A new bijective proof of the cubic q-binomial identity with applications in the quantum group

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The 93rd Séminaire Lotharingien de Combinatoire, 25th March 2025

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Introduction

Theorem (The cubic binomial identity; Gould '72)

Let m, e, s, t be non-negative integers. Then

$$\begin{bmatrix} m \\ t \end{bmatrix}_q \begin{bmatrix} m+e \\ s \end{bmatrix}_q = \sum_{j \geqslant 0} q^{(s-j)(t+e-j)} \begin{bmatrix} t+e \\ j \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m+j \\ s+t \end{bmatrix}_q .$$

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- All *cubic q-binomial identities*

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  in literature are equivalent;
- New, shorter bijective proof using vector space interpretation of q-binomials;
- As a corollary, we give the complete multiplication rule for the generators of Lusztig's integral form in $U_q(\mathfrak{sl}_2)$.

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A common origin...

A common origin

Székely ('85) showed that the identity

$$\binom{a+c+d+e}{a+c}\binom{b+c+d+e}{c+e} = \sum_{k\geqslant 0}\binom{a+b+c+d+e-k}{a+b+c+d}\binom{a+d}{k+d}\binom{b+c}{k+c}$$

implies all of the *cubic binomial identities*

$$\binom{\cdot}{\cdot}\binom{\cdot}{\cdot} = \sum \binom{\cdot}{\cdot}\binom{\cdot}{\cdot}\binom{\cdot}{\cdot}$$

in literature (these are due to Nanjundiah, Stanley, Bizley and Takács)

Cubic q-binomial identities

- Nanjundiah (another form):

$$\begin{bmatrix} x \\ m \end{bmatrix}_q \begin{bmatrix} y \\ n \end{bmatrix}_q = \sum_{K\geqslant 0} q^{(m-x+y-K)(n-K)} \begin{bmatrix} m-x+y \\ K \end{bmatrix}_q \begin{bmatrix} n+x-y \\ n-K \end{bmatrix}_q \begin{bmatrix} x+K \\ m+n \end{bmatrix}_q$$

- Stanley:

$$\begin{bmatrix} x+A \\ B \end{bmatrix}_q \begin{bmatrix} y+B \\ A \end{bmatrix}_q = \sum_{K\geqslant 0} q^{(A-K)(B-K)} \begin{bmatrix} x+y+K \\ K \end{bmatrix}_q \begin{bmatrix} y \\ A-K \end{bmatrix}_q \begin{bmatrix} x \\ B-K \end{bmatrix}_q$$

- Székely:

$$\begin{bmatrix} a+c+d+e \\ a+c \end{bmatrix}_q \begin{bmatrix} b+c+d+e \\ c+e \end{bmatrix}_q$$

$$= \sum_{k\geqslant 0} q^{(k+d)(k+c)} \begin{bmatrix} a+b+c+d+e-k \\ a+b+c+d \end{bmatrix}_q \begin{bmatrix} a+d \\ k+d \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q$$

Cubic q-binomial identities (ctd.)

- Bizely:

$$\begin{bmatrix} A \\ B-D \end{bmatrix}_q \begin{bmatrix} A+D \\ C+D \end{bmatrix}_q = \sum_{K\geqslant 0} q^{(C+D-K)(B-K)} \begin{bmatrix} B \\ K \end{bmatrix}_q \begin{bmatrix} C \\ K-D \end{bmatrix}_q \begin{bmatrix} A+K \\ B+C \end{bmatrix}_q$$

- Takács:

$$\begin{bmatrix} t \\ m-s \end{bmatrix}_q \begin{bmatrix} t-m+s+r \\ s \end{bmatrix}_q = \sum_{j\geqslant 0} q^{(r-j)(s-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q \begin{bmatrix} m-r \\ s-j \end{bmatrix}_q \begin{bmatrix} t+j \\ m \end{bmatrix}_q$$

The cubic q-binomial identity

Theorem (Gutiérrez, Martínez, S., Wildon, '25+)

All cubic binomial identities in literature are equivalent up to simultaneous change of variables. The same holds for their q-analogues.

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A new bijective proof

Proofs available in literature

- Algebraic proof – Gould ('72);
- Partitions inside a rectangle – Andrews, Bressoud ('84); Yee ('08);
- Inversions in a word – Goulden ('85); Zeilberger ('87).

