrightarrow & Z_* & \longrightarrow & C_* & \xrightarrow{\partial} & (\Sigma B)_* & \longrightarrow & 0
\end{array}$$

where $(\Sigma B)_q := B_{q-1}$. What do we get for $\varphi = \operatorname{id}_{C_*}$?

45 (Nine Lemma). Consider a commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows. Show that if two of the three columns are exact, then so is the third.

46 (Simplicial homology). Let Δ be a finite abstract simplicial complex. Let $S_q(\Delta)$ denote the (finite) set of q-simplices of Δ and let $C_q(\Delta) := \mathbb{Z}[S_q(\Delta)]$ denote the free abelian group generated by $S_q(\Delta)$. For each q, fix an ordering of the vertices of each q-simplex $\sigma \in S_q(\Delta)$, i.e. $v_0^\sigma, \dots, v_q^\sigma$ shall be all vertices of σ. For $\tau \in S_{q-1}(\Delta)$ with $\tau \subseteq \sigma$ we put $\varepsilon(\sigma, \tau) := 1$ if $v_0^\tau, \dots, v_{q-1}^\tau$ followed by the one missing vertex of σ coincides with the ordering of the vertices $v_0^\sigma, \dots, v_q^\sigma$ up to an even permutation, and $\varepsilon(\sigma, \tau) := -1$ otherwise.[8] Suppose $\rho \in S_{q-2}(\Delta)$ and $\sigma \in S_q(\Delta)$ such that $\rho \subseteq \sigma$. Show that there exist precisely two simplices $\tau \in S_{q-1}$ such that $\rho \subseteq \tau \subseteq \sigma$. Denoting these two simplicies by τ' and τ'', respectively, show that

$$\varepsilon(\sigma, \tau')\varepsilon(\tau', \rho) + \varepsilon(\sigma, \tau'')\varepsilon(\tau'', \rho) = 0.$$

Define a homomorphism $\partial\colon C_q(\Delta) \to C_{q-1}(\Delta)$ on basis elements $\sigma \in S_q(\Delta)$ by

$$\partial\sigma := \sum_{\tau \in S_{q-1}(\Delta),\, \tau \subseteq \sigma} \varepsilon(\sigma, \tau)\tau.$$

Show that $\partial^2 = 0$ and define simplicial homology of Δ as the homology of this (simplicial) complex, $H_q(\Delta) := H_q(C_*(\Delta), \partial)$. Conclude that

$$\sum_q (-1)^q \operatorname{rank} H_q(\Delta) = \sum_q (-1)^q \sharp S_q(\Delta).$$

One can show that $H_q(|\Delta|) = H_q(\Delta)$ and thus $\chi(|\Delta|) = \sum_q (-1)^q \sharp S_q(\Delta)$.

47. Compute the simplicial homology of the simplicial complexes Δ^n and $\dot{\Delta}^n$ in Problem 28. Hint: Consider the chain homotopy $h\colon C_q(\Delta^n) \to C_{q+1}(\Delta^n)$ defined on generators $\sigma \in S_q(\Delta^n)$ by

$$h(\sigma) := \sum_{\rho \in S_{q+1}(\Delta^n),\, \sigma \subseteq \rho} \varepsilon(\rho, \sigma)\rho$$

and show that $\partial h + h\partial = (n+1)\operatorname{id}$ on $C_q(\Delta^n)$, for all $q \geq 1$.

48. Choose an abstract simplicial complex Δ such that $|\Delta| \cong \mathbb{R}\mathrm{P}^2$ and compute $H_*(\Delta)$.

49 (Finite dimensional Hodge decomposition). Let

$$\cdots \leftarrow V_{q-1} \xleftarrow{\partial_q} V_q \xleftarrow{\partial_{q+1}} V_{q+1} \leftarrow \cdots$$

be a complex of finite dimensional real/complex vector spaces, i.e. each V_q is finite dimensional and $\partial_q \partial_{q+1} = 0$. Suppose each V_q is equipped with a Euklidean/Hermitian inner product, and let $\partial_q^*\colon V_{q-1} \to V_q$ denote the adjoint of

[8] Note that these signs $\varepsilon(\sigma, \tau)$ only depend on the orientation of the simplices σ and τ spezified by the ordering of their vertices.

$\partial_q \colon V_q \to V_{q-1}$. Moreover, put $\Delta_q := \partial_{q+1}\partial_{q+1}^* + \partial_q^*\partial_q$. Show that we have an orthogonal decomposition

$$V_q = \operatorname{img}(\partial_q^*) \oplus \underbrace{\ker(\Delta_q) \oplus \operatorname{img}(\partial_{q+1})}_{\ker(\partial_q)}.$$

Conclude that each homology class in $H_q(V) := \ker(\partial_q)/\operatorname{img}(\partial_{q+1})$ has a unique (harmonic) representative in $\ker(\Delta_q)$, that is, $H_q(V) = \ker(\Delta_q)$.