Algebraic invariants of the Eulerian ideal via (T, p)-joins

UC|UP Joint PhD Program in Mathematics

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Combinatorial Commutative Algebra — Introduction

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Neves, Vaz Pinto, Villarreal (2020). *J. Algebra*

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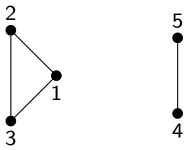

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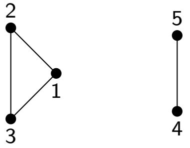

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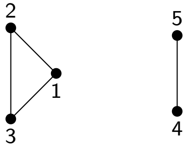

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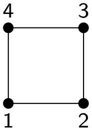

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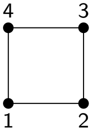

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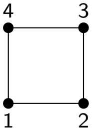

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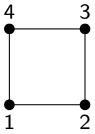

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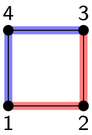

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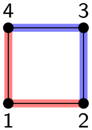

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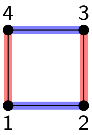

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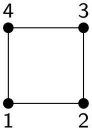

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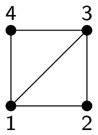

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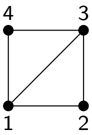

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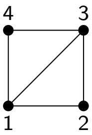

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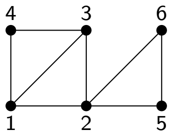

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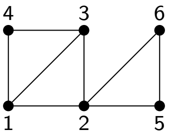

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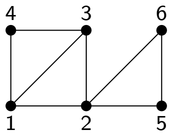

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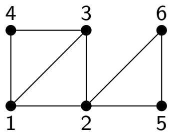

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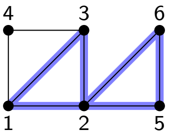

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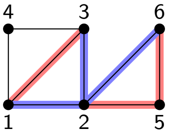

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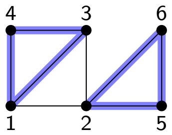

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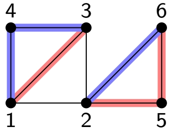

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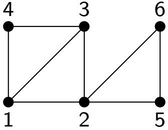

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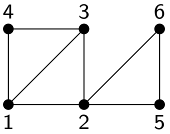

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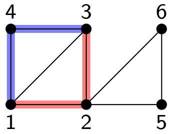

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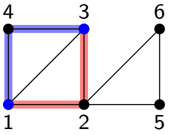

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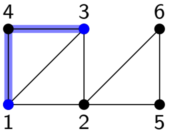

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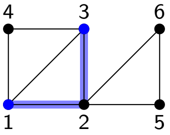

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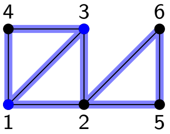

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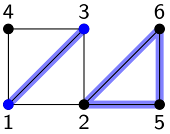

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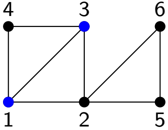

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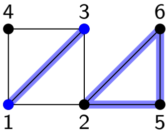

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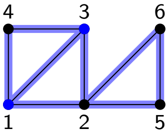

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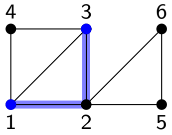

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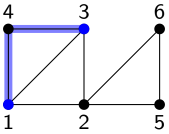

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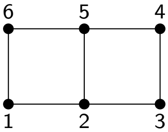

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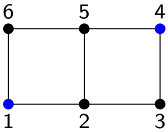

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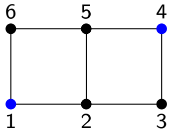

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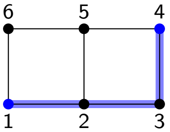

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- J, L minimum cardinality (T, p)-joins, with $(T, p) \in V_G \times \mathbb{Z}_2$;

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If $\exists\ J_0, \ldots, J_r$ minimum cardinality (T, p)-joins:

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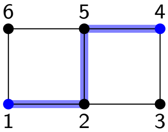

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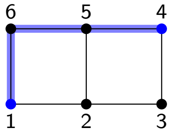

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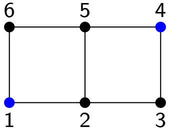

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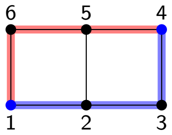

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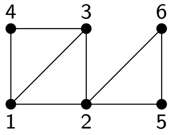

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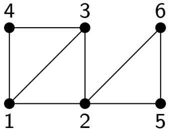

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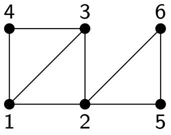

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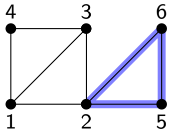

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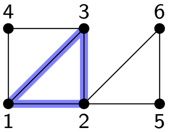

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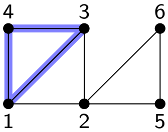

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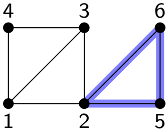

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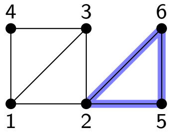

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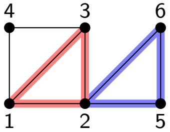

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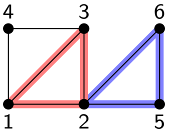

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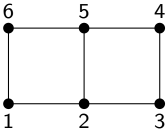

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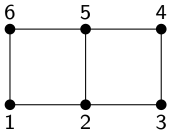

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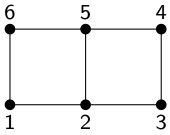

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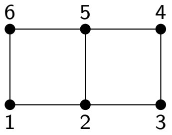

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Neves, Varejão (2024). *arXiv*.

The end!

Thank you for your attention !

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