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Our proof

- **Shorter** than existing proofs and a natural extension of the binomial case;
- **Another interpretation** of q-binomials: if q is a prime power, then $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of k-dimensional subspaces of $\mathbb{F}_q^n$;
- **Illustrative example** of lifting bijective proofs (based on counting subsets) of binomial identities to their q-analogues.

Vector space interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q$

Let $0 \leqslant k \leqslant n$ be natural numbers and q be a prime power. Then $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number k-dimensional subspaces of $\mathbb{F}_q^n$.

Vector space interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q$

Let $0 \leqslant k \leqslant n$ be natural numbers and q be a prime power. Then $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number k-dimensional subspaces of $\mathbb{F}_q^n$.

For example, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = q + 1$. Equivalently, there are $q^2 - 1$ non-zero points in $\mathbb{F}_q^2$, and each line contains $q - 1$ non-zero points.

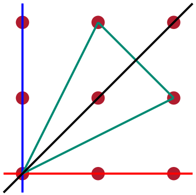

Figure: Case $q = 3$

Outline of the new bijective proof – initial transformation

We first transform the identity

$$\begin{bmatrix} m \\ t \end{bmatrix}_q \begin{bmatrix} m+e \\ s \end{bmatrix}_q = \sum_{j \geqslant 0} q^{(s-j)(t+e-j)} \begin{bmatrix} t+e \\ j \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m+j \\ s+t \end{bmatrix}_q$$

into

$$\begin{bmatrix} m \\ t \end{bmatrix}_q \begin{bmatrix} m+e \\ s \end{bmatrix}_q = \sum_{j \geqslant 0} \sum_{k \geqslant 0} q^{(s-j)(t+e-j)} q^{(m-s-t+k)k} \begin{bmatrix} t+e \\ k \end{bmatrix}_q \begin{bmatrix} t+e-k \\ j-k \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m \\ s+t-k \end{bmatrix}_q.$$

Outline of the new bijective proof – binomial intuition

To illustrate the strategy, we prove the binomial identity

$$\binom{m}{t}\binom{m+e}{s} = \sum_{j\geqslant 0}\sum_{k\geqslant 0}\binom{t+e}{k}\binom{t+e-k}{j-k}\binom{s-e}{s-j}\binom{m}{s+t-k}.$$

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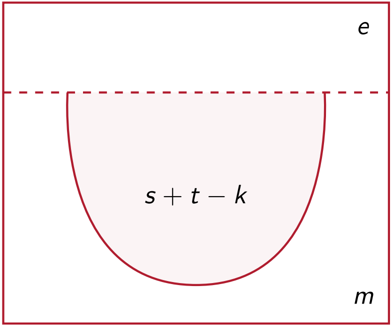

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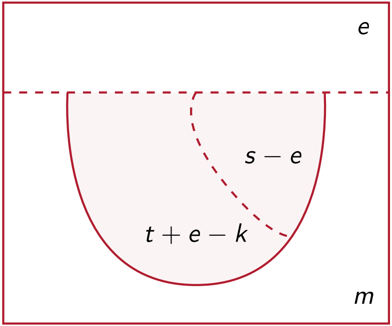

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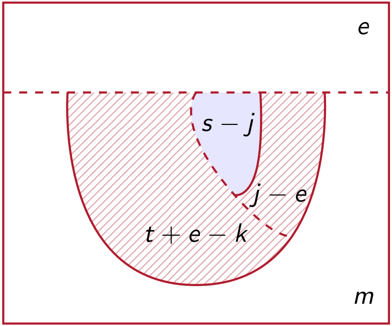

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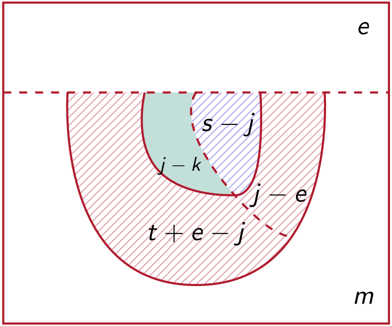

