Commuting graphs of completely 0-simple semigroups

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Commuting graphs of semigroups

Let S be a finite non-commutative semigroup.

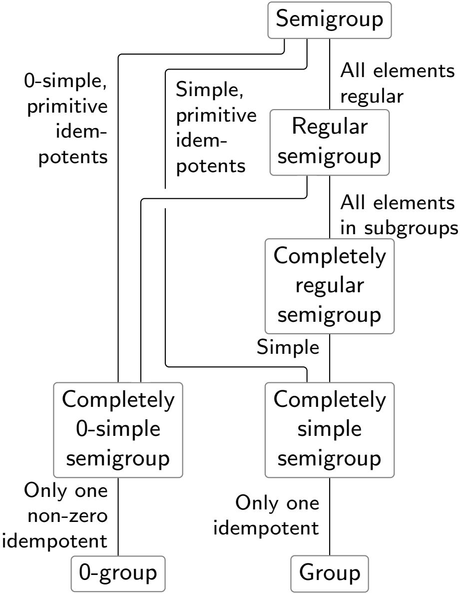
The **commuting graph** of S, denoted $\mathcal{G}(S)$, is the simple graph such that:

- $S \setminus Z(S)$ is the set of vertices, where

$$Z(S) = \{x \in S : xy = yx \text{ for all } y \in S\}.$$

- $\{x, y\}$ is an edge if and only if $x \neq y$ and $xy = yx$.

Completely 0-simple semigroups



Theorem (Rees–Suschkewitsch Theorem)

A semigroup S is completely 0-simple if and only if there exist a group G, index sets I and Λ, and a regular $\Lambda \times I$ matrix P with entries from G^0 such that $S \simeq \mathcal{M}_0[G; I, \Lambda; P]$.

0-Rees matrix semigroup over a group

Let G be a group, I and Λ be index sets, and P be a regular $\Lambda \times I$ matrix with entries from G^0.

Let $p_{\lambda i}$ be the (λ, i)-th entry of P.

A 0-**Rees matrix semigroup over a group**, denoted $\mathcal{M}_0[G; I, \Lambda; P]$, is the set $(I \times G \times \Lambda) \cup \{0\}$ with the multiplication

$$(i, x, \lambda)(j, y, \mu) = \begin{cases} (i, x p_{\lambda j} y, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$

$$0(i, x, \lambda) = (i, x, \lambda)0 = 00 = 0.$$

Zeros are important

Lemma

Let P and Q be regular $\Lambda \times I$ matrices with entries from G^0. If for all $i \in I$ and $\lambda \in \Lambda$ we have $p_{\lambda i} = 0$ if and only if $q_{\lambda i} = 0$, then the graphs $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ and $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; Q])$ are isomorphic.

Example

Let $e, g, h \in G$.

$$\begin{bmatrix} e & 0 & 0 \\ 0 & g & h \\ e & 0 & g \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ \times & 0 & \times \end{bmatrix}$$

Reordering columns and rows implies isomorphism

Lemma

Let Q be the matrix obtained from P by reordering the columns and rows of P. Then the graphs $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ and $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; Q])$ are isomorphic.

Example

$$\begin{bmatrix} \times & 0 & \times & \times \\ 0 & \times & 0 & \times \\ \times & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \times & \times & \times & 0 \\ 0 & \times & 0 & \times \\ \times & 0 & 0 & 0 \end{bmatrix}$$

$|I| = |\Lambda| = 1$: characterization of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

- $\mathcal{M}_0[G; I, \Lambda; P] \simeq G^0$.
- Suppose that G is non-abelian. Then the graphs $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ and $\mathcal{G}(G)$ are isomorphic.

$|I| > 1$ or $|\Lambda| > 1$, P has no zeros: connectedness

- $Z(\mathcal{M}_0[G; I, \Lambda; P]) = \{0\}$.
- $\mathcal{M}_0[G; I, \Lambda; I]$ is non-commutative.

Theorem (P., 2024)

- *$\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is not connected.*
- *The connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ are the graphs induced by $\{i\} \times G \times \{\lambda\}$, $i \in I$, $\lambda \in \Lambda$.*
- *Let $\mathcal{C}$ be a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$. Then*

$$\mathcal{C} \simeq \begin{cases} K_{|G|} & \text{if } G \text{ is abelian,} \\ K_{|Z(G)|} \nabla \mathcal{G}(G) & \text{if } G \text{ is non-abelian.} \end{cases}$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: 0-closure method

Example

$$\begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: 0-closure method

Example

$$\begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: 0-closure method

Example

$$\begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: 0-closure method

Example

$$\begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \boxed{0} & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \boxed{0} & 0 & \boxed{\times} \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{\times} \\ \times & 0 & \boxed{0} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \boxed{\times} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{\times} \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix}$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: 0-closure method