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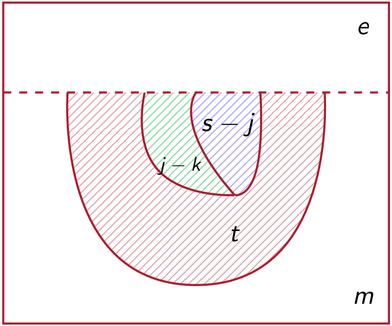

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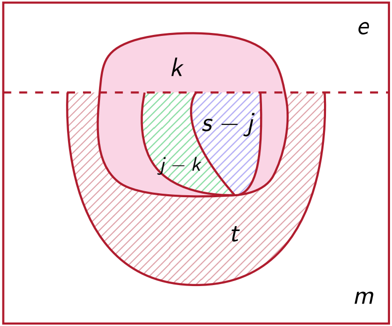

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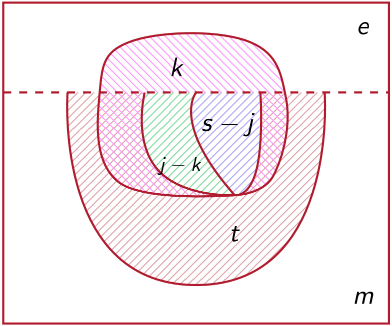

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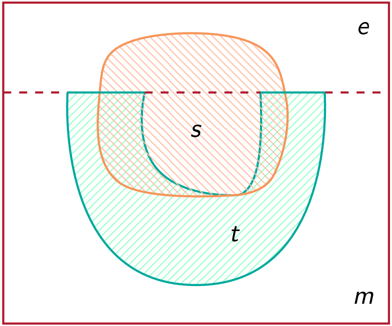

Bijective q-identities (1/2)

Let $l \leqslant k \leqslant n$ be non-negative integers.

Subspace of a subspace identity

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ l \end{bmatrix}_q = \begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} n-l \\ k-l \end{bmatrix}_q$$

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Complements are not unique

A k-dimensional subspace of $\mathbb{F}_q^n$ has $q^{k(n-k)}$ distinct complements.

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Symmetry identity

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$$

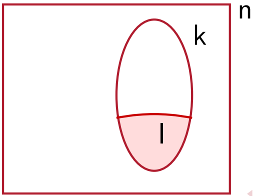
Bijective q-identities (2/2)

In a direct sum of vector spaces we lose control of the vectors which belong to neither of the summands. Instead we consider extesions:

Counting extensions

Let V_k be a k-dimensional subspace of $V = \mathbb{F}_q^n$ and V_l an l-dimensional subspace of V_k. The number of distinct m-dimensional extensions V_m of V_l inside of V, such that $V_m \cap V_k = V_l$, equals

$$q^{(m-l)(k-l)} \begin{bmatrix} n-k \\ m-l \end{bmatrix}_q .$$



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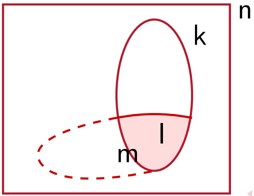
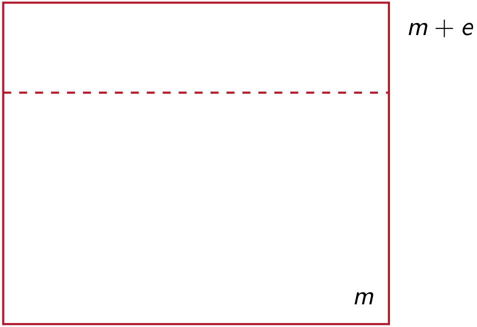


Outline of the new bijective proof

$$\begin{bmatrix} m \\ t \end{bmatrix}_q \begin{bmatrix} m+e \\ s \end{bmatrix}_q = \sum_{j \geqslant 0} \sum_{k \geqslant 0} q^{\cdots} \begin{bmatrix} t+e \\ k \end{bmatrix}_q \begin{bmatrix} t+e-k \\ j-k \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m \\ s+t-k \end{bmatrix}_q$$



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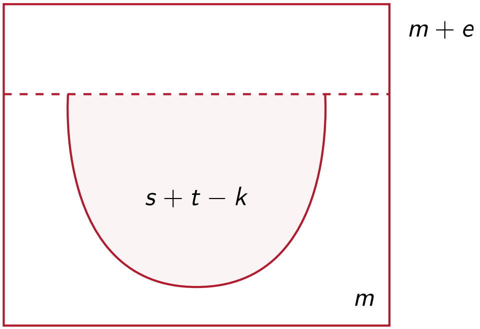