Example

$$\begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \boxed{0} & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \boxed{0} & 0 & \boxed{\times} \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{\times} \\ \times & 0 & \boxed{0} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \boxed{\times} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{\times} \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix}$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: 0-closure method

Example

$$
\begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix}
\rightsquigarrow
\begin{bmatrix} \boxed{0} & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix}
\rightsquigarrow
\begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \boxed{0} & 0 & \times \\ \times & 0 & 0 \end{bmatrix}
$$

$$
\rightsquigarrow
\begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \boxed{0} & 0 & \boxed{\times} \\ \times & 0 & 0 \end{bmatrix}
\rightsquigarrow
\begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{\times} \\ \times & 0 & \boxed{0} \end{bmatrix}
\rightsquigarrow
\begin{bmatrix} \boxed{0} & \boxed{\times} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{\times} \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix}
$$

Example

$$\begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{array}{cc} & i \\ \lambda & \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\zeta(\lambda, i) = (\#\ \text{submatrices constructed}) - 1 = 2$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: connectedness

Theorem (P., 2024)

The following conditions are equivalent:

- $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *is connected.*
- *For all* $i \in I$ *and* $\lambda \in \Lambda$ *the* 0*-closure starting at* (λ, i) *is* P*.*
- *For some* $i \in I$ *and* $\lambda \in \Lambda$ *the* 0*-closure starting at* (λ, i) *is* P*.*

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \end{array} &
\begin{bmatrix}
\times & \times & \times & 0 & \times \\
0 & \times & \times & \times & \times \\
\times & 0 & \times & \times & \times \\
\times & 0 & \times & 0 & \times \\
\times & \times & 0 & \times & 0 \\
\times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times
\end{bmatrix}
\end{array}
\rightsquigarrow
\begin{array}{cc}
 & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \end{array} &
\begin{bmatrix}
\times & \times & \times & 0 & \times \\
0 & \times & \times & \times & \times \\
\times & 0 & \times & \times & \times \\
\times & 0 & \times & 0 & \times \\
\times & \times & 0 & \times & 0 \\
\times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times
\end{bmatrix}
\end{array}
$$

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\ \begin{bmatrix} \times & \times & \times & 0 & \times \\ 0 & \times & \times & \times & \times \\ \times & 0 & \times & \times & \times \\ \times & 0 & \times & 0 & \times \\ \times & \times & 0 & \times & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\ \begin{bmatrix} \times & \times & \times & 0 & \times \\ 0 & \times & \times & \times & \times \\ \times & 0 & \times & \times & \times \\ \times & 0 & \times & 0 & \times \\ \times & \times & 0 & \times & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times \end{bmatrix} \end{array}$$

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

$$\rightsquigarrow \begin{array}{cc} & \begin{array}{ccccc} 2 & 4 & 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 4 \\ 2 \\ 5 \\ 6 \\ 7 \end{array} & \left[\begin{array}{cc|ccc} \times & 0 & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ \hline \times & \times & 0 & \times & \times \\ \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times \end{array}\right] \end{array}$$

$$\rightsquigarrow \begin{array}{cc} & \begin{array}{ccccc} 2 & 4 & 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 4 \\ 2 \\ 5 \\ 6 \\ 7 \end{array} & \left[\begin{array}{cc|c|cc} \times & 0 & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ \hline \times & \times & 0 & \times & \times \\ \hline \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times \end{array}\right] \end{array}$$

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

$$\rightsquigarrow \quad \begin{array}{cc} & \begin{array}{cc|ccc} \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{3} & \mathbf{5} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{2} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \end{array} & \left[\begin{array}{cc|ccc} \times & 0 & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ \hline \times & \times & 0 & \times & \times \\ \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times \end{array}\right] \end{array} \qquad \rightsquigarrow \quad \begin{array}{cc} & \begin{array}{cc|c|cc} \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{3} & \mathbf{5} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{2} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \end{array} & \left[\begin{array}{cc|c|cc} \times & 0 & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ \hline \times & \times & 0 & \times & \times \\ \hline \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times \end{array}\right] \end{array}$$

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

$$
\rightsquigarrow \quad
\begin{array}{c}
\\ 1 \\ 3 \\ 4 \\ 2 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{3} & \mathbf{5} \end{array} \\
\left[\begin{array}{cc|c|cc}
\times & 0 & \times & \times & \times \\
0 & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times \\
\hline
\times & \times & 0 & \times & \times \\
\hline
\times & \times & \times & 0 & 0 \\
\times & \times & \times & \times & 0 \\
\hline
\times & \times & \times & \times & \times
\end{array}\right]
\end{array}
$$