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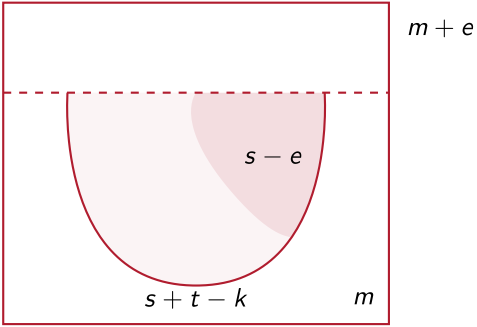

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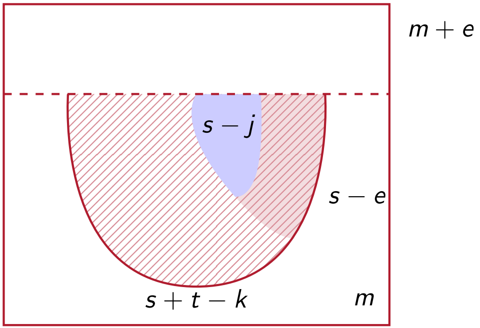

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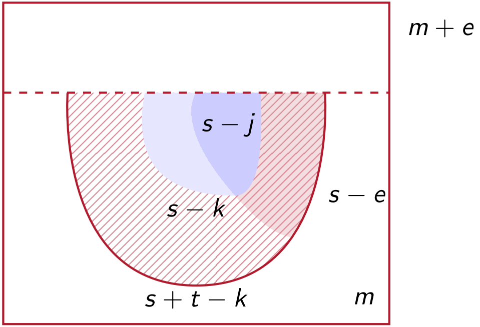

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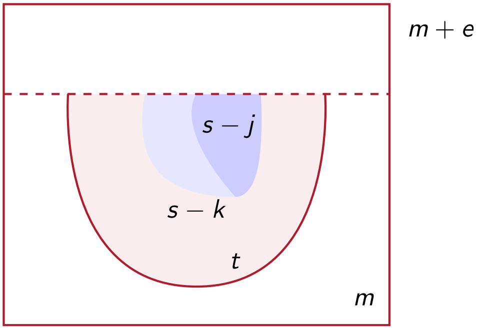

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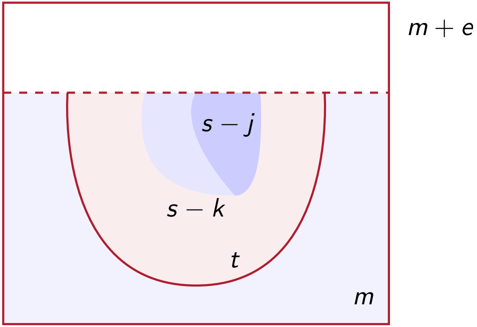

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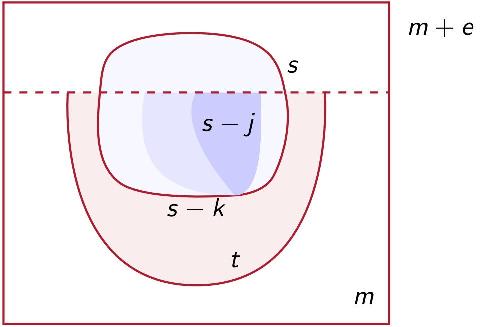

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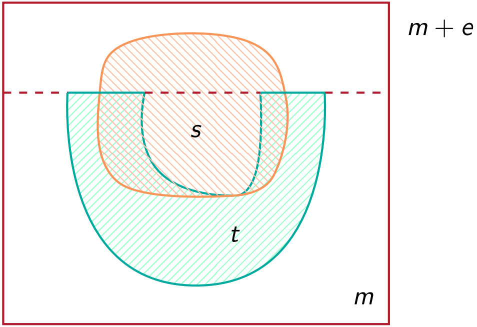

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Quantum group $U_q(\mathfrak{sl}_2)$

Definition (Quantum group $U_q(\mathfrak{sl}_2)$)

The quantum group $U_q(\mathfrak{sl}_2)$ is the unital associative algebra over $\mathbb{Q}(q)$ on the generators E, F, K, K^{-1} subject to the relations

$$KK^{-1} = K^{-1}K = 1, KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F, EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

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Proposition (Triangular decomposition)

The quantum group admits the following decomposition

$$U_q(\mathfrak{sl}_2) = U_q^-(\mathfrak{sl}_2) \otimes_{\mathbb{Q}(q)} U_q^0(\mathfrak{sl}_2) \otimes_{\mathbb{Q}(q)} U_q^+(\mathfrak{sl}_2)$$

where

$$U_q^-(\mathfrak{sl}_2) = \mathbb{Q}(q)[F], \quad U_q^+(\mathfrak{sl}_2) = \mathbb{Q}(q)[E], \quad U_q^0(\mathfrak{sl}_2) = \mathbb{Q}(q)[K, K^{-1}].$$

Lusztig's integral form – multiplication rule for generators

In the defining relations of the integral form, Lusztig gave the multiplication rules for all pairs of generators except one. The following result completes the study of the multiplicative structure of Lusztig's integral form.

Multiplication rule of $\begin{bmatrix} K;c \\ t \end{bmatrix}$ (Gutierréz, Martínez, S., Wildon '25+)

Let b, c, s, t be non-negative integers satisfying $0 \leqslant c \leqslant t$ and $0 \leqslant b \leqslant s$. The generators respect the following multiplication rule:

$$\begin{bmatrix} K;c \\ t \end{bmatrix} \begin{bmatrix} K;b \\ s \end{bmatrix} = \sum_{i \geq 0} \begin{Bmatrix} t-c+b \\ i-c \end{Bmatrix}_q \begin{Bmatrix} s-b+c \\ i-b \end{Bmatrix}_q \begin{bmatrix} K;i \\ t+s \end{bmatrix}.$$

Lusztig's integral form – a new basis

Corollary (Gutierréz, Martínez, S., Wildon '25+)

1. The elements $F^{(a)}\begin{bmatrix} K;c \\ t \end{bmatrix}E^{(b)}$ for $a, b, t \geq 0$ and $c \in \{0, \min(1, t)\}$, form a basis of $U_{\mathcal{A}}(\mathfrak{sl}_2)$ as a free $\mathcal{A}$-module.
2. The $\mathcal{A}$-algebra $U^0_{\mathcal{A}}(\mathfrak{sl}_2)$ has an $\mathcal{A}$-basis

$$\mathcal{B} = \left\{ \begin{bmatrix} K \\ t \end{bmatrix} : t \geq 0 \right\} \cup \left\{ \begin{bmatrix} K; 1 \\ t \end{bmatrix} : t \geq 1 \right\}$$

Lusztig's integral form – a new presentation

Corollary (Gutierréz, Martínez, S., Wildon '25+)

The Cartan subalgebra $U^0_{\mathcal{A}}(\mathfrak{sl}_2)$ has a presentation given by the generators $\begin{bmatrix} K;c \\ t \end{bmatrix}$ for $c \in \mathbb{Z}$, $t \geq 0$, the multiplication relation

$$\begin{bmatrix} K;c \\ t \end{bmatrix} \begin{bmatrix} K;b \\ s \end{bmatrix} = \sum_{i \geq 0} \begin{Bmatrix} t-c+b \\ i-c \end{Bmatrix}_q \begin{Bmatrix} s-b+c \\ i-b \end{Bmatrix}_q \begin{bmatrix} K;i \\ t+s \end{bmatrix},$$

and the relation

$$\begin{bmatrix} K;c+2 \\ t \end{bmatrix} = (q^t + q^{-t}) \begin{bmatrix} K;c+1 \\ t \end{bmatrix} - \begin{bmatrix} K;c \\ t \end{bmatrix} + \begin{bmatrix} K;c \\ t-2 \end{bmatrix}.$$

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The corollary above gives a new presentation of entire integral form.

Thank you! Any questions?