The subgraph induced by $\{2, 4\} \times G \times \{1, 3, 4\}$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

$$\rightsquigarrow \quad \begin{array}{cc}
 & \begin{array}{cc|c|cc} \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{3} & \mathbf{5} \end{array} \\
\begin{array}{c} \mathbf{1} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{2} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \end{array} &
\left[\begin{array}{cc|c|cc}
\times & 0 & \times & \times & \times \\
0 & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times \\ \hline
\times & \times & 0 & \times & \times \\ \hline
\times & \times & \times & 0 & 0 \\
\times & \times & \times & \times & 0 \\ \hline
\times & \times & \times & \times & \times
\end{array}\right]
\end{array}$$

The subgraph induced by $\{2, 4\} \times G \times \{1, 3, 4\}$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

$$\begin{array}{cc} & \begin{array}{ccccc} \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{3} & \mathbf{5} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{2} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \end{array} & \left[\begin{array}{cc|c|cc} \times & 0 & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ \hline \times & \times & 0 & \times & \times \\ \hline \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ \hline \times & \times & \times & \times & \times \end{array}\right] \end{array}$$

The subgraph induced by $(\{1\} \times G \times \{1,3,4\}) \cup (\{2,4\} \times G \times \{2\})$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

$$\begin{array}{cc}
 & \begin{array}{cc|c|cc} \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{3} & \mathbf{5} \end{array} \\
\begin{array}{c} \mathbf{1} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{2} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \end{array} &
\left[\begin{array}{cc|c|cc}
\times & 0 & \times & \times & \times \\
0 & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times \\
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\times & \times & \times & \times & 0 \\
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\times & \times & \times & \times & \times
\end{array}\right]
\end{array}$$

The subgraph induced by $(\{1\} \times G \times \{1,3,4\})$ $\cup(\{2,4\} \times G \times \{2\})$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

Example (cont.)

$$
\begin{array}{c}
\begin{array}{cc|c|cc} 2 & 4 & 1 & 3 & 5 \end{array} \\
\end{array}
$$

$$
\begin{array}{c c}
 & \begin{array}{cc|c|cc} \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{3} & \mathbf{5} \end{array} \\
\begin{array}{c} \mathbf{1} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{2} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \end{array} &
\left[\begin{array}{cc|c|cc}
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0 & \times & \times & \times & \times \\
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\times & \times & \times & \times & \times
\end{array}\right]
\end{array}
$$

The subgraph induced by $\{2\} \times G \times \{7\}$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

$$\begin{array}{c} \\ \mathbf{1} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{2} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{3} & \mathbf{5} \end{array} \\ \left[\begin{array}{cc|c|cc} \times & 0 & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ \hline \times & \times & 0 & \times & \times \\ \hline \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ \hline \boxed{\times} & \times & \times & \times & \times \end{array}\right] \end{array}$$

The subgraph induced by $\boxed{\{2\} \times G \times \{7\}}$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: diameter

Let $H = (V, E)$ be a simple graph.

- The **diameter** of H, denoted $\mathrm{diam}(H)$, corresponds to the maximum distance between vertices of H, that is,

$$\max\{d(u, v) : u, v \in V\}.$$

- The diameter of H is finite if and only if H is connected.

Theorem (P., 2024)

Suppose that $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected. Then

$$\mathrm{diam}(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = \max\{\, \zeta(\lambda, i) : i \in I,\ \lambda \in \Lambda \text{ and } p_{\lambda i} = 0 \,\}.$$

$|I| > 1$ or $|\Lambda| > 1$, P has no zeros: clique number

Let $H = (V, E)$ be a simple graph.

- A **clique** is a subset $K \subseteq V$ such that $\{u, v\} \in E$, for all distinct $u, v \in K$.
- The **clique number** of H, denoted $\omega(H)$, is the largest integer r such that H has a clique K such that $|K| = r$.

Theorem (P., 2024)

$$\omega(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = \begin{cases} |G| & \textit{if } G \textit{ is abelian,} \\ \omega(\mathcal{G}(G)) + |Z(G)| & \textit{if } G \textit{ is non-abelian.} \end{cases}$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: clique number

Theorem (P., 2024)

- *Suppose that G is abelian. If*

$$\begin{bmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{bmatrix}$$

is a submatrix of P and

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are not submatrices of P, then

$$\omega(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 3|G|.$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: clique number

Theorem (cont.) (P., 2024)

- *Suppose that G is abelian. If*

$$\begin{bmatrix} \times & 0 \\ 0 & \times \end{bmatrix}$$

is a submatrix of P and

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

are not submatrices of P, then

$$\omega(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 2|G|.$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: clique number

Theorem (cont.) (P., 2024)

- *For the remaining cases, we have*

$$\omega(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = |G| \cdot \max\{nm : 0_{n\times m} \textit{ is a submatrix of } P\}.$$

$|I| > 1$ or $|\Lambda| > 1$, P has no zeros: chromatic number

Let $H = (V, E)$ be a simple graph.