Nanjundiah's identity is equivalent to Székely's identity

$$\begin{bmatrix} m \\ t \end{bmatrix}_q \begin{bmatrix} m+e \\ s \end{bmatrix}_q = \sum_{j \geqslant 0} q^{(s-j)(t+e-j)} \begin{bmatrix} t+e \\ j \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m+j \\ s+t \end{bmatrix}_q$$

$$\Updownarrow$$

$$\begin{bmatrix} a+c+d+e \\ a+c \end{bmatrix}_q \begin{bmatrix} b+c+d+e \\ c+e \end{bmatrix}_q$$

$$= \sum_{k \geqslant 0} q^{(k+d)(k+c)} \begin{bmatrix} a+b+c+d+e-k \\ a+b+c+d \end{bmatrix}_q \begin{bmatrix} a+d \\ k+d \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q$$

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The equivalence follows from simultaneous substitution of variables:

$$(m, e, s, t, j) \mapsto (a+c+d+e, b-a, b+d, a+c, b-k),$$

$$(a, b, c, d, e, i) \mapsto (0, e, t, s-e, e-s-t, e-j).$$

Nanjundiah's identity is equivalent to Stanley's identity

$$\begin{bmatrix} m \\ t \end{bmatrix}_q \begin{bmatrix} m+e \\ s \end{bmatrix}_q = \sum_{j\geqslant 0} q^{(s-j)(t+e-j)} \begin{bmatrix} t+e \\ j \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m+j \\ s+t \end{bmatrix}_q$$

$$\Updownarrow$$

$$\begin{bmatrix} x+A \\ B \end{bmatrix}_q \begin{bmatrix} y+B \\ A \end{bmatrix}_q = \sum_{K\geqslant 0} q^{(A-K)(B-K)} \begin{bmatrix} x+y+K \\ K \end{bmatrix}_q \begin{bmatrix} y \\ A-K \end{bmatrix}_q \begin{bmatrix} x \\ B-K \end{bmatrix}_q$$

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$$\Updownarrow$$

$$\begin{bmatrix} x+A \\ B \end{bmatrix}_q \begin{bmatrix} y+B \\ A \end{bmatrix}_q = \sum_{K \geqslant 0} q^{(A-K)(B-K)} \begin{bmatrix} x+y+K \\ K \end{bmatrix}_q \begin{bmatrix} y \\ A-K \end{bmatrix}_q \begin{bmatrix} x \\ B-K \end{bmatrix}_q$$

The equivalence follows from simultaneous substitution of variables:

$$(m, e, s, t, j) \mapsto (B+y, A+x-B-y, A+x-B, B+y-A, K-B+x),$$

$$(A, B, x, y, K) \mapsto (m-t, m+e-s, t+e, s-e, m-s-t+j).$$

Basic properties (3/3)

Let $m \leqslant n$ be non-negative integers. Then for any non-negative integer $k \leqslant n$:

Vandermonde's convolution (1)

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \sum_{l=0}^{m} q^{(m-l)(k-l)} \begin{bmatrix} k \\ l \end{bmatrix}_q \begin{bmatrix} n-k \\ m-l \end{bmatrix}_q$$

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Vandermonde's convolution (2)

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \sum_{l=0}^{m} q^{l(n-k-m+l)} \begin{bmatrix} k \\ l \end{bmatrix}_q \begin{bmatrix} n-k \\ m-l \end{bmatrix}_q .$$

Lusztig's integral form – generators

Let $[K; a] = \frac{q^a K - q^{-a} K^{-1}}{q - q^{-1}}$.

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$$E^{(n)} = \frac{E^n}{\{n\}_q!} \qquad \text{and} \qquad F^{(n)} = \frac{F^n}{\{n\}_q!},$$

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as well as the elements

$$\begin{bmatrix} K;c \\ t \end{bmatrix} = \frac{[K;c][K;c-1]\cdots[K;c-t+1]}{\{t\}_q!}$$

with $\begin{bmatrix} K;c \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} K \\ t \end{bmatrix} = \begin{bmatrix} K;0 \\ t \end{bmatrix}$.

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with $\begin{bmatrix} K; c \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} K \\ t \end{bmatrix} = \begin{bmatrix} K; 0 \\ t \end{bmatrix}$.
The element $\begin{bmatrix} K; c \\ t \end{bmatrix}$ should be thought of as a quantization of the element

$$\binom{H+c}{t} = \frac{(H+c)(H+c-1) \cdots (H+c-t+1)}{t!},$$

which belongs to Kostant's $\mathbb{Z}$-form of the enveloping algebra of $\mathfrak{sl}_2$.