- The **chromatic number** of H, denoted $\chi(H)$, is the minimum number of colours necessary to colour the vertices of H in such a way that no adjacent vertices have the same colour.

Theorem (P., 2024)

$$\chi(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = \begin{cases} |G| & \textit{if } G \textit{ is abelian,} \\ \chi(\mathcal{G}(G)) + |Z(G)| & \textit{if } G \textit{ is non-abelian.} \end{cases}$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: chromatic number

Theorem (P., 2025)

- *Suppose that* $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *is connected. Then*

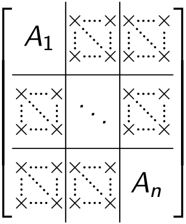
$$\chi(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) \leqslant |G| \cdot \big|\{ (i, \lambda) \in I \times \Lambda : p_{\lambda i} = 0 \}\big|.$$

- *Suppose that* $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *is not connected. Then*

$$\chi(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) \leqslant |G| \cdot \max\{2, z_1, \ldots, z_n\},$$

where for each $i \in \{1, \ldots, n\}$

$$z_i = \big|\{(i, \lambda) \in I \times \Lambda : p_{\lambda i} = 0 \text{ and } p_{\lambda i} \text{ is an entry of } A_i\}\big|.$$



$|I| > 1$ or $|\Lambda| > 1$, P has no zeros: girth

Let $H = (V, E)$ be a simple graph.

- The **girth** of H, denoted $\mathrm{girth}(H)$, is the length of a shortest cycle contained in H.

Theorem (P., 2024)

$\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *has cycles if and only if* $|G| \geqslant 3$, *in which case*

$$\mathrm{girth}(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 3.$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: girth

Theorem (P., 2024)

- *Suppose that* $|G| \geqslant 3$. *Then*

$$girth(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 3.$$

- *Suppose that* $|G| = 2$. *Then* $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *contains cycles if and only if* P *contains more than one zero entry, in which case*

$$girth(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 3.$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: girth

Theorem (cont.) (P., 2024)

- *Suppose that* $|G| = 1$. *Then* $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *contains cycles if and only if at least one of the following matrices is a submatrix of* P.

$$\mathrm{girth}(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 3$$

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{bmatrix}$$

$$\mathrm{girth}(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 4$$

$$\begin{bmatrix} 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \times \\ 0 & \times \\ \times & 0 \\ \times & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \times \\ 0 & 0 \\ \times & 0 \end{bmatrix}$$

$|I| > 1$ or $|\Lambda| > 1$, P has no zeros: knit degree

Let S be a finite non-commutative semigroup.

- A path $a_1 - a_2 - \cdots - a_k$ in $\mathcal{G}(S)$ is called a **left path** if $a_1 \neq a_k$ and $a_1 a_i = a_k a_i$, for all $i \in \{1, \ldots, k\}$.
- Suppose $\mathcal{G}(S)$ has a left path. The **knit degree** of S, denoted $\mathrm{kd}(S)$, is the length of a shortest left path in $\mathcal{G}(S)$.

Theorem (P., 2024)

$\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *contains no left paths.*

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: knit degree

Theorem (P., 2024)

$\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ contains left paths if and only if at least one of the following conditions is satisfied:

- $|G| > 1$.
- *At least one of the following matrices is a submatrix of P.*

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In this case,

$$kd(\mathcal{M}_0[G; I, \Lambda; P]) = 1.$$

Completely 0-simple semigroups

Theorem (Rees–Suschkewitsch Theorem)

A semigroup S is completely 0-simple if and only if there exist a group G, index sets I and Λ, and a regular $\Lambda \times I$ matrix P with entries from G^0 such that $S \simeq \mathcal{M}_0[G; I, \Lambda; P]$.

Commuting graph of completely 0-simple semigroups

Theorem (P., 2024)

- *For each $n \in \mathbb{N}$, there is a completely 0-simple semigroup S such that $\omega(\mathcal{G}(S)) = n$.*
- *For each $n \in \mathbb{N}$, there is a completely 0-simple semigroup S such that $\chi(\mathcal{G}(S)) = n$.*
- *For each $n \in \mathbb{N}$ such that $n \geqslant 2$, there is a completely 0-simple semigroup S such that $\operatorname{diam}(\mathcal{G}(S)) = n$.*
- *Let S be a finite non-commutative completely 0-simple semigroup. If $\mathcal{G}(S)$ contains cycles, then $\operatorname{girth}(\mathcal{G}(S)) \in \{3, 4\}$.*
- *Let S be a finite non-commutative completely 0-simple semigroup. If $\mathcal{G}(S)$ contains left paths, then $\operatorname{kd}(S) = 1$.*

Thank you!