Lusztig's integral form – definition and relations

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Lusztig (88') defined an integral form $U_{\mathcal{A}}(\mathfrak{sl}_2)$ for $U_q(\mathfrak{sl}_2)$ as the $\mathcal{A}$-algebra generated by the divided powers $E^{(n)}$ and $F^{(n)}$, the elements K, K^{-1} and the elements $\begin{bmatrix} K \\ t \end{bmatrix}$ for $t \geq 1$.

Theorem (Lusztig, 88')

The generators of the integral form satisfy the following relations

$$E^{(n)}E^{(m)} = \begin{Bmatrix} n+m \\ n \end{Bmatrix}_q E^{(n+m)}, \qquad F^{(n)}F^{(m)} = \begin{Bmatrix} n+m \\ n \end{Bmatrix}_q F^{(n+m)},$$

$$\begin{bmatrix} K; c \\ t \end{bmatrix} E^{(n)} = E^{(n)} \begin{bmatrix} K; c+2n \\ t \end{bmatrix}, \qquad \begin{bmatrix} K; c \\ t \end{bmatrix} F^{(n)} = F^{(n)} \begin{bmatrix} K; c-2n \\ t \end{bmatrix},$$

$$E^{(n)}F^{(m)} = \sum_{t \geqslant 0} F^{(m-t)} \begin{bmatrix} K; 2t-m-n \\ t \end{bmatrix} E^{(n-t)}.$$

Lusztig's integral form – basis

Proposition (Lusztig, 90')

The $\mathcal{A}$-algebras $U_{\mathcal{A}}(\mathfrak{sl}_2)$ and $U^0_{\mathcal{A}}(\mathfrak{sl}_2)$ are free as $\mathcal{A}$-modules.

1. The $\mathcal{A}$-algebra $U_{\mathcal{A}}(\mathfrak{sl}_2)$ has an $\mathcal{A}$-basis given by the elements
$$F^{(a)} K^{\delta} \begin{bmatrix} K \\ t \end{bmatrix} E^{(b)}$$
for $a, b, t \geq 0$ and $\delta \in \{0, \min(1, t)\}$.
2. The elements $K^{\delta} \begin{bmatrix} K \\ t \end{bmatrix}$ for $t \geq 0$ and $\delta \in \{0, \min(1, t)\}$ form a basis for the Cartan subalgebra $U^0_{\mathcal{A}}(\mathfrak{sl}_2)$.

Lusztig's integral form – presentation (ctd.)

Corollary (Gutierréz, Martínez, S., Wildon 25'+)

Lusztig's integral form $U_{\mathcal{A}}(\mathfrak{sl}_2)$ has a presentation given by the monomials $E^{(n)} \begin{bmatrix} K;c \\ t \end{bmatrix} F^{(m)}$ for $n, m, t \geq 0$, $c \in \mathbb{Z}$, relations

$$E^{(n)} E^{(m)} = \begin{Bmatrix} n+m \\ n \end{Bmatrix}_q E^{(n+m)}, \qquad F^{(n)} F^{(m)} = \begin{Bmatrix} n+m \\ n \end{Bmatrix}_q F^{(n+m)},$$

$$\begin{bmatrix} K;c \\ t \end{bmatrix} E^{(n)} = E^{(n)} \begin{bmatrix} K;c+2n \\ t \end{bmatrix}, \qquad \begin{bmatrix} K;c \\ t \end{bmatrix} F^{(n)} = F^{(n)} \begin{bmatrix} K;c-2n \\ t \end{bmatrix},$$

$$E^{(n)} F^{(m)} = \sum_{t \geqslant 0} F^{(m-t)} \begin{bmatrix} K;2t-m-n \\ t \end{bmatrix} E^{(n-t)},$$

with the multiplication relation

$$\begin{bmatrix} K;c \\ t \end{bmatrix} \begin{bmatrix} K;b \\ s \end{bmatrix} = \sum_{i \geq 0} \begin{Bmatrix} t-c+b \\ i-c \end{Bmatrix}_q \begin{Bmatrix} s-b+c \\ i-b \end{Bmatrix}_q \begin{bmatrix} K;i \\ t+s \end{bmatrix},$$

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